ÇANKAYA UNIVERSITY
Department of Mathematics and Computer Science
MATH 155 Calculus for Engineering I
Summer 2008

Problems and Solutions for Recitation 5

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(p.474)

1. Derivatives of Inverse Functions

31. Let $f(x) = x^3 - 3x^2, x \geq 2$. Find the value of df^{-1}/dx at the point $x = -1 = f(3)$.
Solution:
$$\frac{df}{dx} = 3x^2 - 6x \Longrightarrow \frac{df^{-1}}{dx}\Big|_{x=f(3)} = \frac{1}{\frac{df}{dx}}\Big|_{x=3} = \frac{1}{9}$$

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32. Let $f(x) = x^2 - 4x, x > 2$. Find the value of df^{-1}/dx at the point $x = 0 = f(5)$.
Solution:
$$\frac{df}{dx} = 2x - 4 \Longrightarrow \frac{df^{-1}}{dx}\Big|_{x=f(5)} = \frac{1}{\frac{df}{dx}}\Big|_{x=5} = \frac{1}{6}$$

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33. Suppose that the differentiable function $y = f(x)$ has an inverse and that the graph of f passes through the point $(2, 4)$ and has a slope of $1/3$ there. Find the value of df^{-1}/dx at the point $x = 4$.
Solution:
$$\frac{df^{-1}}{dx}\Big|_{x=4} = \frac{df^{-1}}{dx}\Big|_{x=f(2)} = \frac{1}{\frac{df}{dx}}\Big|_{x=2} = \frac{1}{\left(\frac{1}{3}\right)} = 3.$$

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34. Suppose that the differentiable function $y = g(x)$ has an inverse and that the graph of g passes through the origin with slope 2. Find the slope of the graph of g^{-1} at the origin.
Solution:
$$\frac{dg^{-1}}{dx}\Big|_{x=0} = \frac{dg^{-1}}{dx}\Big|_{x=f(0)} = \frac{1}{\frac{dg}{dx}}\Big|_{x=0} = \frac{1}{2}.$$

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2. Using Properties of Logarithms

(p.484)
1. Express the following logarithms in terms of $\ln 2$ and $\ln 3$.
a. $\ln 0.75$ **b.** $\ln(4/9)$ **c.** $\ln(1/2)$

d. $\ln \sqrt[3]{9}$ **e.** $\ln 3\sqrt{2}$ **f.** $\ln \sqrt{13.5}$

Solution:

(a) $\ln 0.75 = \ln \frac{3}{4} = \ln 3 - \ln 4 = \ln 3 - \ln 2^2 = \ln 3 - 2\ln 2$

(b) $\ln(4/9) = \ln 4 - \ln 9 = \ln 2^2 - \ln 3^2 = 2\ln 2 - 2\ln 3$

(c) $\ln(1/2) = \ln 1 - \ln 2 = -\ln 2$

(d) $\ln \sqrt[3]{9} = \frac{1}{3}\ln 9 = \frac{1}{3}\ln 3^2 = \frac{2}{3}\ln 3$

(e) $\ln 3\sqrt{2} = \ln 3 + \ln 2^{1/2} = \ln 3 + \frac{1}{2}\ln 2$

(f) $\ln \sqrt{13.5} = \frac{1}{2}\ln 13.5 = \frac{1}{2}\ln \frac{27}{2} = \frac{1}{2}\left(\ln 3^3 - \ln 2\right) = \frac{1}{2}(3\ln 3 - \ln 2).$

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2. Express the following logarithms in terms of $\ln 5$ and $\ln 7$.

a. $\ln(1/125)$ **b.** $\ln 9.8$ **c.** $\ln 7\sqrt{7}$

d. $\ln 1225$ **e.** $\ln 0.056$ **f.** $(\ln 35 + \ln(1/7))/\ln 25$

Solution:

(a) $\ln(1/125) = \ln 1 - 3\ln 5 = -3\ln 5$

(b) $\ln 9.8 = \ln \frac{49}{5} = \ln 7^2 - \ln 5 = 2\ln 7 - \ln 5$

(c) $\ln 7\sqrt{7} = \ln 7^{3/2} = \frac{3}{2}\ln 7$

(d) $\ln 1225 = \ln 35^2 = 2\ln 35 = 2\ln 5 + 2\ln 7$

(e) $\ln 0.056 = \ln \frac{7}{125} = \ln 7 - \ln 5^3 = \ln 7 - 3\ln 5$

(f) $(\ln 35 + \ln(1/7))/\ln 25 = \dfrac{\ln 5 + \ln 7 - \ln 7}{2\ln 5} = \dfrac{1}{2}.$

Use properties of logarithms to simplify the expressions in Exercises 3 and 4.

