ÇANKAYA UNIVERSITY

Department of Mathematics and Computer Science

MATH 155 Calculus for Engineering I

Summer 2008

Problems and Solutions for Recitation 6

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(p.558)

1. Basic Substitutions

Evaluate each integral in Exercises 1-36 by using a substitution to reduce it to standard form.

1. $\displaystyle\int \frac{16x\ dx}{\sqrt{8x^2+1}}$

Solution:

$$\int \frac{16x}{\sqrt{8x^2+1}}\ dx; \left[\begin{array}{c} u = 8x^2+1 \\ du = 16x\ dx \end{array}\right] \longrightarrow \int \frac{du}{\sqrt{u}} = 2\sqrt{u} + C = 2\sqrt{8x^2+1} + C.$$

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2. $\displaystyle\int \frac{3\cos x}{\sqrt{1+3\sin x}}.$

Solution:

$$\int \frac{3\cos x}{\sqrt{1+3\sin x}}\ dx; \left[\begin{array}{c} u = 1+3\sin x \\ du = 3\cos x\ dx \end{array}\right] \longrightarrow \int \frac{du}{\sqrt{u}} = 2\sqrt{u} + C = 2\sqrt{1+3\sin x} + C.$$

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3. $\displaystyle\int 3\sqrt{\sin v}\cos v\ dv.$

Answer:

$2\left(\sin v\right)^{3/2} + C.$

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4. $\displaystyle\int \cot^3 y \csc^2 y\ dy$

Solution:

$$\int \cot^3 y \csc^2 y\ dy; \left[\begin{array}{c} u = \cot y \\ du = -\csc^2 y\ dy \end{array}\right] \longrightarrow \int u^3\ \ (-du) = -\frac{u^4}{4} + C = \frac{-\cot^4 y}{4} + C.$$

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5. $\displaystyle\int_0^1 \frac{16x\ dx}{8x^2+2}$

Solution:

$$\int_0^1 \frac{16x\ dx}{8x^2+2}; \left[\begin{array}{c} u = 8x^2+2 \\ du = 16x\ dx \\ x=0 \Longrightarrow u=2, x=1 \Longrightarrow u=10 \end{array}\right] \longrightarrow \int_2^{10} \frac{du}{u} = \left[\ln|u|\right]_2^{10} = \ln 10 - \ln 2 =$$

$\ln 5.$

6. $\displaystyle\int_{\pi/4}^{\pi/3} \frac{\sec^2 z\ dz}{\tan z}$

Solution:

$$\int_{\pi/4}^{\pi/3} \frac{\sec^2 z\ dz}{\tan z}; \left[\begin{array}{c} u = \tan z \\ du = \sec^2 z\ dz \\ z = \frac{\pi}{4} \Longrightarrow u = 1, z = \frac{\pi}{3} \Longrightarrow u = \sqrt{3} \end{array}\right] \longrightarrow \int_1^{\sqrt{3}} \frac{1}{u}\ du = [\ln|u|]_1^{\sqrt{3}}$$

$$= \ln\sqrt{3} - \ln 1 = \ln\sqrt{3}.$$

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7. $\displaystyle\int \frac{dx}{\sqrt{x}\left(\sqrt{x}+1\right)}$

Solution:

$$\int \frac{dx}{\sqrt{x}\left(\sqrt{x}+1\right)}; \left[\begin{array}{c} u = \sqrt{x}+1 \\ du = \frac{1}{2\sqrt{x}} \\ 2du = \frac{dx}{\sqrt{x}} \end{array}\right] \longrightarrow \int \frac{2\ du}{u} = 2\ln|u| + C = 2\ln\left(\sqrt{x}+1\right) + C.$$

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8. $\displaystyle\int \frac{dx}{x-\sqrt{x}}$

