ÇANKAYA UNIVERSITY
Department of Mathematics and Computer Science

MATH 365
Elementary Number Theory I
1st Midterm
SOLUTIONS
November 12, 2007
16:40-18:00

Surname : ____________________
Name : ____________________
ID # : ____________________
Department : ____________________
Section : ____________________
Instructor : ____________________
Signature : ____________________

- The exam consists of 6 questions.
- Please read the questions carefully and write your answers under the corresponding questions. Be neat.
- Show all your work. Correct answers without sufficient explanation might not get full credit.
- Calculators are not allowed.

GOOD LUCK!

Please do not write below this line.

Q1	Q2	Q3	Q4	Q5	Q6	TOTAL
20	20	20	20	20	10	110

1. Find the greatest common divisor g of 291 and 573 in two different ways.

Solution:

First way:
$573 = 291\,(1) + 282$
$291 = 282\,(1) + 9$
$282 = 9\,(31) + 3$
$9 = 3\,(3) + 0$

Thus by the Euclidean Algorithm, $\gcd(573, 282) = 3$.

Second way: we factorize both numbers

Divisors of $573 : 1, -1, 3, -3, 191, -191$
Divisors of $291 : 1, -1, 3, -3, 97, -97$
Common divisors of 291 and $573 : 1, -1, 3, -3$

Thus the greatest common divisor of 291 and 573 is $3 = \gcd(573, 291)$.

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2. Find all solutions to the linear Diophantine equation

$$34x + 24y = 1000$$

in nonegative integers x and y.

Solution: We note that $\gcd(34, 24) = 2$ and $2 \mid 1000$, so solutions do exist. In fact $x_0 = 2500$ and $y_0 = -3500$ is one solution.

We know that all solutions are of the form $x = x_0 + \frac{b}{d}, y = y_0 - \frac{a}{d}t$.

Thus $x = 2500 + 12t, y = -3500 - 17t$.

Since we want $x \geq 0$ and $y \geq 0$, we must have that $2500 + 12t \geq 0$ and $-3500 - 17t \geq 0$.

But these imply that $t \geq -\frac{625}{3}$ and $t \leq -\frac{3500}{17}$.

Hence as t runs from -208 to -206.

Thus there are only 3 solutions and we ist them in the following table:

t	-208	-207	-206
x	4	16	28
y	36	19	2

3. Find all solutions to $15x + 16y = -1,000$ with
a) $x \leq 0$ and $y \leq 0$.
b) $x \geq 0$ and $y \geq 0$.

Solution:

a) We first note that $\gcd(15, 16) = 1$, and one solution to the linear equation $15x+16y = -1,000$ is $x_0 = 1,000$ and $y_0 = -1,000$.

Hence from the formula $x = x_0 + \frac{b}{d}, y = y_0 - \frac{a}{d}t$ (with $a = 15, b = 16, d = 1$).

We obtain $x = 1,000 + 16t$ and $y = -1,000 - 15t$.

Since we want $x \leq 0$ and $y \leq 0$, we obtain that $1,000 + 16t \leq 0$, and $-1,000 - 15t \leq 0$, both of which imply that $\leq -\frac{1,000}{6}$, and $t \geq -\frac{1,000}{15}$.

Thus t must run from -66 to -63.

Therefore we list all possible solutions in the following table:

t	-66	-65	-64	-63
x	-56	-40	-24	-8
y	-10	-25	-40	-55

b) In this case there are no values of $x \geq 0$ and $y \geq 0$ such that $15x + 16y = -1,000$.

4. Describe the solutions (if any) in x to the congruence $10x \equiv 15 \pmod{35}$.

Solution: An integer x satisfies $10x \equiv 15 \pmod{35}$ if and only if there is an integer k such that

$10x - 15 = 35k$, if and only if there is an integer k such that $10x + 35k = 15$.

This is a linear Diophantine equation.

To solve it we first apply the Euclidean algorithm to 10 and 35, giving $\gcd(10, 35) = 5 = 10\,(-3) + 35\,(1)$.

Multiplying by 3,

$$10\,(-9) + 35\,(3) = 15.$$

So $x = -9, k = 3$ is a solution to the linear diophantine equation.

The general solution to the congruence is $x = -9 - 7m, k = 3 + 2m$ for any x satisfying $x \equiv -9 \pmod{7}$.

Finally, it is usual to reduce the right hand side of a congruence like this to its least nonnegative residue; in this instance we replace -9 by the least nonnegative residue of -9 modulo 7, which is 5.

So the solution is $x \equiv 5 \pmod{7}$. (Now check this!)

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5. What is the least residue of
a) $33 \cdot 26^2 \pmod{31}$?
b) $50^{99} \pmod 7$?

Solution:

a) We have $33 \equiv 2 \pmod{31}$ and $26 \equiv -5 \pmod{31}$. Then by one of our theorems, we conclude that

$$33 \cdot 26^2 \equiv 2\,(-5)^2 \equiv 50 \equiv 19 \pmod{31}.$$

b) We have $50 \equiv 1 \pmod 7$ and $(50)^{99} \equiv (1)^{99} = 1 \pmod 7$. Then, we conclude that

$$(50)^{99} \equiv 1 \pmod 7.$$

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6. (Bonus) What is the remainder when $3^{202} + 5^9$ is divided by 8?

Solution:

We have $3^4 \equiv 1 \pmod{8}$. Then $3^{202} = 3^{4(50)+2} = \left(3^4\right)^{50} 3^2 \equiv (1)^{50} 3^2 \equiv 3^2 \equiv 1 \pmod{8}$.

Similarly, $5^2 \equiv 1 \pmod{8}$, then $5^9 = 5^{2(4)+1} = \left(5^2\right)^4 5 \equiv (1)^4 5 = 5 \pmod{8}$.

Thus the remainder is 6, since $3^{202} + 5^9 \equiv 1 + 5 = 6 \pmod{8}$.