3. **a.** $\ln \sin\theta - \ln\left(\frac{\sin\theta}{5}\right)$ **b.** $\ln\left(3x^2 - 9x\right) + \ln\left(\frac{1}{3x}\right)$

c. $\frac{1}{2}\ln\left(4t^4\right) - \ln 2$

Solution:

(a) $\ln \sin\theta - \ln\left(\frac{\sin\theta}{5}\right) = \ln\left(\frac{\sin\theta}{\frac{\sin\theta}{5}}\right) = \ln 5$

(b) $\ln\left(3x^2 - 9x\right) + \ln\left(\frac{1}{3x}\right) = \ln\left(\frac{3x^2 - 9x}{3x}\right) = \ln(x-3)$

(c) $\frac{1}{2}\ln\left(4t^4\right) - \ln 2 = \ln\sqrt{4t^4} - \ln 2 = \ln 2t^2 - \ln 2 = \ln\left(\frac{2t^2}{2}\right) = \ln\left(t^2\right)$

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4. **a.** $\ln \sec\theta + \ln\cos\theta$ **b.** $\ln(8x+4) - 2\ln 2$

c. $3\ln\sqrt[3]{t^2-1} - \ln(t+1)$

Solution:

(a) $\ln\sec\theta + \ln\cos\theta = \ln[(\sec\theta)(\cos\theta)] = \ln 1 = 0$

(b) $\ln(8x+4) - 2\ln 2 = \ln(8x+4) - \ln 4 = \ln\left(\frac{8x+4}{4}\right) = \ln(2x+1)$

(c) $3\ln\sqrt[3]{t^2-1} - \ln(t+1) = 3\ln\left(t^2-1\right)^{1/3} - \ln(t+1) = 3\left(\frac{1}{3}\right)\ln\left(t^2-1\right) - \ln(t+1) = \ln\left(\frac{(t+1)(t-1)}{(t+1)}\right) = \ln(t-1).$

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3. Derivatives of Logarithms

In Exercises 5-36, find the derivative of y with respect to x, t, or θ, as appropriate.

5. $y = \ln 3x$
Solution:
$$y = \ln 3x \Longrightarrow y' = \left(\frac{1}{3x}\right)(3) = \frac{1}{x}.$$

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6. $y = \ln kx$, k constant
Solution:
$$y = \ln kx \Longrightarrow y' = \left(\frac{1}{kx}\right)(k) = \frac{1}{x}.$$

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7. $y = \ln\left(t^2\right)$
Solution:
$$y = \ln\left(t^2\right) \Longrightarrow y' = \left(\frac{1}{t^2}\right)(2t) = \frac{2}{t}.$$

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8. $y = \ln\left(t^{3/2}\right)$
Solution:
$$y = \ln\left(t^{3/2}\right) \Longrightarrow \frac{dy}{dt} = \left(\frac{1}{t^{3/2}}\right)\left(\frac{3}{2}t^{1/2}\right) = \frac{3}{2t}.$$

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9. $y = \ln\frac{3}{x}$
Solution:
$$y = \ln\frac{3}{x} = \ln 3x^{-1} \Longrightarrow \frac{dy}{dx} = \left(\frac{1}{3x^{-1}}\right)\left(-3x^{-2}\right) = -\frac{1}{x}.$$

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10. $y = \ln\frac{3}{x}$
Solution:
$$y = \ln\frac{3}{x} = \ln 3 - \ln x \Longrightarrow y' = -\frac{1}{x}$$

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24. $y = \ln(\ln(\ln x))$
Solution:
$$y = \ln(\ln(\ln x)) \Longrightarrow y' = \frac{1}{\ln(\ln x)}\frac{d}{dx}(\ln(\ln x)) = \frac{1}{\ln(\ln x)}\frac{1}{\ln x}\frac{d}{dx}(\ln x) = \frac{1}{x\ln x\ln(\ln x)}$$