Solution:

$$\int \frac{dx}{x-\sqrt{x}} = \int \frac{dx}{\sqrt{x}\left(\sqrt{x}-1\right)}; \left[\begin{array}{c} u = \sqrt{x}-1 \\ du = \frac{1}{2\sqrt{x}}\ dx \\ 2\ du = \frac{dx}{\sqrt{x}} \end{array}\right] \longrightarrow \int \frac{2\ du}{u} = 2\ln|u| + C = 2\ln\left|\sqrt{x}-1\right| + C.$$

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9. $\displaystyle\int \cot(3-7x)\ dx$

Solution:

$$\int \cot(3-7x)\ dx; \left[\begin{array}{c} u = 3-7x \\ du = -7\ dx \end{array}\right] \longrightarrow -\frac{1}{7}\int \cot u\ du = -\frac{1}{7}\ln|\sin u| + C$$

$$= -\frac{1}{7}\ln|\sin(3-7x)| + C$$

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10. $\displaystyle\int \csc(\pi x - 1)\ dx$

Solution:

$$\int \csc(\pi x - 1)\ dx; \left[\begin{array}{c} u = \pi x - 1 \\ du = \pi\ dx \end{array}\right] \longrightarrow \int \csc y\ \frac{du}{\pi} = \frac{-1}{\pi}\ln|\csc u + \cot u| + C$$

$$= \frac{-1}{\pi}\ln|\csc(\pi x - 1) + \cot(\pi x - 1)| + C.$$

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11. $\displaystyle\int e^\theta \csc\left(e^\theta + 1\right)\ d\theta$

Solution:

$$\int e^\theta \csc\left(e^\theta + 1\right)\ d\theta; \left[\begin{array}{c} u = e^\theta + 1 \\ du = e^\theta \end{array}\right] \longrightarrow \int \csc u\ du$$

$$= -\ln|\csc u + \cot u| + C = -\ln\left|\csc\left(e^\theta + 1\right) + \cot\left(e^\theta + 1\right)\right| + C.$$

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12. $\displaystyle\int \frac{\cot(3+\ln x)}{x}\ dx$

Solution:

$$\int \frac{\cot(3+\ln x)}{x}\,dx \begin{bmatrix} u = 3+\ln x \\ du = \frac{dx}{x} \end{bmatrix} \longrightarrow \int \cot u\; du = \ln|\sin u| + C = \ln|\sin(3+\ln x)| + C.$$

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13. $\displaystyle\int \sec\frac{t}{3}\; dt$

Solution:

$$\int \sec\frac{t}{3}\; dt; \begin{bmatrix} u = \frac{t}{3} \\ du = \frac{dt}{3} \end{bmatrix} \longrightarrow \int 3\sec u\; du = 3\ln|\sec u + \tan u| + C$$

$$= 3\ln\left|\sec\frac{t}{3} + \tan\frac{t}{3}\right| + C.$$

---

14. $\displaystyle\int x\sec\left(x^2-5\right)\; dx$

Solution:

$$\int x\sec\left(x^2-5\right)\; dx; \begin{bmatrix} u = x^2-5 \\ du = 2x\; dx \end{bmatrix} \longrightarrow \int \frac{1}{2}\sec u\; du = \frac{1}{2}\ln|\sec u + \tan u| + C$$

$$= \frac{1}{2}\ln\left|\sec\left(x^2-5\right) + \tan\left(x^2-5\right)\right| + C.$$

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15. $\displaystyle\int \csc(s-\pi)\; ds$

Solution:

$$\int \csc(s-\pi)\; ds; \begin{bmatrix} u = s-\pi \\ du = ds \end{bmatrix} \longrightarrow \int \csc u\; du = -\ln|\csc u + \cot u| + C$$

$$= -\ln|\csc(s-\pi) + \cot(s-\pi)| + C.$$

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16. $\displaystyle\int \frac{1}{\theta^2}\csc\frac{1}{\theta}\; d\theta$

