ÇANKAYA UNIVERSITY
Department of Mathematics and Computer Science
MATH 155 Calculus for Engineering I
Summer 2008

Problems and Solutions for Recitation 4

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(p.383)

1. Area

Find the areas of the regions enclosed by the lines and curves in Exercises 41-50.

41. $y = x^2 - 2$ and $y = 2$

Solution:

$a = -2, b = 2;$

$f(x) - g(x) = 2 - (x^2 - 2) = 4 - x^2$

$\Longrightarrow A = \int_{-2}^{2} (4 - x^2)\ dx = \left[4x - \frac{x^3}{3}\right]_{-2}^{2} = \left(8 - \frac{8}{3}\right) - \left(-8 + \frac{8}{3}\right)$

$= 2 \cdot \left(\frac{24}{3} - \frac{8}{3}\right) = \frac{32}{3}$

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42. $y = 2x - x^2$ and $y = -3$

Solution:

$a = -1, b = 3;$

$f(x) - g(x) = (2x - x^2) - (-3) = 2x - x^2 + 3$

$\Longrightarrow A = \int_{-1}^{3} (2x - x^2 + 3)\ dx = \left[x^2 - \frac{x^3}{3} + 3x\right]_{-1}^{3} = \frac{32}{3}$

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43. $y = x^4$ and $y = 8x$

Solution:

$a = 0, b = 2;$

$f(x) - g(x) = 8x - x^4$

$\Longrightarrow A = \int_{0}^{2} (8x - x^4)\ dx = \left[\frac{8x^2}{2} - \frac{x^5}{5}\right]_{0}^{2} = 16 - \frac{32}{5} = \frac{48}{5}$

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44. $y = x^2 - 2x$ and $y = x$

Solution:

Limits of integration: $x^2 - 2x = x \Longrightarrow x^2 = 3x$

$\Longrightarrow x(x - 3) = 0 \Longrightarrow a = 0$ and $b = 3;$

$f(x) - g(x) = x - (x^2 - 2x) = 3x - x^2$
$\Longrightarrow A = \int_0^3 (3x - x^2)\ dx = \left[\frac{3x^2}{2} - \frac{x^3}{3}\right]_0^3 = \frac{27}{2} - 9 = \frac{9}{2}$

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45. $y = x^2$ and $y = -x^2 + 4x$
Solution:
Limits of Integration: $x^2 = -x^2 + 4x \Longrightarrow 2x^2 - 4x = 0$
$\Longrightarrow 2x(x-2) = 0 \Longrightarrow a = 0$ and $b = 2$;
$f(x) - g(x) = (-x^2 + 4x) - x^2 = -2x^2 + 4x$
$\Longrightarrow A = \int_0^2 (-2x^2 + 4x)\ dx = \left[\frac{-2x^3}{3} - \frac{4x^2}{2}\right]_0^2 = -\frac{16}{3} + \frac{6}{2} = \frac{8}{3}$

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46. $y = 7 - 2x^2$ and $y = x^2 + 4$
Solution:
Limits of Integration: $7 - 2x^2 = x^2 + 4 \Longrightarrow 3x^2 - 3 = 0$
$\Longrightarrow 3(x-1)(x+1) = 0 \Longrightarrow a = -1$ and $b = 1$;
$f(x) - g(x) = (7 - 2x^2) - (x^2 + 4) = 3 - 3x^2$
$\Longrightarrow A = \int_{-1}^1 (3 - 3x^2)\ dx = 3\left[x - \frac{x^3}{3}\right]_{-1}^1 = 4$