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25. $y = \theta(\sin(\ln\theta) + \cos(\ln\theta))$
Solution:
$$y = \theta(\sin(\ln\theta) + \cos(\ln\theta)) \Longrightarrow \frac{dy}{d\theta} = [\sin(\ln\theta) + \cos(\ln\theta)] + \theta\left(\cos(\ln\theta)\cdot\frac{1}{\theta} - \sin(\ln\theta)\cdot\frac{1}{\theta}\right)$$
$$= \sin(\ln\theta) + \cos(\ln\theta) + \cos(\ln\theta) - \sin(\ln\theta) = 2\cos(\ln\theta)$$

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26. $y = \ln(\sec\theta + \tan\theta)$

Answer:
$y' = \sec\theta$

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27. $y = \ln \dfrac{1}{x\sqrt{x+1}}$
Answer:
$y' = -\ln x - \dfrac{1}{2}\ln(x+1) \Longrightarrow y' = -\dfrac{3x+2}{2x(x+1)}.$

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28. $y = \dfrac{1}{2}\ln\dfrac{1+x}{1-x}$
Answer:
$y' = \dfrac{1}{1-x^2}.$

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29. $y = \dfrac{1+\ln t}{1-\ln t}$
Answer:
$y' = \dfrac{2}{t(1-\ln t)^2}.$

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30. $y = \sqrt{\ln\sqrt{t}}$
Solution:
$y = \sqrt{\ln\sqrt{t}} = \left(\ln t^{1/2}\right)^{1/2} \Longrightarrow \dfrac{dy}{dt} = \dfrac{1}{2}\left(\ln t^{1/2}\right)^{-1/2} \cdot \dfrac{d}{dt}\left(\ln t^{1/2}\right) = \dfrac{1}{2}\left(\ln t^{1/2}\right)^{-1/2} \cdot \dfrac{1}{t^{1/2}} \cdot \dfrac{d}{dt}\left(t^{1/2}\right)$
$y' = \dfrac{1}{4t\sqrt{\ln\sqrt{t}}}.$

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31. $y = \ln(\sec(\ln\theta))$
Solution:
$y = \ln(\sec(\ln\theta)) \Longrightarrow \dfrac{dy}{d\theta} = \dfrac{1}{\sec(\ln\theta)} \cdot \dfrac{d}{d\theta}(\sec(\ln\theta)) = \dfrac{\tan(\ln\theta)}{\theta}.$

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32. $y = \ln\dfrac{\sqrt{\sin\theta\cos\theta}}{1+2\ln\theta}$
Solution:
$y = y = \ln\dfrac{\sqrt{\sin\theta\cos\theta}}{1+2\ln\theta} = \dfrac{1}{2}(\ln\sin\theta + \ln\cos\theta) - \ln(1+2\ln\theta) \Longrightarrow \dfrac{dy}{d\theta} = \dfrac{1}{2}\left(\dfrac{\cos\theta}{\sin\theta} - \dfrac{\sin\theta}{\cos\theta}\right) -$
$\dfrac{\frac{2}{\theta}}{1+2\ln\theta}$
$\Longrightarrow y' = \dfrac{1}{2}\left[\cot\theta - \tan\theta - \dfrac{4}{\theta(1+2\ln\theta)}\right].$

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33. $y = \ln\left(\dfrac{(x^2+1)^5}{\sqrt{1-x}}\right)$
Solution:

$y = \ln\left(\frac{(x^2+1)^5}{\sqrt{1-x}}\right) = 5\ln\left(x^2+1\right) - \frac{1}{2}\ln(1-x) \Longrightarrow y' = \frac{5\cdot 2x}{x^2+1} - \frac{1}{2}\left(\frac{1}{1-x}\right)(-1) = \frac{10x}{x^2+1} + \frac{1}{2(1-x)}$.

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34. $y = \ln\sqrt{\frac{(x+1)^5}{(x+2)^{20}}}$

Solution:

$y = \ln\sqrt{\frac{(x+1)^5}{(x+2)^{20}}} = \frac{1}{2}\left[5\ln(x+1) - 20\ln(x+2)\right] \Longrightarrow y' = \frac{1}{2}\left(\frac{5}{x+1} - \frac{20}{x+2}\right)$.