Solution:

$$\int \frac{1}{\theta^2}\csc\frac{1}{\theta}\; d\theta; \begin{bmatrix} u = \frac{1}{\theta} \\ du = -\frac{1}{\theta^2}\; d\theta \end{bmatrix} \longrightarrow \int -\csc u\; du = \ln|\csc u + \cot u| + C$$

$$= \ln\left|\csc\frac{1}{\theta} + \cot\frac{1}{\theta}\right| + C$$

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17. $\displaystyle\int_0^{\sqrt{\ln 2}} 2xe^{x^2}\; dx$

Solution:

$$\int_0^{\sqrt{\ln 2}} 2xe^{x^2}\; dx : \begin{bmatrix} u = x^2 \\ du = 2x\; dx \\ x = 0 \Longrightarrow u = 0, x = \sqrt{\ln 2} \Longrightarrow u = \ln 2 \end{bmatrix} \longrightarrow \int_0^{\ln 2} e^u \;\; du = [e^u]_0^{\ln 2}$$

$$= e^{\ln 2} - e^0 = 2 - 1 = 1.$$

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18. $\displaystyle\int_{\pi/2}^{\pi} \sin(y)\, e^{\cos y}\; dy$

Solution:

$$\int_{\pi/2}^{\pi} \sin(y)\, e^{\cos y}\; dy; \begin{bmatrix} u = \cos y \\ du = -\sin y\; dy \\ y = \frac{\pi}{2} \Longrightarrow u = 0, y = \pi \Longrightarrow u = -1 \end{bmatrix} \longrightarrow \int_0^{-1} -e^u\; du = \int_{-1}^{0} e^u\; du$$

$$= [e^u]_{-1}^{0} = \frac{e-1}{e}.$$

19. $\int e^{\tan v} \sec^2 v \, dv$

Solution:

$$\int e^{\tan v} \sec^2 v \, dv; \begin{bmatrix} u = \tan v \\ du = \sec^2 v \, dv \end{bmatrix} \longrightarrow \int e^u \, du = e^u + C = e^{\tan v} + C.$$

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20. $\int \frac{e^{\sqrt{t}}}{\sqrt{t}} \, dt$

Solution:

$$\int \frac{e^{\sqrt{t}}}{\sqrt{t}} \, dt; \begin{bmatrix} u = \sqrt{t} \\ du = \frac{dt}{2\sqrt{t}} \end{bmatrix} \longrightarrow \int 2e^u \, du = 2e^u + C = 2e^{\sqrt{t}} + C.$$

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21. $\int 3^{x+1} \, dx$

Solution:

$$\int 3^{x+1} \, dx; \begin{bmatrix} u = x+1 \\ du = dx \end{bmatrix} \longrightarrow \int 3^u \, du = \left(\frac{1}{\ln 3}\right) 3^u + C = \frac{3^{x+1}}{\ln 3} + C.$$

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22. $\int \frac{2^{\ln x}}{x} \, dx$

Solution:

$$\int \frac{2^{\ln x}}{x} \, dx; \begin{bmatrix} u = \ln x \\ du = \frac{dx}{x} \end{bmatrix} \longrightarrow \int 2^u \, du = \frac{2^u}{\ln 2} + C = \frac{2^{\ln x}}{\ln 2} + C.$$

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23. $\int \frac{2^{\sqrt{w}}}{2\sqrt{w}} \, dw$

Solution:

$$\int \frac{2^{\sqrt{w}}}{2\sqrt{w}} \, dw; \begin{bmatrix} u = \sqrt{w} \\ du = \frac{dw}{2\sqrt{w}} \end{bmatrix} \longrightarrow \int 2^u \, du = \frac{2^u}{\ln 2} + C = \frac{2^{\sqrt{w}}}{\ln 2} + C.$$

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24. $\int 10^{2\theta} \, d\theta$

Solution:

$$\int 10^{2\theta} \, d\theta; \begin{bmatrix} u = 2\theta \\ du = 2 \, d\theta \end{bmatrix} \longrightarrow \int \frac{1}{2} 10^u \, du = \frac{10^u}{2 \ln 10} + C = \frac{1}{2}\left(\frac{10^{2\theta}}{\ln 10}\right) + C.$$