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47. $y = x^4 - 4x^2 + 4$ and $y = x^2$
Solution:
Limits of Integration: $x^4 - 4x^2 + 4 = x^2 \Longrightarrow x^4 - 5x^2 + 4 = 0$
$\Longrightarrow (x^2 - 4)(x^2 - 1) = 0 \Longrightarrow (x+2)(x-2)(x+1)(x-1) = 0$
$\Longrightarrow x = -2, -1, 1, 2$;
$f(x) - g(x) = (x^4 - 4x^2 + 4) - x^2 = x^4 - 5x^2 + 4$ and
$g(x) - f(x) = x^2 - (x^4 - 4x^2 + 4) = -x^4 + 5x^2 - 4$
$\Longrightarrow A = \int_{-2}^{-1} (-x^4 + 5x^2 - 4)\ dx + \int_{-1}^1 (x^4 - 5x^2 + 4)\ dx$
$+ \int_1^2 (-x^4 + 5x^2 - 4)\ dx$
$\left[-\frac{x^5}{5} + \frac{5x^3}{3} - 4x\right]_{-2}^{-1} + \left[\frac{x^5}{5} - \frac{5x^3}{3} + 4x\right]_{-1}^1 + \left[-\frac{x^5}{5} + \frac{5x^3}{3} - 4x\right]_1^2 = 8$

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48. $y = x\sqrt{a^2 - x^2}, a > 0$ and $y = 0$
Solution:
Limits of Integration: $x\sqrt{a^2 - x^2} = 0 \Longrightarrow x = 0$ or
$\sqrt{a^2 - x^2} = 0 \Longrightarrow x = 0$ or $a^2 - x^2 = 0 \Longrightarrow x = -a, 0, a$;
$A = \int_{-a}^0 -x\sqrt{a^2 - x^2}\ dx + \int_0^a x\sqrt{a^2 - x^2}\ dx$

$$= \frac{1}{2}\left[\frac{2}{3}\left(a^2 - x^2\right)^{3/2}\right]_{-a}^{0} - \frac{1}{2}\left[\frac{2}{3}\left(a^2 - x^2\right)^{3/2}\right]_{0}^{a}$$
$$= \frac{1}{3}\left(a^2\right)^{3/2} - \left[-\frac{1}{3}\left(a^2\right)^{3/2}\right] = \frac{2a^2}{3}$$

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49. $y = \sqrt{|x|}$ and $5y = x + 6$ (How many intersection points are there?)
Solution:
Limits of Integration: $y = \sqrt{|x|} = f(x) = \begin{cases} \sqrt{-x} & \text{if } x \le 0 \\ \sqrt{x} & \text{if } x > 0 \end{cases}$ and
$5y = x + 6$ or $y = \frac{x}{5} + \frac{6}{5}$; for $x \le 0$: $\sqrt{-x} = \frac{x}{5} + \frac{6}{5}$
$\Longrightarrow 5\sqrt{-x} = x + 6 \Longrightarrow 25(-x) = x^2 + 12x + 36$
$\Longrightarrow x^2 + 37x + 36 = 0 \Longrightarrow (x+1)(x+36) = 0$
$\Longrightarrow x = -1, -36$ (but $x = -36$ is not a solution);
for $x \ge 0 : 5\sqrt{x} = x + 6 \Longrightarrow 25x = x^2 + 12x + 36$
$\Longrightarrow x^2 - 13x + 36 = 0 \Longrightarrow (x-4)(x-9) = 0$
$\Longrightarrow x = 4, 9$; there are three intersection points and
$$A = \int_{-1}^{0}\left(\frac{x+6}{5} - \sqrt{-x}\right)\,dx + \int_{0}^{4}\left(\frac{x+6}{5} - \sqrt{x}\right)\,dx + \int_{4}^{9}\left(\sqrt{x} - \frac{x+6}{5}\right)\,dx$$
$$= \left[\frac{(x+6)^2}{10} + \frac{2}{3}(-x)^{3/2}\right]_{-1}^{0} + \left[\frac{(x+6)^2}{10} - \frac{2}{3}x^{3/2}\right]_{0}^{4} + \left[\frac{2}{3}x^{3/2} - \frac{(x+6)^2}{10}\right]_{4}^{9} = \frac{5}{3}$$

