DEPARTMENT OF ELECTRICAL & ELECTRONICS ENGINEERING

DIGITAL COMMUNICATION
Spring 2010
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OUTLINE

- Probability of Error due to Noise
- Intersymbol Interference
PROBABILITY OF ERROR DUE TO NOISE

- Since the matched filter is the optimum detector of a known pulse in additive noise, the probability of error rate due to the noise can be calculated.
- Let's consider polar NRZ signaling, where symbols 1 and 0 are represented by positive and negative pulses by equal amplitude.
- The received signal in additive noise is

\[ x(t) = \begin{cases} 
  + A + w(t) & \text{symbol 1 was sent} \\
  - A + w(t) & \text{symbol 0 was sent} 
\end{cases} \]

PROBABILITY OF ERROR DUE TO NOISE

- Let's assume that the receiver knows the starting and ending times of each transmitted pulse.
- The receiver samples the output of the matched filter at \( t = T_b \) and decides whether the transmitted symbol is 0 or 1 from the received noisy signal \( x(t) \)
PROBABILITY OF ERROR DUE TO NOISE

- In decision device, the sampled value is compared with threshold value $\lambda$.
  - If the sampled signal is above $\lambda$ the receiver assumes that the transmitted symbol is 1.
  - If the sampled signal is below $\lambda$ the receiver assumes that the transmitted symbol is 0.
  - If the sampled signal is equal $\lambda$ the receiver makes a guess so that the outcome does not alter the average probability of error.

- There are two possible kinds of errors
  - Symbol 1 is chosen when actually 0 was transmitted.
  - Symbol 0 is chosen when actually 1 was transmitted.

- To determine the probability of error, these two situations should be analyzed separately.

PROBABILITY OF ERROR DUE TO NOISE

- Suppose that symbol 0 was sent, then the received signal is
  \[ x(t) = -A + w(t) \quad 0 \leq t \leq T_b \]

- The signal at the matched filter output sampled at $t=T_b$ becomes
  \[ y = \int_{0}^{T_b} x(t) dt = -A + \frac{1}{T_b} \int_{0}^{T_b} w(t) dt \]

- which represents the random variable $Y$
PROBABILITY OF ERROR DUE TO NOISE

Since the additive noise is Gaussian distributed, the variance of the random variable $Y$ is

$$
\sigma_Y^2 = E[(Y + A)^2] \\
= \frac{1}{T_b} E \left[ \int_0^{T_b} \int_0^{T_b} w(t)w(u) dt du \right] \\
= \frac{1}{T_b} \int_0^{T_b} \int_0^{T_b} E[w(t)w(u)] dt du \\
= \frac{1}{T_b} \int_0^{T_b} \int_0^{T_b} \rho_{WW}(t,u) dt du
$$

$\rho_{WW}(t,u)$ is the autocorrelation function of the white noise $w(t)$.

PROBABILITY OF ERROR DUE TO NOISE

Since the power spectral density of the white noise is $N_0/2$, $\rho_{WW}(t,u)$ is

$$
\rho_{WW}(t,u) = \frac{N_0}{2} \delta(t-u)
$$

The variance of $Y$ becomes

$$
\sigma_Y^2 = \frac{1}{T_b} \int_0^{T_b} \int_0^{T_b} \frac{N_0}{2} \delta(t-u) dt du \\
= \frac{N_0}{2T_b}
$$
PROBABILITY OF ERROR DUE TO NOISE

- The probability density function of the random variable X which has a Gaussian distribution is

\[ f_X(x) = \frac{1}{\sqrt{2\pi}\sigma_X} e^{-\frac{(x-\mu_X)^2}{2\sigma_X^2}} \]

- Therefore the probability density function of the random variable Y, given that symbol 0 has sent

\[ f_Y(y | 0) = \frac{1}{\sqrt{\pi N_0 / T_b}} e^{-\frac{(y+\Delta)^2}{N_0 / T_b}} \]

PROBABILITY OF ERROR DUE TO NOISE

- The probability density function is plotted below. 

\[ P_{e|0} \] denotes the conditional probability of error, given that symbol 0 has sent.

\[ f_Y(y | 0) \quad f_Y(y | 1) \]

- The shaded area from \( \lambda \) to infinity corresponds to the range of values that are assumed as symbol 1.

