ÇANKAYA UNIVERSITY
Department of Mathematics and Computer Science

MATH 155 Calculus for Engineering I
2^{nd} Midterm
SOLUTIONS
August 4, 2008
13:40-15:10

Surname : ______________________
Name : ______________________
ID # : ______________________
Department : ______________________
Section : ______________________
Instructor : ______________________
Signature : ______________________

- The exam consists of 6 questions.
- Please read the questions carefully and write your answers under the corresponding questions. Be neat.
- Show all your work. Correct answers without sufficient explanation might not get full credit.
- Calculators are not allowed.

GOOD LUCK!

Please do not write below this line.

Q1	Q2	Q3	Q4	Q5	Q6	TOTAL
18	21	15	16	15	20	105

1. Evaluate the following limits

$$\text{a)}\ \lim_{x\to+\infty}(e^x+x)^{1/x},\qquad \text{b)}\lim_{x\to 0}\frac{e^x-1}{\sin x},\qquad \text{c)}\ \lim_{x\to 3^-}\left(\frac{1}{\ln(x-2)}-\frac{1}{x-3}\right)$$

Solution:

a)

$\lim_{x\to+\infty}(e^x+x)^{1/x}\quad [\infty^0]$

Let $y=(e^x+x)^{1/x}$.

$$\lim_{x\to\infty}\ln y=\lim_{x\to\infty}\frac{\ln(e^x+x)}{x}\quad\left[\frac{\infty}{\infty}\right]$$

Therefore L'Hopital's Rule applies, and so we have

$$=\lim_{x\to\infty}\frac{\frac{e^x+1}{e^x+x}}{1}=\lim_{x\to\infty}\frac{e^x}{e^x+1}=\lim_{x\to\infty}\frac{e^x}{e^x}=\lim_{x\to\infty}1=1.$$

Thus,

$=\lim_{x\to\infty}(e^x+x)^{1/x}=e.$

b)

$$\lim_{x\to 0}\frac{e^x-1}{\sin x}\quad\left[\frac{0}{0}\right]$$

Therefore L'Hopital's Rule applies, and so we have

$$\lim_{x\to 0}\frac{e^x-1}{\sin x}=\lim_{x\to 0}\frac{e^x}{\cos x}=\frac{e^0}{\cos 0}=1$$

c)

$$\lim_{x\to 3^-}\left(\frac{1}{\ln(x-2)}-\frac{1}{x-3}\right)\quad[\infty-\infty]$$

$$\lim_{x\to 3^-}\left(\frac{1}{\ln(x-2)}-\frac{1}{x-3}\right)=\lim_{x\to 3^-}\frac{x-3-\ln(x-2)}{(x-3)\ln(x-2)}\quad\left[\frac{0}{0}\right]$$

Therefore L'Hopital's Rule applies, and so we have

$$=\lim_{x\to 3^-}\frac{1-\frac{1}{x-2}}{\ln(x-2)+\frac{x-3}{x-2}}=\lim_{x\to 3^-}\frac{\frac{x-2-1}{x-2}}{\ln(x-2)+\frac{x-3}{x-2}}=\lim_{x\to 3^-}\frac{x-3}{(x-2)\ln(x-2)+x-3}$$

$$=\lim_{x\to 3^-}\frac{1}{\ln(x-2)+(x-2)\frac{1}{x-2}+1}=\frac{1}{2}$$

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2. For each of the following functions, calculate the derivative. Do not simplify your answers.

$$\text{a) } y = \left(\frac{x}{x+9}\right)^{3x+2}, \qquad \text{b) } y = \ln\frac{\sin^{-1}x}{\sin x}, \qquad \text{c) } f(x) = \int_{\sqrt{x}}^{x^2} \frac{dt}{1+t+\sin t}$$