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50. $y = \left|x^2 - 4\right|$ and $y = \left(x^2/2\right) + 4$
Solution:
Limits of Integration:
$y = \left|x^2 - 4\right| = \begin{cases} x^2 - 4 & \text{if } x \le -2 \text{ or } x \ge 2 \\ 4 - x^2 & \text{if } -2 \le x \le 2 \end{cases}$
for $x \le -2$ and $x \ge 2$: $x^2 - 4 = \frac{x^2}{2} + 4$
$\Longrightarrow 2x^2 - 8 = x^2 + 8 \Longrightarrow x^2 = 16 \Longrightarrow x = \pm 4$;
for $-2 \le x \le 2$: $4 - x^2 = \frac{x^2}{2} + 4 \Longrightarrow 8 - 2x^2 = x^2 + 8$
$\Longrightarrow x^2 = 0 \Longrightarrow x = 0$
$$A = 2\int_{0}^{2}\left[\left(\frac{x^2}{2} + 4\right) - \left(4 - x^2\right)\right]\,dx + 2\int_{2}^{4}\left[\left(\frac{x^2}{2} + 4\right) - \left(x^2 - 4\right)\right]\,dx = 2\left[\frac{x^2}{2}\right]_{0}^{2} + 2\left[8x - \frac{x^3}{6}\right]_{2}^{4} = \frac{64}{3}$$

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51. $x = 2y^2, x = 0$, and $y = 3$
Solution:
Limits of Integration: $c = 0$ and $d = 3$;
$f(y) - g(y) = 2y^2 - 0 = 2y^2$
$$\Longrightarrow A = \int_{0}^{3} 2y^2\,dy = \left[\frac{2y^3}{3}\right]_{0}^{3} = 18$$

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52. $x = y^2$ and $x = y + 2$
Solution:

Limits of Integration: $y^2 = y + 2 \Longrightarrow (y+1)(y-2) = 0$
$c = -1, d = 2;$
$f(y) - g(y) = (y+2) - y^2$
$\Longrightarrow A = \int_{-1}^{2} \left[(y+2) - y^2\right] \, dy = \left[\frac{y^2}{2} + 2y - y^2\right]_{-1}^{2} = \frac{9}{2}$

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53. $y^2 - 4x = 4$ and $4x - y = 16$
Solution:
Limits of Integration: $4x = y^2 - 4$ and $4x = 16 + y \Longrightarrow y^2 - 4 = 16 + y$
$\Longrightarrow y^2 - y - 20 = 0 \Longrightarrow (y-5)(y+4) = 0 \Longrightarrow c = -4, d = 5;$
$f(y) - g(y) = \left(\frac{16+y}{4}\right) - \left(\frac{y^2-4}{4}\right) = \frac{-y^2+y+20}{4}$
$\Longrightarrow A = \frac{1}{4}\int_{-4}^{5} \left(-y^2 + y + 20\right) \, dy = \frac{243}{8}$

(p.414)

2. Revolution About the y-Axis

Use the shell method to find the volumes of the solids generated by revolving the regions bounded by the curves and lines in Exercises 7-14 about the y-axis.
7. $y = x, y = -x/2, x = 2$
Solution:
$a = 0, b = 2;$
$$V = \int_a^b 2\pi \begin{pmatrix} \text{shell} \\ \text{radius} \end{pmatrix} \begin{pmatrix} \text{shell} \\ \text{height} \end{pmatrix} \, dx = \int_0^2 2\pi x \left[x - \left(-\frac{x}{2}\right)\right] \, dx = \int_0^2 2\pi x^2 \cdot \frac{3}{2} \, dx$$
$$= \pi \int_0^2 3x^2 dx = \pi \left[x^3\right]_0^2 = 8\pi.$$

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8. $y = 2x, y = x/2, x = 1$
Solution:
$a = 0, b = 1;$
$$V = \int_a^b 2\pi \begin{pmatrix} \text{shell} \\ \text{radius} \end{pmatrix} \begin{pmatrix} \text{shell} \\ \text{height} \end{pmatrix} \, dx = \int_0^1 2\pi x \left[2x - \frac{x}{2}\right] \, dx = \pi \int_0^1 2\left(\frac{3x^2}{2}\right) \, dx$$
$$= \pi \int_0^1 3x^2 dx = \pi \left[x^3\right]_0^1 = \pi.$$

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9. $y = x^2, y = 2 - x, x = 0$, for $x \geq 0$
Solution:
$a = 0, b = 1;$
$$V = \int_a^b 2\pi \begin{pmatrix} \text{shell} \\ \text{radius} \end{pmatrix} \begin{pmatrix} \text{shell} \\ \text{height} \end{pmatrix} \, dx = \int_0^1 2\pi x \left[(2-x) - x^2\right] \, dx = 2\pi \int_0^1 \left(2x - x^2 - x^3\right) \, dx$$
$$= 2\pi \left[x^2 - \frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 = 2\pi \left(1 - \frac{1}{3} - \frac{1}{4}\right) = \frac{5\pi}{6}.$$