- In the absence of noise the decision is always symbol 0 for the sampled value of \(-A\).
**Probability of Error due to Noise**

- The probability of error on the condition that symbol 0 has sent is

$$P_{e0} = P(y > \lambda | \text{symbol 0 was sent})$$

$$= \int_{\lambda}^{\infty} f_y(y | 0) \, dy$$

$$= \frac{1}{\sqrt{\pi N_0 / T_b}} \int_{\lambda}^{\infty} e^{-\frac{(y+4\lambda)^2}{N_0 / T_b}} \, dy$$

**Probability of Error due to Noise**

- The value of the threshold $\lambda$ can be assigned with the probabilities of symbols 0 and 1 denoted by $p_0$ and $p_1$.
- It is clear that the sum of $p_0$ and $p_1$ must be 1.
- If we assume that symbols 0 and 1 are occurred with equal probabilities ($p_0=p_1=0.5$), it reasonable to assume the threshold halfway between $+A$ (symbol 1) and $-A$ (symbol 0).

$$\lambda = 0$$
PROBABILITY OF ERROR DUE TO NOISE

- The conditional probability of error when symbol 0 was sent becomes

\[ P_{e0} = \frac{1}{\sqrt{\pi N_0/T_b}} \int_0^\infty e^{-\frac{(y+\Delta)^2}{2N_0/T_b}} \, dy \]

- Let’s define a new variable \( z \)

\[ z = \frac{y + \Delta}{\sqrt{N_0/2T_b}} \]

PROBABILITY OF ERROR DUE TO NOISE

- Let’s reformulate \( P_{e0} \)

\[ P_{e0} = \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-\frac{z^2}{2E_b/N_0}} \, dz \]

- where \( E_b \) is the transmitted signal energy per bit

\[ E_b = A^2 T_b \]
**THE Q-FUNCTION**

- The Q-function is used by communication engineers to determine the area under the tails of the Gaussian distribution.
- Error function is the twice of the area under a normalized Gaussian from 0 to \( u \). The complementary error function is defined as one minus the error function

\[
\text{erfc}(u) = 2 \int_{u}^{\infty} e^{-z^2} \, dz
\]

The Q-function and complementary error function is related to each other

\[
Q(u) = \frac{1}{2} \text{erfc} \left( \frac{u}{\sqrt{2}} \right)
\]

**PROBABILITY OF ERROR DUE TO NOISE**

- The conditional probability of error \( P_{e0} \) can be written in terms of Q-function.

\[
P_{e0} = Q \left( \frac{2E_b}{N_0} \right)
\]

- For the conditional probability of error \( P_{e1} \), the mean is \( +A \)

\[
f_y(y|1) = \frac{1}{\sqrt{\pi N_0 / T_b}} e^{\frac{(y-A)^2}{N_0 / T_b}}
\]
**Probability of Error due to Noise**

- The probability of error $P_{e1}$ is area from $-\infty$ to $\lambda$.
  
  $\quad \quad P_{e1} = P(y > \lambda \mid \text{symbol 1 was sent})$
  
  $\quad = \int_{-\infty}^{\lambda} f_Y(y \mid 1)dy$
  
  $\quad = \frac{1}{\sqrt{\pi N_0 / T_b}} \int_{-\infty}^{\lambda} e^{-(y-A)^2 / N_0 / T_b} dy$

- Setting the threshold to $\lambda=0$ and putting
  
  $\quad z = \frac{y - A}{\sqrt{N_0 / 2T_b}}$

- results in $P_{e1}=P_{e0}$

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**Probability of Error due to Noise**