Solution:

a)

$$y = \left(\frac{x}{x+9}\right)^{3x+2} \Longrightarrow \ln y = (3x+2)\left[\ln x - \ln(x+9)\right]$$

$$\Longrightarrow \frac{1}{y}y' = 3\left[\ln x - \ln(x+9)\right] + (3x+2)\left[\frac{1}{x} - \frac{1}{x+9}\right]$$

$$\Longrightarrow y' = 3y\left[\ln x - \ln(x+9) + (3x+2)\left(\frac{1}{x} - \frac{1}{x+9}\right)\right]$$

$$\Longrightarrow y' = 3\left(\frac{x}{x+9}\right)^{3x+2}\left[\ln x - \ln(x+9) + (3x+2)\left(\frac{1}{x} - \frac{1}{x+9}\right)\right]$$

b)

$$y = \ln\frac{\sin^{-1}x}{\sin x} = \ln\sin^{-1}x - \ln\sin x$$

$$\Longrightarrow y' = \frac{1}{\sin^{-1}x}\frac{1}{\sqrt{1-x^2}} - \frac{1}{\sin x}\cos x$$

c)

$$f(x) = \int_{\sqrt{x}}^{x^2} \frac{dt}{1+t+\sin t} \Longrightarrow f'(x) = \frac{1}{1+x^2+\sin(x^2)} \cdot \frac{d}{dx}\left(x^2\right) - \frac{1}{1+\sqrt{x}+\sin(\sqrt{x})} \cdot \frac{d}{dx}\left(\sqrt{x}\right)$$

$$\Longrightarrow f'(x) = \frac{1}{1+x^2+\sin(x^2)} \cdot (2x) - \frac{1}{1+\sqrt{x}+\sin(\sqrt{x})} \cdot \left(\frac{1}{2\sqrt{x}}\right)$$

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3. Given $f(x) = x^5 + 3x^3 + 2x + 1$,
a) show that f has an inverse $g(x)$.
b) compute $g'(7)$.

Solution:

a)
$f'(x) = 5x^4 + 9x^2 + 2 > 0$ for all $x \in \mathbb{R} \Longrightarrow f(x)$ is increasing for all $x \in \mathbb{R}$ which implies that $f(x)$ is increasing for all $x \in \mathbb{R}$, and so $f(x)$ is one-to-one for all $x \in \mathbb{R}$. Therefore f has an inverse $g(x)$.

b)
To compute $g'(7)$: we have

$$g'(7) = \frac{1}{f'(g(7))}$$

But to use this we also need the value of $g(7)$.

If we write $x = g(7)$, then $x = f^{-1}(7)$.

By trial and error, however it is not hard to see that $f(1) = 7$, so that $g(7) = 1$.

Hence we have

$$g'(7) = \frac{1}{f'(1)} = \frac{1}{16}.$$

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4. Evaluate the following integrals

$$\text{(a)} \int \frac{x^3 - 4x^2 + 3x - 1}{\sqrt{x}}\, dx, \quad \text{(b)} \int x^3 \left(x^2+1\right)^{-1/2}\, dx, \quad \text{(c)} \int_{\pi/4}^{\pi/3} \left(\sin\theta + \frac{1}{\sin^2\theta}\right)\, d\theta$$

Solution:

a)

$$\int \frac{x^3 - 4x^2 + 3x - 1}{\sqrt{x}}\, dx = \int \left(x^{5/2} - 4x^{3/2} + 3x^{1/2} - x^{-1/2}\right)\, dx$$

$$= \left[\frac{x^{7/2}}{7/2} - 4\frac{x^{5/2}}{5/2} + 3\frac{x^{3/2}}{3/2} - \frac{x^{1/2}}{1/2}\right] + C$$

$$= \frac{2}{7}x^{7/2} - \frac{8}{5}x^{5/2} + 2x^{3/2} - 2x^{1/2} + C$$

b)

$$\int x^3 \left(x^2+1\right)^{-1/2}\, dx = 2\int x^2 \left(x^2+1\right)^{-1/2} \left(\frac{1}{2}x\, dx\right)$$

$$\left[\begin{array}{c} u = x^2+1 \\ du = 2x\ dx \\ dx = \frac{1}{2}\, du \end{array}\right] \longrightarrow 2\int (u-1)\, u^{-1/2}\, du = 2\int \left(u^{1/2} - u^{-1/2}\right)\, du = 2\left[\frac{u^{3/2}}{3/2} - \frac{u^{1/2}}{1/2}\right] + C$$