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25. $\int \frac{9 \, du}{1 + 9u^2}$

Solution:

$$\int \frac{9 \, du}{1 + 9u^2}; \begin{bmatrix} x = 3u \\ dx = 3 \, du \end{bmatrix} \longrightarrow \int \frac{3 \, dx}{1 + x^2} = 3 \tan^{-1} x + C = 3 \tan^{-1} (3u) + C.$$

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26. $\int \frac{4 \, dx}{1 + (2x+1)^2}$

Solution:

$$\int \frac{4\,dx}{1+(2x+1)^2};\left[\begin{array}{c} u=2x+1 \\ du=2\,dx \end{array}\right] \longrightarrow \int \frac{2\,du}{1+u^2}=2\tan^{-1}u+C=2\tan^{-1}(2x+1)+C.$$

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27. $\displaystyle\int_0^{1/6} \frac{dx}{\sqrt{1-9x^2}}$

Solution:

$$\int_0^{1/6} \frac{dx}{\sqrt{1-9x^2}};\left[\begin{array}{c} u=3x \\ du=3\,dx \\ x=0 \Longrightarrow u=0, x=\frac{1}{6} \Longrightarrow u=\frac{1}{2} \end{array}\right] \longrightarrow \int_0^{1/2} \frac{1}{3}\frac{du}{\sqrt{1-u^2}}=\left[\frac{1}{3}\sin^{-1}u\right]_0^{1/2}$$
$$=\frac{1}{3}\left(\frac{\pi}{6}-0\right)=\frac{\pi}{18}.$$

---

28. $\displaystyle\int_0^{1} \frac{dt}{\sqrt{4-t^2}}$

Solution:

$$\int_0^{1} \frac{dt}{\sqrt{4-t^2}}=\left[\sin^{-1}\frac{t}{2}\right]_0^1=\sin^{-1}\left(\frac{1}{2}\right)-0=\frac{\pi}{6}.$$

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29. $\displaystyle\int \frac{2s\,ds}{\sqrt{1-s^4}}$

Solution:

$$\int \frac{2s\,ds}{\sqrt{1-s^4}};\left[\begin{array}{c} u=s^2 \\ du=2s\,ds \end{array}\right] \longrightarrow \int \frac{du}{\sqrt{1-u^2}}=\sin^{-1}u+C=\sin^{-1}s^2+C.$$

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30. $\displaystyle\int \frac{2\,dx}{x\sqrt{1-4\ln^2 x}}$

Solution:

$$\int \frac{2\,dx}{x\sqrt{1-4\ln^2 x}};\left[\begin{array}{c} u=2\ln x \\ du=\frac{2\,dx}{x} \end{array}\right] \longrightarrow \int \frac{du}{\sqrt{1-u^2}}=\sin^{-1}u+C=\sin^{-1}(2\ln x)+C.$$

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31. $\displaystyle\int \frac{6\,dx}{x\sqrt{25x^2-1}}$

Solution:

$$\int \frac{6\,dx}{x\sqrt{25x^2-1}}=\frac{6}{5}\cdot 5\sec^{-1}|5x|+C=6\sec^{-1}|5x|+C$$

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32. $\displaystyle\int \frac{dr}{r\sqrt{r^2-9}}$

Solution:

$$\int \frac{dr}{r\sqrt{r^2-9}}=\frac{1}{3}\sec^{-1}\left|\frac{r}{3}\right|+C$$

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33. $\displaystyle\int \frac{dx}{e^x+e^{-x}}$

Solution:

$$\int \frac{dx}{e^x+e^{-x}}=\int \frac{e^x\,dx}{e^{2x}+1};\left[\begin{array}{c} u=e^x \\ du=e^x\,dx \end{array}\right] \longrightarrow \int \frac{du}{u^2+1}=\tan^{-1}u+C=\tan^{-1}e^x+C.$$

