ÇANKAYA UNIVERSITY
Department of Mathematics and Computer Science

MATH 155 Calculus for Engineering I

1^{st} Midterm

SOLUTIONS

July 14, 2008

13:40-15:10

Surname	:	________________
Name	:	________________
ID #	:	________________
Department	:	________________
Section	:	________________
Instructor	:	________________
Signature	:	________________

- The exam consists of 6 questions.
- Please read the questions carefully and write your answers under the corresponding questions. Be neat.
- Show all your work. Correct answers without sufficient explanation might not get full credit.
- Calculators are not allowed.

GOOD LUCK!

Please do not write below this line.

Q1	Q2	Q3	Q4	Q5	Q6	TOTAL
18	21	15	16	15	20	105

1. Calculate the following limits (if they exist). Make your answer as informative as possible: if a limit does not exist, say so; if appropriate one-sided limits exist instead, state them explicitly; if any limits are infinite, state this explicitly as well. (DO NOT USE L'HOPITAL'S RULE)

$$\text{a)}\ \lim_{x\to-\infty}\left(\sqrt{x^2+5x+1}+x\right),\qquad \text{b)}\ \lim_{x\to0}\frac{\sin(\tan x)}{\sin x},\qquad \text{c)}\ \lim_{x\to5}\frac{|2x-10|}{3x+15}$$

Solution:

(a)

$$\lim_{x\to-\infty}\left(\sqrt{x^2+5x+1}+x\right)=\lim_{x\to-\infty}\left(\sqrt{x^2+5x+1}+x\right)\frac{\sqrt{x^2+5x+1}-x}{\sqrt{x^2+5x+1}-x}$$

$$=\lim_{x\to-\infty}\frac{x^2+5x+1-x^2}{\sqrt{x^2+5x+1}-x}=\lim_{x\to-\infty}\frac{5x+1}{\sqrt{x^2+5x+1}-x}=\lim_{x\to-\infty}\frac{x\left(5+\frac{1}{x}\right)}{\sqrt{x^2}\sqrt{1+\frac{5}{x}+\frac{1}{x^2}}-x}$$

$$=\lim_{x\to-\infty}\frac{x\left(5+\frac{1}{x}\right)}{|x|\sqrt{1+\frac{5}{x}+\frac{1}{x^2}}-x}=\lim_{x\to-\infty}\frac{x\left(5+\frac{1}{x}\right)}{-x\sqrt{1+\frac{5}{x}+\frac{1}{x^2}}-x}=\lim_{x\to-\infty}\frac{x\left(5+\frac{1}{x}\right)}{-x\left(\sqrt{1+\frac{5}{x}+\frac{1}{x^2}}+1\right)}$$

$$=\lim_{x\to-\infty}\frac{5+\frac{1}{x}}{-\left(\sqrt{1+\frac{5}{x}+\frac{1}{x^2}}+1\right)}=-\frac{5}{2}.$$

(b)

$$\lim_{x\to0}\frac{\sin(\tan x)}{\sin x}=\lim_{x\to0}\frac{\sin(\tan x)}{\sin x}\frac{\tan x}{\tan x}=\left(\lim_{x\to0}\frac{\sin(\tan x)}{\tan x}\right)\left(\lim_{x\to0}\frac{1}{\cos x}\right)$$

$$=\left(\lim_{u\to0}\frac{\sin u}{u}\right)(1)=1 \text{ where } u=\tan x \text{ and } u\to0 \text{ as } u\to0.$$

(c)

$$\lim_{x\to5}\frac{|2x-10|}{3x+15}=\frac{|2(5)-10|}{3(5)+15}=\frac{0}{30}=0$$

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2. For each of the following functions, calculate the derivative. Do not simplify your answers.

a) $y = \sin(\cos(\tan t))$, b) $f(x) = \sqrt{5x + \sqrt{2x + \sqrt{3x}}}$, c) $h(x) = \cot^3(2x+3)\sin^2(x - 2x^3)$

Solution:

(a)

$y = \sin(\cos(\tan t))$

$\Longrightarrow \dfrac{dy}{dt} = \cos(\cos(\tan t))(-\sin(\tan t))\sec^2 t = -\cos(\cos(\tan t))\sin(\tan t)\sec^2 t$

(b)

$$f(x) = \sqrt{5x + \sqrt{2x + \sqrt{3x}}}$$

$$f'(x) = \frac{1}{2\sqrt{5x + \sqrt{2x + \sqrt{3x}}}}\left(5 + \frac{1}{2\sqrt{2x + \sqrt{3x}}}\left(2 + \frac{1}{2\sqrt{3x}}(3)\right)\right)$$

(c)

$h'(x) = \sin^2(x - 2x^3)(3)\cot^2(2x+3)\left(-\csc^2(2x+3)\right)(2)$
$+\cot^3(2x+3)(2)\sin(x - 2x^3)\left(\cos(x - 2x^3)\right)(1 - 6x^2)$

and so

$h'(x) = -6\sin^2(x - 2x^3)\cot^2(2x+3)\csc^2(2x+3)$
$+2(1 - 6x^2)\cot^3(2x+3)\sin(x - 2x^3)\cos(x - 2x^3)$

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3. Determine the values of the constants A and B so that the function f is continuous at every real number.

$$f(x) = \begin{cases} 3x & \text{if } \ x \leq 2 \\ Ax + B & \text{if } \ 2 < x < 5 \\ -6x & \text{if } \ x \geq 5 \end{cases}$$

Solution:

Note that f is continuous everywhere iff it is continuous at both $x = 2$ and $x = 5$. Then f is continuous at $x = 2$ implies $\lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x) = f(2)$ i.e., $\lim_{x \to 2^-} (3x) = \lim_{x \to 2^+} (Ax + B) = f(2)$ so we get $2A + B = 6$.

