ÇANKAYA UNIVERSITY
Department of Mathematics and Computer Science

MATH 155 Calculus for Engineering I
Final
SOLUTIONS
August 08, 2008
9:00-11:00

Surname : ____________________
Name : ____________________
ID # : ____________________
Department : ____________________
Section : ____________________
Instructor : ____________________
Signature : ____________________

- The exam consists of 6.
- Please read the questions carefully and write your answers under the corresponding questions. Be neat.
- Show all your work. Correct answers without sufficient explanation might not get full credit.
- Calculators are not allowed.

GOOD LUCK!

Please do not write below this line.

Q1	Q2	Q3	Q4	Q5	Q6	TOTAL
12	27	18	18	18	12	105

1. Find the area of the region enclosed by the curve $y = x \sin x$ and the x-axis for $8\pi \leq x \leq 9\pi$.
Solution:
We know that $y = x \sin x \geq 0$ for $8\pi \leq x \leq 9\pi$. Therefore, by integrating by parts, we have

$$\begin{aligned} \text{AREA} &= \int_{8\pi}^{9\pi} x \sin x \; dx = [-x \cos x + \sin x]_{8\pi}^{9\pi} \\ &= [-9\pi \cos(9\pi) + \sin(9\pi)] - [-8\pi \cos(8\pi) + \sin(8\pi)] \\ &= -9\pi(-1) + 0 + 8\pi(1) - 0 \\ & \quad 17\pi \end{aligned}$$

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2. Evaluate.

a) $\displaystyle\int \frac{5e^{1/y}}{3y^2}\ dy$, b) $\displaystyle\int \frac{2x+2}{(x^2+1)(x-1)^2}\ dx$, c) $\displaystyle\int_2^4 6x\ln x\ dx$.

Solution:

a) $\displaystyle\int \frac{5e^{1/y}}{3y^2}\ dy$

$$\begin{bmatrix} u = \frac{1}{y} \\ du = -\frac{1}{y^2}\ dy \end{bmatrix} \longrightarrow \int \frac{5e^{1/y}}{3y^2}\ dy = -\frac{5}{3}\int e^u\ du = -\frac{5}{3}e^u + C = -\frac{5}{3}e^{1/y} + C$$

b) $\displaystyle\int \frac{2x+2}{(x^2+1)(x-1)^2}\ dx$

$$\frac{2x+2}{(x^2+1)(x-1)^2} = \frac{Ax+B}{x^2+1} + \frac{C}{x-1} + \frac{D}{(x-1)^2}$$

$$\Longrightarrow 2x+2 = (Ax+B)(x-1)^2 + C(x-1)\left(x^2+1\right) + D\left(x^2+1\right)$$

$$\Longrightarrow 2x+2 = (A+C)x^3 + (-2A+B-C+D)x^2 + (A-2B+C)x + B - C + D$$

$$\begin{bmatrix} A+C=0 \\ -2A+B-C+D=0 \\ A-2B+C=2 \\ B-C+D=2 \end{bmatrix} \longrightarrow A=1, B=-1, C=-1, D=2$$

$$\begin{aligned}
\frac{2x+2}{(x^2+1)(x-1)^2} &= \frac{x-1}{x^2+1} - \frac{1}{x-1} + \frac{2}{(x-1)^2} \\
\int \frac{2x+2}{(x^2+1)(x-1)^2}\ dx &= \int \left(\frac{x-1}{x^2+1} - \frac{1}{x-1} + \frac{2}{(x-1)^2}\right) dx \\
&= \frac{1}{2}\int \frac{2x}{x^2+1}\ dx - \int \frac{1}{x+1}\ dx - \int \frac{1}{x-1}\ dx + 2\int \frac{1}{(x-1)^2}\ dx \\
&= \frac{1}{2}\ln\left(x^2+1\right) - \ln|x+1| - \ln|x-1| - 2\frac{1}{x-1} + C
\end{aligned}$$

c) $\displaystyle\int_2^4 6x\ln x\ dx$; we apply integration by parts

$$\begin{aligned}
u &= \ln x, \quad dv = 6x\ dx \\
du &= \frac{1}{x}\ dx, \quad v = 3x^2
\end{aligned}$$