34. $\int \frac{dy}{\sqrt{e^{2y}-1}}$

Solution:

$\int \frac{dy}{\sqrt{e^{2y}-1}} = \int \frac{e^y\ dy}{e^y\sqrt{(e^y)^2-1}}; \begin{bmatrix} u = e^y \\ du = e^y\ dy \end{bmatrix} \longrightarrow \int \frac{du}{u\sqrt{u^2-1}} = \sec^{-1}|u| + C = \sec^{-1}|e^y| + C.$

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35. $\int_1^{e^{\pi/3}} \frac{dx}{x\cos(\ln x)}$

Solution:

$\int_1^{e^{\pi/3}} \frac{dx}{x\cos(\ln x)}; \begin{bmatrix} u = \ln x \\ du = \frac{dx}{x} \\ x = 1 \Longrightarrow u = 0, x = \frac{\pi}{3} \Longrightarrow u = \frac{\pi}{3} \end{bmatrix} \longrightarrow \int_0^{\pi/3} \frac{du}{\cos u} = \int_0^{\pi/3} \sec u\ du$

$= [\ln|\sec u + \tan u|]_0^{\pi/3} = \left[\ln\left|\sec\frac{\pi}{3} + \tan\frac{\pi}{3}\right|\right] - [\ln|\sec 0 + \tan 0|] = \ln\left(2+\sqrt{3}\right) - \ln(1) = \ln\left(2+\sqrt{3}\right).$

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36. $\int \frac{\ln x\ dx}{x + 4x\ln^2 x}$

Solution:

$\int \frac{\ln x\ dx}{x\left(1 + x\ln^2 x\right)}; \begin{bmatrix} u = \ln^2 x \\ du = \frac{2}{x}\ln x\ dx \end{bmatrix} \longrightarrow \int \frac{1}{2}\frac{du}{1+4u} = \frac{1}{8}\ln|1+4u| + C = \frac{1}{8}\ln\left(1 + 4\ln^2 x\right) + C.$

---

37. $\int_1^2 \frac{8\ dx}{x^2 - 2x + 2}$

Solution:

$\int_1^2 \frac{8\ dx}{x^2 - 2x + 20} = 8\int_1^2 \frac{dx}{1 + (x-1)^2}; \begin{bmatrix} u = x - 1 \\ du = dx \\ x = 1 \Longrightarrow u = -1, x = 2 \Longrightarrow u = 1 \end{bmatrix} \longrightarrow 8\int_0^1 \frac{du}{1+u^2} =$

$8\left[\tan^{-1} u\right]_0^1$

$= 8\left(\tan^{-1} 1 - \tan^{-1} 0\right) = 8\left(\frac{\pi}{4} - 0\right) = 2\pi.$

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38. $\int_2^4 \frac{2\ dx}{x^2 - 6x + 10}$

Solution:

$\int_2^4 \frac{2\ dx}{x^2 - 6x + 10} = 2\int_2^4 \frac{2\ dx}{(x-3)^2 + 1}; \begin{bmatrix} u = x - 3 \\ du = dx \\ x = 2 \Longrightarrow u = -1, x = 4 \Longrightarrow u = 1 \end{bmatrix} \longrightarrow 2\int_{-1}^1 \frac{du}{u^2+1} =$

$2\left[\tan^{-1} u\right]_{-1}^1$

$= 2\left[\tan^{-1} 1 - \tan^{-1}(-1)\right] = 2\left[\frac{\pi}{4} - \left(-\frac{\pi}{4}\right)\right] = \pi.$

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39. $\int \frac{dt}{\sqrt{-4t^2 + 4t - 3}}$

Solution:
$\int \frac{dt}{\sqrt{-4t^2+4t-3}} = \int \frac{dt}{\sqrt{1-(t-2)^2}}; \begin{bmatrix} u=t-2 \\ du=dt \end{bmatrix} \longrightarrow \int \frac{du}{\sqrt{1-u^2}} = \sin^{-1} u + C = \sin^{-1}(t-2) + C.$