$$= \frac{4}{3}\left(x^2+1\right)^{3/2} - 4\left(x^2+1\right)^{1/2} + C$$

c)

$$\int_{\pi/4}^{\pi/3} \left(\sin\theta + \frac{1}{\sin^2\theta}\right)\, d\theta = \left[-\cos\theta - \cot\theta\right]_{\pi/4}^{\pi/3} = \left(-\cos\frac{\pi}{3} - \cot\frac{\pi}{3}\right) - \left(-\cos\frac{\pi}{4} - \cot\frac{\pi}{4}\right)$$

$$= \left(-\frac{1}{2} - \frac{1}{\sqrt{3}}\right) - \left(-\frac{1}{\sqrt{2}} - 1\right) = 1 - \frac{1}{\sqrt{3}}$$

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5. Find the area bounded by the graphs $y = x^2$ and $y = 2 - x^2$ for $0 \le x \le 2$.
Solution:
To find the point of intersection, we solve

$$x^2 = 2 - x^2,$$

so that

$$2x^2 = 2 \text{ or } x^2 = 1 \text{ or } x = \pm 1.$$

Since $x = -1$ is outside the interval of interest, the only intersection note is at $x = 1$.

Note that

$$2 - x^2 \ge x^2 \text{ for } 0 \le x \le 1$$

and we have

$$x^2 \ge 2 - x^2 \text{ for } 1 \le x \le 2.$$

Now the area is

$$\begin{aligned}
\text{AREA} &= \int_0^1 \left[\left(2 - x^2\right) - x^2\right] \, dx + \int_1^2 \left[x^2 - \left(2 - x^2\right)\right] \, dx \\
&= \int_0^1 \left(2 - 2x^2\right) \, dx + \int_1^2 \left(2x^2 - 2\right) \, dx \\
&= \left[2x - \frac{2x^3}{3}\right]_0^1 + \left[\frac{2x^3}{3} - 2x\right]_1^2 \\
&= \left(2 - \frac{2}{3}\right) - (0 - 0) + \left(\frac{16}{3} - 4\right) - \left(\frac{2}{3} - 2\right) \\
&= \frac{4}{3} + \frac{4}{3} + \frac{4}{3} = 4.
\end{aligned}$$

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6. Let R be the region bounded by the graphs of $y = x^2$ $(x \geq 0)$, $y = 2 - x$ and $x = 0$. Compute the volume of the solid formed by revolving R about
a) the x-axis
b) the y-axis.

Solution:

a)
By using the method of washers, we have

$$\text{VOLUME} = \int_0^1 \pi \left[(2-x)^2 - \left(x^2\right)^2\right] \, dx = \int_0^1 \pi \left[4 - 4x + x^2 - x^4\right] \, dx$$

$$= \pi \left[4x - 2x^2 + \frac{1}{3}x^3 - \frac{1}{5}x^5\right]_0^1 = \pi \left(4(1) - 2(1)^2 + \frac{1}{3}(1)^3 - \frac{1}{5}(1)^5\right) = \pi \left(+\frac{1}{3} - \frac{1}{5}\right)$$

$$= \pi \left(2 + \frac{1}{3} - \frac{1}{5}\right) = \frac{32\pi}{15}$$

b)
By using the method of cylindrical shells, we have

$$\text{VOLUME} = \int_0^1 2\pi x \left[(2-x) - \left(x^2\right)\right] \, dx = \int_0^1 2\pi \left[2x - x^2 - x^3\right] \, dx$$

$$= 2\pi \left[x^2 - \frac{1}{3}x^3 - \frac{1}{4}x^4\right]_0^1 = 2\pi \left((1)^2 - \frac{1}{3}(1)^3 - \frac{1}{4}(1)^4\right) = 2\pi \left(1 - \frac{1}{3} - \frac{1}{4}\right)$$

$$= \frac{5\pi}{6}$$