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35. $y = \int_{x^2/2}^{x^2} \ln\sqrt{t}\ dt$

Solution:

$y = \int_{x^2/2}^{x^2} \ln\sqrt{t}\ dt \Longrightarrow \frac{dy}{dx} = \left(\ln\sqrt{x^2}\right)\cdot\frac{d}{dx}\left(x^2\right) - \left(\ln\sqrt{\frac{x^2}{2}}\right)\cdot\frac{d}{dx}\left(\frac{x^2}{2}\right) = 2x\ln|x| - x\ln\frac{|x|}{\sqrt{2}}$

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36. $y = \int_{\sqrt{x}}^{\sqrt[3]{x}} \ln t\ dt$

Solution:

$y = \int_{\sqrt{x}}^{\sqrt[3]{x}} \ln t\ dt \Longrightarrow \frac{dy}{dx} = \left(\ln\sqrt[3]{x}\right)\cdot\frac{d}{dx}\left(\sqrt[3]{x}\right) - \left(\ln\sqrt{x}\right)\cdot\frac{d}{dx}\left(\sqrt{x}\right) = \left(\ln\sqrt[3]{x}\right)\left(\frac{1}{3}x^{-2/3}\right) - \left(\ln\sqrt{x}\right)\left(\frac{1}{2}x^{-1/2}\right)$

$y' = \frac{\ln\sqrt[3]{x}}{3\sqrt[3]{x^2}} - \frac{\ln\sqrt{x}}{2\sqrt{x}}$.

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4. Integration

Evaluate the integrals in Exercises 37-54.

37. $\int_{-3}^{-2} \frac{dx}{x}$

Solution:

$\int_{-3}^{-2} \frac{dx}{x} = \left[\ln|x|\right]_{-3}^{-2} = \ln\frac{2}{3}$.

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38. $\int_{-1}^{0} \frac{3\ dx}{3x-2}$

Solution:

$\int_{-1}^{0} \frac{3\ dx}{3x-2} = \left[\ln|3x-2|\right]_{-1}^{0} = \ln\frac{2}{5}$.

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39. $\displaystyle\int \frac{2y\ dy}{y^2-25}$

Solution:

$$\int \frac{2y\ dy}{y^2-25} = \ln\left|y^2-25\right| + C.$$

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40. $\displaystyle\int \frac{8y\ dy}{4y^2-5}$

Solution:

$$\int \frac{8y\ dy}{4y^2-5} = \ln\left|4y^2-5\right| + C.$$

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41. $\displaystyle\int_0^{\pi} \frac{\sin t\ dt}{2-\cos t}$

Solution:

$$\int_0^{\pi} \frac{\sin t\ dt}{2-\cos t} = \ln\left[2-\cos t\right]_0^{\pi} = \ln 3.$$

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53. $\displaystyle\int \frac{dx}{2\sqrt{x}+2x}$

Solution:

$$\int \frac{dx}{2\sqrt{x}+2x} = \int \frac{dx}{2\sqrt{x}\left(1+\sqrt{x}\right)}; \text{ let } u = 1+\sqrt{x} \Longrightarrow du = \frac{1}{2\sqrt{x}}\ dx;$$

$$\int \frac{dx}{2\sqrt{x}\left(1+\sqrt{x}\right)} = \int \frac{du}{u} = \ln|u| + C = \ln\left|1+\sqrt{x}\right| + C = \ln\left(1+\sqrt{x}\right) + C$$

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5. Logarithmic Differentiation

In Exercises 55-68, use logarithmic differentiation to find the derivative of y with respect to the given independent variable.