$$\int_2^4 6x\ln x\ dx = \left[3x^2\ln x\right]_2^4 - \int_2^4 3x^2\frac{1}{x}\ dx = \left[3x^2\ln x\right]_2^4 - \int_2^4 3x\ dx$$

$$= \left[3x^2\ln x\right]_2^4 - \left[\frac{3}{2}x^2\right]_2^4 = \left(3(4)^2\ln 4 - 3(2)^2\ln 2\right) - \frac{3}{2}\left((4)^2 - (2)^2\right)$$

$$= 48\ln 4 - 12\ln 2 - \frac{3}{2}(12) = 96\ln 2 - 12\ln 2 - 18$$

$$= 84\ln 2 - 18$$

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3. Evaluate.

a) $\displaystyle\int_0^1 \frac{x^2}{\sqrt{4-x^2}}\,dx$, b) $\displaystyle\int_{\pi/12}^{\pi/2} (1-\cos 4x)\cos 2x\,dx.$

Solution:

a) $\displaystyle\int_0^1 \frac{x^2}{\sqrt{4-x^2}}\,dx$

$$\begin{bmatrix} x = 2\sin\theta \\ dx = 2\cos\theta\,d\theta \\ x=0 \Longrightarrow \theta = 0 \\ x = 1 \Longrightarrow \theta = \frac{\pi}{6} \end{bmatrix} \longrightarrow \int_0^1 \frac{x^2}{\sqrt{4-x^2}}\,dx = \int_0^{\pi/6} \frac{4\sin^2\theta}{\sqrt{4-4\sin^2\theta}} 2\cos\theta\,d\theta = \int_0^{\pi/6} 4\sin^2\theta\,d\theta$$

$$\int_0^{\pi/6} 4\sin^2\theta\,d\theta = 4\int_0^{\pi/6} \frac{1}{2}(1-\cos(2\theta))\,d\theta = 2\left[\theta - \frac{1}{2}\sin(2\theta)\right]_0^{\pi/6}$$

$$= 2\left[\frac{\pi}{6} - \frac{1}{2}\sin\left(\frac{\pi}{3}\right)\right] = \frac{\pi}{3} - \sin\left(\frac{\pi}{3}\right) = \frac{\pi}{3} - \frac{\sqrt{3}}{3} = \frac{\pi-\sqrt{3}}{3}.$$

b) $\displaystyle\int_{\pi/12}^{\pi/2} (1-\cos 4x)\cos 2x\,dx;$

Since $\cos 4x = \cos(2x+2x) = \cos^2(2x) - \sin^2(2x) = 1 - 2\sin^2(2x)$

$$\int_{\pi/12}^{\pi/2} (1-\cos 4x)\cos 2x\,dx = \int_{\pi/12}^{\pi/2} \left(1 - \left(1-2\sin^2(2x)\right)\right)\cos 2x\,dx$$

$$= \int_{\pi/12}^{\pi/2} 2\sin^2(2x)\cos 2x\,dx$$

$$\begin{bmatrix} u = \sin(2x) \\ du = 2\cos(2x)\,dx \\ x = \frac{\pi}{12} \Longrightarrow u = \sin\left(\frac{\pi}{6}\right) = \frac{1}{2} \\ x = \frac{\pi}{2} \Longrightarrow u = \sin(\pi) = 0 \end{bmatrix} \longrightarrow \int_{\pi/12}^{\pi/2} 2\sin^2(2x)\cos 2x\,dx = \int_{1/2}^{0} u^2\,du$$

$$= \left[\frac{1}{3}u^3\right]_{1/2}^{0} = -\frac{1}{3}\left(\frac{1}{2}\right)^3 = -\frac{1}{24}$$

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4. Determine whether the following improper integrals converge or diverge. Give reasons for your answer.

a) $\displaystyle\int_1^e \frac{dx}{x\sqrt[3]{\ln x}}$, b) $\displaystyle\int_1^\infty \frac{1}{x^2+\sin^2 x}\,dx$.