- The average probability of symbol error $P_e$ is
  
  $\quad P_e = p_0 P_{e0} + p_1 P_{e1}$

- Since $P_{e0}=P_{e1}$ and $p_0=p_1=0.5$
  
  $\quad P_e = P_{e0} = P_{e1}$

  $\quad P_e = Q \left( \sqrt{\frac{2E_b}{N_0}} \right)$

- The average probability of symbol error with binary signaling only depends on $E_b/N_0$, the ratio of the transmitted signal energy per bit to the noise spectral density.
PROBABILITY OF ERROR DUE TO NOISE

Matlab Code:

clear all;
close all;

EbN0_dB=0:1e-3:10;
EbN0=10.^(EbN0_dB/10);

Pe=0.5*erfc(sqrt(2*EbN0)/sqrt(2));

figure(1);
semilogy(EbN0_dB,Pe,'linewidth',2);

xlabel('E_b/N_0 (dB)');
ylabel('Probability of Error P_e');
grid;

This waterfall curve makes the digital systems superior compared to analog systems if E_b/N_0 is above few dBs.

INTERSYMBOL INTERFERENCE

- Intersymbol Interference (ISI) happens when the channel has a frequency dependent amplitude spectrum.
- The simplest case is band-limited channel.
  - For example, a brick-wall band-limited channel passes all frequencies |f|<W without distortion and blocks all frequencies |f|>W.
- Consider a polar signaling where

\[ a_k = \begin{cases} 
+1 & \text{symbol 1 was sent} \\
-1 & \text{symbol 0 was sent} 
\end{cases} \]
**INTERSYMBOL INTERFERENCE**

- Once the impulse modulator output is passed through the transmit filter with the impulse response of the pulse \( g(t) \), the transmitted signal becomes

\[
s(t) = \sum_k a_k g(t - kT_b)
\]

- \( s(t) \) passes through the channel with impulse response of \( h(t) \) added white Gaussian noise and received by the receiver.

- The receiver passes the received signal through the receive filter, which is the matched filter for the optimum noise performance.

**INTERSYMBOL INTERFERENCE**

- The output of the receive filter is

\[
y(t) = \mu \sum_k a_k p(t - kT_b) + n(t)
\]

sampled at \( t=T_b \) and the decision device decides the bits.

- The receive filter output is

\[
y(t_i) = \mu \sum_{k=-\infty}^{\infty} a_k p[(i-k)T_b] + n(t_i)
\]

\[=
\mu a_{i} + \sum_{k=-\infty}^{\infty} a_k p[(i-k)T_b] + n(t_i)\]
INTERSYMBOL INTERFERENCE

- The first term is the contribution of the \( i \)th transmitted bit.
- The second term represents the residual effect of all other transmitted bits on the decoding of \( i \)th bit.
- This residual effect due to the occurrence of pulses before and after the sampling instant \( t_i \) is called intersymbol interference (ISI).
- The last term \( n(t_i) \) represents the noise sample at the time \( t_i \).
- When the signal-to-noise ratio is high, the system is limited by ISI rather than noise.

THE TELEPHONE CHANNEL

- The frequency response of a telephone channel is shown below.
THE TELEPHONE CHANNEL

- The properties of this channel
  - The pass-band of the channel cuts off rapidly above 3.5KHz. Therefore, we should use a line code with a narrow spectrum, so we can maximize the data rate.
  - The channel does not pass DC. It is preferable to use a line code that has no DC component such as bipolar signaling or Manchester code.

- These two requirements are contradictory.
  - The polar NRZ has narrow spectrum but has DC
  - The Manchester has no DC, but requires higher bandwidth.

THE TELEPHONE CHANNEL

- For the data rate 1600bps, the signals are shown below.

- a is NRZ line code and b is Manchester line code.
- Since NRZ requires DC, the signal drifts to zero volt especially when there is long strings of the symbols of the same polarity.
- Manchester code has no DC drift but has distortion.
THE TELEPHONE CHANNEL

- For the data rate 3200bps, the DC drift in NRZ is less evident but there is still considerable signal distortion.
- The Manchester code has significant spectral components outside the band limits of this channel.