55. $y = \sqrt{x\left(x+1\right)}$

Solution:

$$y = \sqrt{x\left(x+1\right)} = \left(x\left(x+1\right)\right)^{1/2} \Longrightarrow \ln y = \frac{1}{2}\ln\left(x\left(x+1\right)\right) \Longrightarrow 2\ln y = \ln x + \ln\left(x+1\right)$$

$$\Longrightarrow \frac{2y'}{y} = \frac{1}{x} + \frac{1}{x+1} \Longrightarrow y' = \left(\frac{1}{2}\right)\sqrt{x\left(x+1\right)}\left(\frac{1}{x}+\frac{1}{x+1}\right) = \frac{2x+1}{2\sqrt{x\left(x+1\right)}}$$

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56. $y = \sqrt{\left(x^2+1\right)\left(x-1\right)^2}$

Solution:

$$y = \sqrt{\left(x^2+1\right)\left(x-1\right)^2} \Longrightarrow \ln y = \frac{1}{2}\left[\ln\left(x^2+1\right) + 2\ln\left(x-1\right)\right]$$

$$\Longrightarrow \frac{y'}{y} = \frac{1}{2}\left(\frac{2x}{x^2+1}+\frac{2}{x-1}\right) \Longrightarrow y' = \sqrt{\left(x^2+1\right)\left(x-1\right)^2}\left(\frac{x}{x^2+1}+\frac{1}{x-1}\right)$$

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57. $y = \sqrt{\dfrac{t}{t+1}}$

Solution:

$$y = \sqrt{\frac{t}{t+1}} = \left(\frac{t}{t+1}\right)^{1/2} \Longrightarrow \ln y = \frac{1}{2}\left[\ln t - \ln(t+1)\right]$$
$$\Longrightarrow \frac{1}{y}\frac{dy}{dt} = \frac{1}{2}\left(\frac{1}{t} - \frac{1}{t+1}\right) \Longrightarrow \frac{dy}{dt} = \frac{1}{2}\sqrt{\frac{t}{t+1}}\left[\frac{1}{t(t+1)}\right]$$

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58. $y = \sqrt{\frac{1}{t(t+1)}}$

Answer:

$$y = \sqrt{\frac{1}{t(t+1)}}$$
$$\Longrightarrow \frac{dy}{dt} = -\frac{2t+1}{2(t^2+1)^{3/2}}.$$

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59. $y = \sqrt{\theta+3}\,(\sin\theta)$

Answer:

$$y = \sqrt{\theta+3}\,(\sin\theta)$$
$$\Longrightarrow \frac{dy}{d\theta} = \sqrt{\theta+3}\,(\sin\theta)\left[\frac{1}{2(\theta+3)} + \cot\theta\right].$$

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60. $y = (\tan\theta)\sqrt{2\theta+1}$

Answer:

$$y = (\tan\theta)\sqrt{2\theta+1}$$
$$\Longrightarrow \frac{dy}{d\theta} = \left(\sec^2\theta\right)\sqrt{2\theta+1} + \frac{\tan\theta}{\sqrt{2\theta+1}}.$$

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61. $y = t(t+1)(t+2)$

Answer:

$$y = t(t+1)(t+2)$$
$$\Longrightarrow \frac{dy}{dt} = 3t^2 + 6t + 2.$$

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62. $y = \frac{1}{t(t+1)(t+2)}$

Solution:

$$y = \frac{1}{t(t+1)(t+2)} \Longrightarrow \ln y = \ln 1 - \ln t - \ln(t+1) - \ln(t+2) \Longrightarrow \frac{1}{y}\frac{dy}{dt} = -\frac{1}{t} - \frac{1}{t+1} - \frac{1}{t+2}$$
$$\Longrightarrow \frac{dy}{dt} = \frac{1}{t(t+1)(t+2)}\left[-\frac{1}{t} - \frac{1}{t+1} - \frac{1}{t+2}\right] = -\frac{3t^2+6t+2}{(t^3+3t^2+2t)^2}.$$

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63. $y = \frac{\theta+5}{\theta\cos\theta}$

Answer:

$$y = \frac{\theta+5}{\theta\cos\theta}$$
$$\Longrightarrow \frac{dy}{d\theta} = \left(\frac{\theta+5}{\theta\cos\theta}\right)\left(\frac{1}{\theta+5} - \frac{1}{\theta} + \tan\theta\right).$$

64. $y = \dfrac{\theta \sin\theta}{\sqrt{\sec\theta}}$

Answer:

$$y = \frac{\theta \sin\theta}{\sqrt{\sec\theta}}$$
$$\Longrightarrow \frac{dy}{d\theta} = \frac{\theta \sin\theta}{\sqrt{\sec\theta}} \left(\frac{1}{\theta} + \cot\theta - \frac{1}{2}\tan\theta\right).$$