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40. $\int \frac{d\theta}{\sqrt{2\theta-\theta^2}}$

Solution:
$\int \frac{d\theta}{\sqrt{2\theta-\theta^2}} = \int \frac{d\theta}{\sqrt{1-(\theta-1)^2}}; \begin{bmatrix} u=\theta-1 \\ du=d\theta \end{bmatrix} \longrightarrow \int \frac{du}{\sqrt{1-u^2}} = \sin^{-1} u + C = \sin^{-1}(\theta-1) + C.$

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2. Integration by Parts

1. $\int x \sin \frac{x}{2}\ dx$

Solution:
$u = x, du = dx; dv = \sin \frac{x}{2}\ dx, v = -2\cos \frac{x}{2};$
$\int x \sin \frac{x}{2}\ dx = -2x\cos\frac{x}{2} - \int \left(-2\cos\frac{x}{2}\right)\ dx = -2x\cos\left(\frac{x}{2}\right) + 4\sin\left(\frac{x}{2}\right) + C.$

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2. $\int \theta \cos \pi\theta\ d\theta$

Solution:
$u = \theta, du = d\theta; dv = \cos \pi\theta\ d\theta, v = \frac{1}{\pi}\sin \pi\theta$
$\int \theta\cos\pi\theta\ d\theta = \frac{\theta}{\pi}\sin\pi\theta - \int \frac{1}{\pi}\sin\pi\theta\ d\theta = \frac{\theta}{\pi}\sin\pi\theta + \frac{1}{\pi^2}\cos\pi\theta + C.$

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3. $\int t^2 \cos t\ dt$

Solution:
$$\begin{aligned} &\cdots \cos t \\ t^2 &\longrightarrow \sin t \\ 2t &\longrightarrow -\cos t \\ 2 &\longrightarrow -\sin t \\ &0 \cdots \end{aligned}$$
$\int t^2\cos t\ dt = t^2\sin t + 2t\cos t - 2\sin t + C.$

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4. $\int x^2 \sin x\ dx$

Solution:
$$\begin{aligned} &\cdots \sin x \\ x^2 &\longrightarrow -\cos x \\ 2x &\longrightarrow -\sin x \\ 2 &\longrightarrow \cos x \\ &0 \cdots \end{aligned}$$
$\int x^2\sin x\ dx = -x^2\cos x + 2x\sin x + 2\cos x + C.$

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5. $\int_1^2 x \ln x \; dx$

Solution:

$u = \ln x, du = \frac{dx}{x}; dv = x \; dx, v = \frac{x^2}{2}$

$\int_1^2 x \ln x \; dx = \left[\frac{x^2}{2} \ln x\right]_1^2 - \int_1^2 \frac{x^2}{2} \frac{1}{x} \; dx = 2 \ln 2 - \left[\frac{x^2}{4}\right]_1^2 = 2 \ln 2 - \frac{3}{4} = \ln 4 - \frac{3}{4}.$

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6. $\int_1^e x^3 \ln x \; dx$

Solution:

$u = \ln x, du = \frac{dx}{x}; dv = x \; dx, v = \frac{x^4}{4};$

$\int_1^e x^3 \ln x \; dx = \left[\frac{x^4}{4} \ln x\right]_1^e - \int_1^e \frac{x^4}{4} \frac{dx}{x} = \frac{e^4}{4} - \left[\frac{x^4}{16}\right]_1^e = \frac{3e^4 + 1}{16}$

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7. $\int \tan^{-1} y \; dy$

Solution:

$u = \tan^{-1} y, du = \frac{dy}{1 + y^2}; dv = dy, v = y;$

$\int \tan^{-1} y \; dy = y \tan^{-1} y - \int \frac{y \; dy}{1 + y^2} = y \tan^{-1} y - \frac{1}{2} \ln\left(1 + y^2\right) + C = y \tan^{-1} y - \ln \sqrt{1 + y^2} + C$