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10. $y = 2 - x^2, y = x^2, x = 0$

Solution:
$a = 0, b = 1;$

$$V = \int_a^b 2\pi \begin{pmatrix} \text{shell} \\ \text{radius} \end{pmatrix} \begin{pmatrix} \text{shell} \\ \text{height} \end{pmatrix} dx = \int_0^1 2\pi x \left[\left(2 - x^2\right) - x^2\right] dx = 2\pi \int_0^1 x\left(2 - 2x^2\right) dx$$
$$= 4\pi \int_0^1 \left(x - x^3\right) dx = \pi.$$

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11. $y = 2x - 1, y = \sqrt{x}, x = 0$
Solution:
$a = 0, b = 1;$

$$V = \int_a^b 2\pi \begin{pmatrix} \text{shell} \\ \text{radius} \end{pmatrix} \begin{pmatrix} \text{shell} \\ \text{height} \end{pmatrix} dx = \int_0^1 2\pi x \left[\sqrt{x} - (2x - 1)\right] dx = 2\pi \int_0^1 \left(x^{3/2} - 2x^2 + x\right) dx$$
$$= 2\pi \left[\frac{2}{5}x^{5/2} - \frac{2}{3}x^3 + \frac{1}{2}x^2\right]_0^1 = \frac{7\pi}{15}.$$

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12. $y = 3/\left(2\sqrt{x}\right), y = 0, x = 1, x = 4$
Solution:
$a = 1, b = 4;$

$$V = \int_a^b 2\pi \begin{pmatrix} \text{shell} \\ \text{radius} \end{pmatrix} \begin{pmatrix} \text{shell} \\ \text{height} \end{pmatrix} dx = \int_1^4 2\pi x \left(\frac{3}{2}x^{-1/2}\right) dx = 3\pi \int_1^4 x^{1/2}\, dx$$
$$= 3\pi \left[\frac{2}{3}x^{3/2}\right]_1^4 = 14\pi.$$

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3. Revolution About the x-Axis

Use the shell method to find the volumes of the solids generated by revolving the regions bounded by the curves and lines in Exercises 15-22 about the x-axis.
15. $x = \sqrt{y}, x = -y, y = 2$
Solution:
$c = 0, d = 2;$

$$V = \int_c^d 2\pi \begin{pmatrix} \text{shell} \\ \text{radius} \end{pmatrix} \begin{pmatrix} \text{shell} \\ \text{height} \end{pmatrix} dy = \int_0^2 2\pi y \left[\sqrt{y} - (-y)\right] dy = 2\pi \int_0^2 \left(y^{3/2} + y^2\right) dy$$
$$= 2\pi \left[\frac{2y^{5/2}}{5} + \frac{y^3}{3}\right]_0^2 = 2\pi \left[\frac{2}{5}\left(\sqrt{2}\right)^5 + \frac{2^3}{3}\right] = 2\pi \left(\frac{8\sqrt{2}}{5} + \frac{8}{3}\right)$$
$$= 16\pi \left(\frac{\sqrt{2}}{5} + \frac{1}{3}\right) = \frac{16\pi}{15}\left(3\sqrt{2} + 5\right).$$

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16. $x = y^2, x = -y, y = 2, y \geq 0$
Solution:
$c = 0, d = 2;$

$$V = \int_c^d 2\pi \begin{pmatrix} \text{shell} \\ \text{radius} \end{pmatrix} \begin{pmatrix} \text{shell} \\ \text{height} \end{pmatrix} dy = \int_0^2 2\pi y \left[y^2 - (-y)\right] dy = 2\pi \int_0^2 \left(y^3 + y^2\right) dy$$
$$= 2\pi \left[\frac{y^4}{4} + \frac{y^3}{3}\right]_0^2 = 16\pi \left[\frac{2}{4} + \frac{1}{3}\right] = \frac{40\pi}{3}.$$

17. $x = 2y - y^2, x = 0$
Solution:

$c = 0, d = 2;$
$V = \int_c^d 2\pi \begin{pmatrix} \text{shell} \\ \text{radius} \end{pmatrix} \begin{pmatrix} \text{shell} \\ \text{height} \end{pmatrix} \, dy = \int_0^2 2\pi y \left[2y - y^2\right] \, dy = 2\pi \int_0^2 \left(2y^2 - y^3\right) \, dy$
$= 2\pi \left[\frac{2y^3}{3} - \frac{y^4}{4}\right]_0^2 = 2\pi \left[\frac{16}{3} - \frac{16}{4}\right] = \frac{8\pi}{3}.$

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18. $x = 2y - y^2, x = y$
Solution:

$c = 0, d = 1;$
$V = \int_c^d 2\pi \begin{pmatrix} \text{shell} \\ \text{radius} \end{pmatrix} \begin{pmatrix} \text{shell} \\ \text{height} \end{pmatrix} \, dy = \int_0^1 2\pi y \left[2y - y^2 - y\right] \, dy = 2\pi \int_0^1 y \left(y - y^2\right) \, dy$
$= 2\pi \int_0^1 \left(y^2 - y^3\right) \, dy = 2\pi \left[\frac{y^3}{3} - \frac{y^4}{4}\right]_0^1 = \frac{\pi}{6}.$

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19. $y = |x|, y = 1$
Solution:
$c = 0, d = 1;$
$V = \int_c^d 2\pi \begin{pmatrix} \text{shell} \\ \text{radius} \end{pmatrix} \begin{pmatrix} \text{shell} \\ \text{height} \end{pmatrix} \, dy = 2\pi \int_0^1 y \left[y - (-y)\right] \, dy = 2\pi \int_0^1 2y^2 \, dy$
$= \frac{4\pi}{3} \left[y^3\right]_0^1 = \frac{4\pi}{3}.$

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20. $y = x, y = 2x, y = 2$
Solution:

$c = 0, d = 2;$
$V = \int_c^d 2\pi \begin{pmatrix} \text{shell} \\ \text{radius} \end{pmatrix} \begin{pmatrix} \text{shell} \\ \text{height} \end{pmatrix} \, dy = 2\pi \int_0^2 y \left[y - \frac{y}{2}\right] \, dy = 2\pi \int_0^2 \frac{y^2}{2} \, dy$
$= \frac{\pi}{3} \left[y^3\right]_0^1 = \frac{8\pi}{3}.$

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21. $y = \sqrt{x}, y = 0, y = x - 2$
Solution:
$c = 0, d = 2;$
$V = \int_c^d 2\pi \begin{pmatrix} \text{shell} \\ \text{radius} \end{pmatrix} \begin{pmatrix} \text{shell} \\ \text{height} \end{pmatrix} \, dy = 2\pi \int_0^2 y \left[(2 + y) - y^2\right] \, dy = 2\pi \int_0^2 \left(2y + y^2 - y^3\right) \, dy$
$= \frac{16\pi}{3}.$

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22. $y = \sqrt{x}, y = 0, y = 2 - x$
Solution:

$c = 0, d = 1;$

$$V = \int_c^d 2\pi \begin{pmatrix} \text{shell} \\ \text{radius} \end{pmatrix} \begin{pmatrix} \text{shell} \\ \text{height} \end{pmatrix} dy = 2\pi \int_0^1 y\left[(2-y) - y^2\right] dy = 2\pi \int_0^2 \left(2y - y^2 - y^3\right) dy$$

$$= \frac{5\pi}{6}.$$

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4. Comparing the Washer and Shell Models

25. Compute the volumes of the solid generated by revolving the region bounded by $y = x$ and $y = x^2$ about each coordinate axis using

a. the shell method. **b.** the washer method.