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65. $y = \dfrac{x\sqrt{x^2+1}}{(x+1)^{2/3}}$

Answer:

$$y = \frac{x\sqrt{x^2+1}}{(x+1)^{2/3}}$$
$$\Longrightarrow \frac{dy}{dx} = \frac{x\sqrt{x^2+1}}{(x+1)^{2/3}} \left[\frac{1}{x} + \frac{x}{x^2+1} - \frac{2}{3\,(x+1)}\right].$$

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66. $y = \sqrt{\dfrac{(x+1)^{10}}{(2x+1)^5}}$

Answer:

$$y = \sqrt{\frac{(x+1)^{10}}{(2x+1)^5}}$$
$$\Longrightarrow \frac{dy}{dx} = \sqrt{\frac{(x+1)^{10}}{(2x+1)^5}} \left[\frac{5}{x+1} - \frac{5}{2x+1}\right].$$

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67. $y = \sqrt[3]{\dfrac{x\,(x-2)}{x^2+1}}$

Answer:

$$y = \sqrt[3]{\frac{x\,(x-2)}{x^2+1}}$$
$$\Longrightarrow \frac{dy}{dx} = \frac{1}{3}\sqrt[3]{\frac{x\,(x-2)}{x^2+1}} \left[\frac{1}{x} + \frac{1}{x-2} - \frac{2x}{x^2+1}\right].$$

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68. $y = \sqrt[3]{\dfrac{x\,(x+1)\,(x-2)}{(x^2+1)\,(2x+3)}}$

Solution:

$$y = \sqrt[3]{\frac{x\,(x+1)\,(x-2)}{(x^2+1)\,(2x+3)}} \Longrightarrow \ln y = \frac{1}{3}\left[\ln x + \ln(x+1) + \ln(x-2) - \ln\left(x^2+1\right) - \ln(2x+3)\right]$$
$$\Longrightarrow y' = \frac{1}{3}\sqrt[3]{\frac{x\,(x+1)\,(x-2)}{(x^2+1)\,(2x+3)}} \left(\frac{1}{x} + \frac{1}{x+1} + \frac{1}{x-2} - \frac{2x}{x^2+1} - \frac{2}{2x+3}\right).$$

In Exercises 17-36, find the derivative of y with respect to x, t, θ as appropriate.

17. $y = e^{-5x}$

Solution:

$y = e^{-5x} \Longrightarrow y' = -5e^{-5x}$.

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18. $y = e^{2x/3}$

Solution:

$y = e^{2x/3} \Longrightarrow y' = \frac{2}{3}e^{2x/3}$.

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36. $y = \int_{e^{4\sqrt{x}}}^{e^{2x}} \ln t \, dt$

Solution:

$y = \int_{e^{4\sqrt{x}}}^{e^{2x}} \ln t \, dt \Longrightarrow y' = \left(\ln e^{2x}\right) \cdot \frac{d}{dx}\left(e^{2x}\right) - \left(\ln e^{4\sqrt{x}}\right) \cdot \frac{d}{dx}\left(e^{4\sqrt{x}}\right) = (2x)\left(e^{2x}\right) - \left(4\sqrt{x}\right)\left(e^{4\sqrt{x}}\right) \cdot \frac{d}{dx}\left(4\sqrt{x}\right)$

$\Longrightarrow y' = 4xe^{2x} - 4\sqrt{x}e^{4\sqrt{x}}\left(\frac{2}{\sqrt{x}}\right) \Longrightarrow y' = 4xe^{2x} - 8e^{4\sqrt{x}}$.

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7. EVALUATING LIMITS

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Evaluate the limits in Exercises 85-96.

85. $\lim_{x \to 0} \frac{10^x - 1}{x}$

Solution:

The limit leads to the indeterminate form $\frac{0}{0}$: $\lim_{x \to 0} \frac{10^x - 1}{x} = \lim_{x \to 0} \frac{(\ln 10)\, 10^x}{1} = \ln 10$.

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86. $\lim_{\theta \to 0} \frac{3^\theta - 1}{\theta}$

Solution:

The limit leads to the indeterminate form $\frac{0}{0}$: $\lim_{\theta \to 0} \frac{3^\theta - 1}{\theta} = \lim_{\theta \to 0} \frac{(\ln 3)\, 3^\theta}{1} = \ln 3$.