---

8. $\int \sin^{-1} y \; dy$

Solution:

$u = \sin^{-1} y, du = \frac{dy}{\sqrt{1 - y^2}}; dv = dy, v = y;$

$\int \sin^{-1} y \; dy = y \sin^{-1} y - \int \frac{y}{\sqrt{1 - y^2}} \; dy = y \sin^{-1} y + \sqrt{1 - y^2} + C$

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9. $\int x \sec^2 x \; dx$

Solution:

$u = x, du = dx; dv = \sec^2 x \; dx, v = \tan x$

$\int x \sec^2 x \; dx = x \tan x - \int \tan x \; dx = x \tan x - \ln|\cos x| + C$

---

10. $\int 4x \sec^2 2x \; dx$

Solution:

$\int 4x \sec^2 2x \; dx; [y = 2x] \longrightarrow \int y \sec^2 y \; dy = y \tan y - \int \tan y \; dy = y \tan y - \ln|\sec y| + C$

$= 2x \tan 2x - \ln|\sec 2x| + C$

---

11. $\int x^3 e^x \; dx$

Solution:

$$\begin{array}{ccc} & & e^x \\ x^3 & \xrightarrow{(+)} & e^x \\ 3x^2 & \xrightarrow{(-)} & e^x \\ 6x & \xrightarrow{(+)} & e^x \\ 6 & \xrightarrow{(-)} & e^x \\ 0 & & \end{array}$$

$$\int x^3 e^x \, dx = x^3 e^x - 3x^2 e^x + 6x e^x - 6e^x + C = \left(x^3 - 3x^2 + 6x - 6\right) e^x + C$$

---

12. $\int p^4 e^{-p} \, dp$

Solution:

$$\begin{array}{ccc} & & e^{-p} \\ p^4 & \xrightarrow{(+)} & -e^{-p} \\ 4p^3 & \xrightarrow{(-)} & e^{-p} \\ 12p^2 & \xrightarrow{(+)} & -e^{-p} \\ 24p & \xrightarrow{(-)} & e^{-p} \\ 24 & \xrightarrow{(+)} & -e^{-p} \\ 0 & & \end{array}$$

$$\int p^4 e^{-p} \, dp = -p^4 e^{-p} - 4p^3 e^{-p} - 12p^2 e^{-p} - 24e^{-p} + C = \left(-p^4 - 4p^3 - 12p^2 - 24p - 24\right) e^{-p} + C$$

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13. $\int \left(x^2 - 5x\right) e^x \, dx$

Solution:

$$\begin{array}{ccc} & & e^x \\ x^2 - 5x & \xrightarrow{(+)} & e^x \\ 2x - 5 & \xrightarrow{(-)} & e^x \\ 2 & \xrightarrow{(+)} & e^x \\ 0 & & \end{array}$$

$$\int \left(x^2 - 5x\right) e^x \, dx = \left(x^2 - 5x\right) e^x - (2x - 5) e^x + 2e^x + C = \left(x^2 - 7x + 7\right) e^x + C$$

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14. $\int \left(r^2 + r + 1\right) e^r \, dr$

Answer:

$\left(r^2 - r + 2\right) e^r + C$

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15. $\int x^5 e^x \, dx$

Answer:

$\left(x^5 - 5x^4 + 20x^3 - 60x^2 + 120x - 120\right) e^x + C$

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16. $\int t^2 e^{4t} \, dt$

Answer:

$\left(\frac{t^2}{4} - \frac{t}{8} + \frac{1}{32}\right) e^{4t} + C$

17. $\displaystyle\int_0^{\pi/2} \theta^2 \sin 2\theta \; d\theta$

Answer:
$\dfrac{\pi^2 - 4}{8}$

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18. $\displaystyle\int_0^{\pi/2} x^3 \cos 2x \; dx$