Solution:

(a) About x-axis: $V = \int_c^d 2\pi \begin{pmatrix} \text{shell} \\ \text{radius} \end{pmatrix} \begin{pmatrix} \text{shell} \\ \text{height} \end{pmatrix} dy = \int_0^1 2\pi y\left(\sqrt{y} - y\right) dy$

$$= 2\pi \int_0^1 \left(y^{3/2} - y^2\right) dy = 2\pi \left[\frac{2}{5}y^{5/2} - \frac{1}{3}y^3\right]_0^1 = 2\pi\left(\frac{2}{5} - \frac{1}{3}\right) = \frac{2\pi}{15}$$

About y-axis: $V = \int_a^b 2\pi \begin{pmatrix} \text{shell} \\ \text{radius} \end{pmatrix} \begin{pmatrix} \text{shell} \\ \text{height} \end{pmatrix} dy = \int_0^1 2\pi x\left(x - x^2\right) dx$

$$= 2\pi \int_0^1 \left(x^2 - x^3\right) dx = 2\pi \left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 = 2\pi\left(\frac{1}{3} - \frac{1}{4}\right) = \frac{\pi}{6}$$

(b) About x-axis: $R(x) = x$ and $r(x) = x^2$

$$\Longrightarrow V = \int_a^b \pi\left[R(x)^2 - r(x)^2\right] dx = \int_0^1 \pi\left(x^2 - x^4\right) dx$$

$$= \pi\left[\frac{x^3}{3} - \frac{x^5}{5}\right]_0^1 = \pi\left(\frac{1}{3} - \frac{1}{5}\right) = \frac{2\pi}{15}$$

About y-axis: $R(y) = \sqrt{y}$ and $r(y) = y$

$$\Longrightarrow V = \int_c^d \pi\left[R(y)^2 - r(y)^2\right] dy = \int_0^1 \pi\left(y - y^2\right) dy$$

$$= \pi\left[\frac{y^2}{2} - \frac{y^3}{3}\right]_0^1 = \pi\left(\frac{1}{2} - \frac{1}{3}\right) = \frac{\pi}{6}$$

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5. Choosing Shells or Washers

In Exercises 27-32, find the volumes of the solids generated by revolving the regions about the given axes. If you think it would be better to use washers in any given instance, feel free to do so.

27. The triangle with vertices $(1,1), (1,2)$, and $(2,2)$ about

a. the x-axis **b.** the y-axis

c. the line $x = 10/3$ **d.** the line $y = 1$

Solution:

(a) $V = \int_c^d 2\pi \begin{pmatrix} \text{shell} \\ \text{radius} \end{pmatrix} \begin{pmatrix} \text{shell} \\ \text{height} \end{pmatrix} dy = \int_1^2 2\pi y\,(y-1)\, dy$

$$= 2\pi \int_1^2 \left(y^2 - y\right) dy = 2\pi\left[\frac{y^3}{3} - \frac{y^2}{2}\right]_0^2 = \frac{5\pi}{3}$$

(b) $V = \int_a^b 2\pi \begin{pmatrix} \text{shell} \\ \text{radius} \end{pmatrix} \begin{pmatrix} \text{shell} \\ \text{height} \end{pmatrix} dx = 2\pi \int_1^2 x\,(2-x)\, dx$

$$= 2\pi \int_1^2 \left(2x - x^2\right) \; dx = 2\pi \left[x^2 - \frac{x^3}{3}\right]_1^2 = \frac{4\pi}{3}$$

(c) $V = \displaystyle\int_a^b 2\pi \begin{pmatrix} \text{shell} \\ \text{radius} \end{pmatrix} \begin{pmatrix} \text{shell} \\ \text{height} \end{pmatrix} \; dx = \int_1^2 2\pi \left(\frac{10}{3} - x\right) (2 - x) \; dx$

$$= 2\pi \int_1^2 \left(\frac{20}{3} - \frac{16}{3}x + x^2\right) \; dx = 2\pi \left[\frac{20}{3}x - \frac{8}{3}x^2 + \frac{1}{3}x^3\right]_1^2 = 2\pi$$

(d) $V = \displaystyle\int_c^d 2\pi \begin{pmatrix} \text{shell} \\ \text{radius} \end{pmatrix} \begin{pmatrix} \text{shell} \\ \text{height} \end{pmatrix} \; dy = \int_1^2 2\pi \left(y - 1\right) \left(y - 1\right) \; dy$

$$= 2\pi \int_1^2 \left(y - 1\right)^2 \; dy = 2\pi \left[\frac{\left(y - 1\right)^3}{3}\right]_1^2 = \frac{2\pi}{3}$$