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87. $\lim_{x \to 0} \frac{2^{\sin x} - 1}{e^x - 1}$

Solution:

The limit leads to the indeterminate form $\frac{0}{0}$: $\lim_{x \to 0} \frac{2^{\sin x} - 1}{e^x - 1} = \lim_{x \to 0} \frac{2^{\sin x}(\ln 2)(\cos x)}{e^x} = \ln 2$.

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88. $\lim_{x \to 0} \frac{2^{-\sin x} - 1}{e^x - 1}$

Solution:

The limit leads to the indeterminate form $\frac{0}{0}$: $\lim_{x\to 0}\frac{2^{-\sin x}-1}{e^x-1} = \lim_{x\to 0}\frac{2^{-\sin x}(\ln 2)(-\cos x)}{e^x} = -\ln 2$.

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89. $\lim_{x\to 0}\frac{5-5\cos x}{e^x-x-1}$

Solution:

The limit leads to the indeterminate form $\frac{0}{0}$: $\lim_{x\to 0}\frac{5-5\cos x}{e^x-x-1} = \lim_{x\to 0}\frac{5\sin x}{e^x-1} = \lim_{x\to 0}\frac{5\cos x}{e^x} = 5$.

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90. $\lim_{x\to 0}\frac{4-4e^x}{xe^x}$

Solution:

The limit leads to the indeterminate form $\frac{0}{0}$: $\lim_{x\to 0}\frac{4-4e^x}{xe^x} = \lim_{x\to 0}\frac{-4e^x}{e^x+xe^x} = -4$.

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91. $\lim_{t\to 0^+}\frac{t-\ln(1+2t)}{t^2}$

Solution:

The limit leads to the indeterminate form $\frac{0}{0}$: $\lim_{t\to 0^+}\frac{t-\ln(1+2t)}{t^2} = \lim_{t\to 0^+}\frac{\left(1-\frac{2}{1+2t}\right)}{2t} = -\infty$.

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92. $\lim_{x\to 4}\frac{\sin^2(\pi x)}{e^{x-4}+3-x}$

Solution:

The limit leads to the indeterminate form $\frac{0}{0}$: $\lim_{x\to 4}\frac{\sin^2(\pi x)}{e^{x-4}+3-x} = \lim_{x\to 4}\frac{2\pi\sin(\pi x)\cos(\pi x)}{e^{x-4}-1}$
$= \lim_{x\to 4}\frac{\pi\sin(2\pi x)}{e^{x-4}-1} = \lim_{x\to 4}\frac{2\pi^2\cos(2\pi x)}{e^{x-4}} = 2\pi^2$.

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93. $\lim_{t\to 0^+}\left(\frac{e^t}{t}-\frac{1}{t}\right)$

Solution:

The limit leads to the indeterminate form $\frac{0}{0}$: $\lim_{t\to 0^+}\left(\frac{e^t}{t}-\frac{1}{t}\right) = \lim_{t\to 0^+}\left(\frac{e^t-1}{t}\right) = \lim_{t\to 0^+}\left(\frac{e^t}{1}\right) = 1$.

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94. $\lim_{y\to 0^+} e^{-1/y}\ln y$

Solution:

The limit leads to the indeterminate form $\frac{\infty}{\infty}$: $\lim_{y\to 0^+} e^{-1/y}\ln y = \lim_{y\to 0^+}\frac{\ln y}{e^{y^{-1}}} = \lim_{y\to 0^+}\frac{y^{-1}}{-e^{y^{-1}}(y^{-2})} = -\lim_{y\to 0^+}\frac{y}{-e^{y^{-1}}} = 0$.

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3. $\lim_{x\to 0^+}\left(\cos\sqrt{x}\right)^{1/x}$

Solution:

$y = \left(\cos\sqrt{x}\right)^{1/x} \implies \ln y = \frac{1}{x}\ln\left(\cos\sqrt{x}\right)$ and

$\lim_{x\to 0^+}\frac{\ln(\cos\sqrt{x})}{x} = \lim_{x\to 0^+}\frac{-\sin(\sqrt{x})}{2\sqrt{x}\cos\sqrt{x}} = \frac{-1}{2}\lim_{x\to 0^+}\frac{\tan\sqrt{x}}{\sqrt{x}} = -\frac{1}{2}\lim_{x\to 0^4}\frac{\frac{1}{2}x^{-1/2}\sec^2\sqrt{x}}{\frac{1}{2}x^{-1/2}} = -\frac{1}{2}$

$\implies \lim_{x\to 0^+}\left(\cos\sqrt{x}\right)^{1/x} = e^{-1/2} = \frac{1}{\sqrt{e}}$.