Similarly, f is continuous at $x = 5$ implies $\lim_{x \to 5^-} f(x) = \lim_{x \to 5^+} f(x) = f(5)$ i.e., $\lim_{x \to 5^-} (Ax + B) = \lim_{x \to 5^+} (-6x) = f(5)$ so we get $5A + B = -30$.

Now solving simultaneously the system

$$\begin{aligned} 2A + B &= 6 \\ 5A + B &= -30 \end{aligned}$$

the only values we get are $A = -12, B = 30$.

Conversely, for the values $A = -12, B = 30$, the function

$$f(x) = \begin{cases} 3x & \text{if } \ x \leq 2 \\ -12x + 30 & \text{if } \ 2 < x < 5 \\ -6x & \text{if } \ x \geq 5 \end{cases}$$

is continuous at every real number

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4. Find an equation of the line that is (a) tangent, (b) normal, to the curve $2x+y-\sqrt{2}\sin(xy) = \frac{\pi}{2}$ at the point $\left(\frac{\pi}{4}, 1\right)$.

Solution:

We first find $\frac{dy}{dx}$ by using implicit differentiation.

For this, $\frac{d}{dx}\left(2x + y - \sqrt{2}\sin(xy)\right) = \frac{d}{dx}\left(\frac{\pi}{2}\right) \Longleftrightarrow 2 + y' - \sqrt{2}\,(y + xy')\cos(xy) = 0$.

Thus $2 + y' - \sqrt{2}y\cos(xy) - \sqrt{2}xy'\cos(xy) = 0$.

$\Longrightarrow y'\left(1 - \sqrt{2}x\cos(xy)\right) = -2 + \sqrt{2}y\cos(xy) \Longrightarrow y' = \dfrac{-2 + \sqrt{2}y\cos(xy)}{1 - \sqrt{2}x\cos(xy)}$

Therefore the slope of the tangent is $m_T = y'\,|_{(x,y)=\left(\frac{\pi}{4},1\right)} = \dfrac{-2 + \sqrt{2}\,(1)\cos\left((1)\left(\frac{\pi}{4}\right)\right)}{1 - \sqrt{2}\left(\frac{\pi}{4}\right)\cos\left((1)\left(\frac{\pi}{4}\right)\right)} = \dfrac{-2+1}{1 - \sqrt{2}\left(\frac{\pi}{4}\right)\frac{1}{\sqrt{2}}} =$
$\dfrac{-1}{1 - \frac{\pi}{4}}$, that is, we have

(a) the tangent at $\left(\frac{\pi}{4}, 1\right)$ has equation $y - 1 = \dfrac{-4}{4-\pi}\left(x - \frac{\pi}{4}\right)$

(b) the normal is perpendicular to tangent and so its slope is $m_N = -\dfrac{1}{m_T} = 1 - \dfrac{\pi}{4}$ and so its equation is $y - 1 = \left(1 - \frac{\pi}{4}\right)\left(x - \frac{\pi}{4}\right)$.

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5. Find all (local and absolute) extrema (maximum and minimum) of the function

$$g(x) = \frac{x^2}{x^2+3}$$

on the closed interval $[-1, 2]$.

Solution:

$g'(x) = \frac{(x^2+3)(2x) - x^2(2x)}{(x^2+3)^2} = \frac{6x}{(x^2+3)^2} = 0 \iff x = 0$ is the only critical point in $[-1, 2]$.

Now we calculate the values of g at $x = -1, 0, 2$.

$g(-1) = \frac{(-1)^2}{(-1)^2+3} = \frac{1}{4}$

$g(0) = 0$

$g(2) = \frac{2^2}{2^2+3} = \frac{4}{7}$.

Therefore the absolute minimum value of g is 0 and occurs at $x = 0$ and the absolute maximum value of $\frac{4}{7}$ at $x = 2$.

6. Sketch the graph of

$$f(x) = x^5 - 5x^4$$

At what points (x, y) does the function have local extrema and points of inflection? On what intervals is the function increasing? On what intervals is the function decreasing? Make sure your graph clearly illustrates all these features.

Solution:

$f(x) = x^5 - 5x^4 \Longrightarrow f'(x) = 5x^4 - 20x^3 = 5x^3(x-4) \Longrightarrow f''(x) = 20x^3 - 60x^2 = 20x^2(x-3)$

$f'(x) = 0 \Longleftrightarrow x = 0, x = 4$ are the critical points of f and also $f''(x) = 0 \Longleftrightarrow x = 0, x = 3$.

Now since $f'(x) > 0$ on $(-\infty, 0) \cup (4, \infty)$, we have that f is increasing on $(-\infty, 0) \cup (4, \infty)$, and also since $f'(x) < 0$ on $(0, 4)$, we have that f is decreasing on $(0, 4)$.

Similarly, $f''(x) > 0$ on $(3, \infty)$ implies that the graph of f is concave up on $(3, \infty)$, and since $f''(x) < 0$ on $(-\infty, 3)$, the graph of f is concave down on $(-\infty, 3)$.

Now $f(0) = 0$ is the only local maximum value of f that occurs at $x = 0$ and the only local minimum value of f is $f(4) = -256$ that occurs at $x = 4$. The only inflection point of the graph is $(3, -162)$.