Solution:

a) $\displaystyle\int_1^e \frac{dx}{x\sqrt[3]{\ln x}}$

$\displaystyle\lim_{b\to 1^+}\int_b^e \frac{dx}{x\sqrt[3]{\ln x}}$

$$\begin{bmatrix} u = \ln x \\ du = \frac{1}{x}\,dx \\ x = b \Longrightarrow u = \ln b \\ x = e \Longrightarrow u = \ln e = 1 \end{bmatrix} \longrightarrow \lim_{b\to 1^+}\int_b^e \frac{dx}{x\sqrt[3]{\ln x}} = \lim_{b\to 1^+}\int_{\ln b}^1 \frac{du}{u^{1/3}}$$

$$= \lim_{b\to 1^+}\int_{\ln b}^1 u^{-1/3}\,du = \lim_{b\to 1^+}\left[\frac{u^{2/3}}{2/3}\right]_{\ln b}^1 = \frac{3}{2}\lim_{b\to 1^+}\left(1^{2/3} - (\ln b)^{2/3}\right) = 1$$

Hence the improper integral converges and has value 1.

b) $\displaystyle\int_1^\infty \frac{1}{x^2+\sin^2 x}\,dx$

We apply the direct comparison test, since it is not possible to integrate the integrand in terms of elementary functions.

$$0 \le \frac{1}{x^2+\sin^2 x} \le \frac{1}{x^2} \text{ for all } x \ge 1$$

Therefore, we see that $\displaystyle\int_1^\infty \frac{1}{x^2+\sin^2 x}\,dx$ converges since $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx$ is a convergent p-integral with $p = 2 > 1$.

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5. For each of the following, state whether the sequence $\{a_n\}$ converges and, if so, to what value.

a) $a_n = \dfrac{\sin^{-1}\left(\frac{1}{n}\right)}{\tan^{-1}\left(\frac{1}{n}\right)}$, b) $a_n = \dfrac{\cos^2 n}{2^n}$.

Solution:

a) $a_n = \dfrac{\sin^{-1}\left(\frac{1}{n}\right)}{\tan^{-1}\left(\frac{1}{n}\right)}$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{\sin^{-1}\left(\frac{1}{n}\right)}{\tan^{-1}\left(\frac{1}{n}\right)} = \lim_{n\to\infty} \frac{\frac{1}{\sqrt{1-\left(\frac{1}{n}\right)^2}}\left(-\frac{1}{n^2}\right)}{\frac{1}{!+\left(\frac{1}{n}\right)^2}\left(-\frac{1}{n^2}\right)} = \lim_{n\to\infty} \frac{\frac{1}{\sqrt{1-\left(\frac{1}{n}\right)^2}}}{\frac{1}{!+\left(\frac{1}{n}\right)^2}} = 1.$$

Hence the sequence converges and has limit 1.

b) $a_n = \dfrac{\cos^2 n}{2^n}$

We have $-1 \le \cos n \le 1 \Longrightarrow 0 \le \dfrac{\cos^2 n}{2^n} \le \dfrac{1}{2^n}$ and $\lim\limits_{n\to\infty} \dfrac{1}{2^n} = 0$. Therefore, by Sandwich Theorem, we see that $\lim\limits_{n\to\infty} \dfrac{\cos^2 n}{2^n} = 0$.

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6. Solve the initial value problem for x as a function of t.

$$\begin{aligned} (t+2)\,\frac{dx}{dt} &= x^2+1, \;\; t>-2, \\ x(2) &= \tan 1 \end{aligned}$$

Solution:

$$\begin{aligned} \frac{dx}{x^2+1} &= \frac{dt}{t+2} \\ \tan^{-1} x &= \ln|t+2| + C \\ \tan^{-1} x &= \ln C(t+2) \;\text{ because } t>-2, \\ \tan^{-1}(\tan 1) &= \ln C(2+2) \\ 1 &= \ln 4C \\ 4C &= e \\ C &= \frac{e}{4} \\ \tan^{-1} x &= \ln \frac{e}{4}(t+2) \\ x &= \tan\left(\ln \frac{e(t+2)}{4}\right) \end{aligned}$$