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4. $\lim_{x\to\infty}(x+e^x)^{2/x}$

Solution:

$y = (x+e^x)^{2/x} \Longrightarrow \ln y = \dfrac{2\ln(x+e^x)}{x} \Longrightarrow \lim_{x\to\infty}\ln y = \lim_{x\to\infty}\dfrac{2(1+e^x)}{x+e^x} = \lim_{x\to\infty}\dfrac{2e^x}{1+e^x} = \lim_{x\to\infty}\dfrac{2e^x}{e^x} = 2$

$\Longrightarrow \lim_{x\to\infty}(x+e^x)^{2/x} = \lim_{x\to\infty}e^y = e^2$

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11. For what $x>0$ does $x^{(x^x)} = (x^x)^x$? Give reasons for your answer.

Solution:

$\ln x^{(x^x)} = x^x\ln x$ and $\ln(x^x)^x = x\ln x^x = x^2\ln x$; then, $x^x\ln x = x^2\ln x \Longrightarrow (x^x-x^2)\ln x = 0 \Longrightarrow x^x = x^2$ or $\ln x = 0$.

$\ln x = 0 \Longrightarrow x = 1$; $x^x = x^2 \Longrightarrow x\ln x = 2\ln x \Longrightarrow x = 2$. Therefore, $x^{(x^x)} = (x^x)^x$ when $x=2$ or $x=1$.

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13. Find $f'(2)$ if $f(x) = e^{g(x)}$ and $g(x) = \displaystyle\int_2^x \frac{t}{1+t^4}\,dt$.

Solution:

$f(x) = e^{g(x)} \Longrightarrow f'(x) = e^{g(x)}g'(x)$, where $g'(x) = \dfrac{x}{1+x^4} \Longrightarrow f'(2) = e^0\left(\dfrac{2}{1+16}\right) = \dfrac{2}{17}$

(p.547)

24. Find the derivative of $y = (1+x^2)e^{\tan^{-1}x}$

Solution:

$y = (1+x^2)e^{\tan^{-1}x} \Longrightarrow y' = 2xe^{\tan^{-1}x} + (1+x^2)\left(\dfrac{e^{\tan^{-1}x}}{1+x^2}\right) = 2xe^{\tan^{-1}x} + e^{\tan^{-1}x}$.

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29. Find the derivative of $y = (\sin\theta)^{\sqrt{\theta}}$.

Solution:

$y = (\sin\theta)^{\sqrt{\theta}} \Longrightarrow \ln y = \sqrt{\theta}\ln(\sin\theta)$

$\Longrightarrow \dfrac{1}{y}\dfrac{dy}{d\theta} = \sqrt{\theta}\left(\dfrac{\cos\theta}{\sin\theta}\right) + \dfrac{1}{2}\theta^{-1/2}\ln(\sin\theta)$

$\Longrightarrow \dfrac{dy}{d\theta} = (\sin\theta)^{\sqrt{\theta}}\left(\sqrt{\theta}\cot\theta + \dfrac{\ln(\sin\theta)}{2\sqrt{\theta}}\right)$

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30. Find the derivative of $y = (\ln x)^{1/\ln x}$.

Solution:

$y = (\ln x)^{1/\ln x} \Longrightarrow \ln y = \left(\dfrac{1}{\ln x}\right)\ln(\ln x)$

$\Longrightarrow \dfrac{y'}{y} = \left(\dfrac{1}{\ln x}\right)\left(\dfrac{1}{\ln x}\right)\left(\dfrac{1}{x}\right) + \ln(\ln x)\left[\dfrac{-1}{(\ln x)^2}\right]\left(\dfrac{1}{x}\right)$

$\Longrightarrow y' = (\ln x)^{1/\ln x}\left[\dfrac{1-\ln(\ln x)}{x(\ln x)^2}\right]$.