Answer:
$\dfrac{3\left(4 - \pi^2\right)}{16}$

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19. $\displaystyle\int_{2/\sqrt{3}}^{2} t \sec^{-1} t \; dt$

Solution:

$u = \sec^{-1} t, du = \dfrac{dt}{t\sqrt{t^2-1}}; dv = t \; dt, v = \dfrac{t^2}{2};$

$\displaystyle\int_{2/\sqrt{3}}^{2} t \sec^{-1} t \; dt = \left[\frac{t^2}{2}\sec^{-1} t\right]_{2/\sqrt{3}}^{2} - \int_{2/\sqrt{3}}^{2} \left(\frac{t^2}{2}\right) \frac{dt}{t\sqrt{t^2-1}} \; dt = \left(2 \cdot \frac{\pi}{3} - \frac{2}{3} \cdot \frac{\pi}{6}\right) - \int_{2/\sqrt{3}}^{2} \frac{t \; dt}{2\sqrt{t^2-1}}$

$\displaystyle= \frac{5\pi}{9} - \left[\frac{1}{2}\sqrt{t^2-1}\right]_{2/\sqrt{3}}^{2} = \frac{5\pi}{9} - \frac{1}{2}\left(\sqrt{3} - \sqrt{\frac{4}{3} - 1}\right) = \frac{5\pi}{9} - \frac{1}{2}\left(\sqrt{3} - \frac{\sqrt{3}}{3}\right) = \frac{5\pi - 3\sqrt{3}}{9}$

---

20. $\displaystyle\int_0^{1/\sqrt{2}} 2x \sin^{-1}\left(x^2\right) \; dx$

Solution:

$u = \sin^{-1}\left(x^2\right), du = \dfrac{2x \; dx}{\sqrt{1-x^4}}; dv = 2x \; dx, v = x^2;$

$\displaystyle\int_0^{1/\sqrt{2}} 2x \sin^{-1}\left(x^2\right) \; dx = \left[x^2 \sin^{-1}\left(x^2\right)\right]_0^{1/\sqrt{2}} - \int_0^{1/\sqrt{2}} x^2 \frac{2x \; dx}{\sqrt{1-x^4}} = \left(\frac{1}{2}\right)\left(\frac{\pi}{6}\right) + \int_0^{1/\sqrt{2}} \frac{d\left(1-x^4\right)}{2\sqrt{1-x^4}}$

$\displaystyle= \frac{\pi}{12} + \left[\sqrt{1-x^4}\right]_0^{1/\sqrt{2}} = \frac{\pi}{12} + \sqrt{\frac{3}{4}} - 1 = \frac{\pi + 6\sqrt{3} - 12}{12}$

---

21. $\displaystyle\int e^{\theta} \sin\theta \; d\theta$

Solution:

$\displaystyle I = \int e^{\theta} \sin\theta \; d\theta; \left[u = \sin\theta, du = \cos\theta \; d\theta; dv = e^{\theta} \; d\theta, v = e^{\theta}\right] \Longrightarrow I = e^{\theta}\sin\theta - \int e^{\theta}\cos\theta \; d\theta;$

$\displaystyle\left[u = \cos\theta, du = -\sin\theta \; d\theta; dv = e^{\theta} \; d\theta, v = e^{\theta}\right] \Longrightarrow I = e^{\theta}\sin\theta - \left(e^{\theta}\cos\theta + \int e^{\theta}\sin\theta \; d\theta\right)$

$\displaystyle= e^{\theta}\sin\theta - e^{\theta}\cos\theta - I + C' \Longrightarrow 2I = \left(e^{\theta}\sin\theta - e^{\theta}\cos\theta\right) + C' \Longrightarrow I = \frac{1}{2}\left(e^{\theta}\sin\theta - e^{\theta}\cos\theta\right) + C$

where $C = \dfrac{C'}{2}$ is another arbitrary constant.

---

22. $\displaystyle\int e^{-y} \sin y \; dy$

Answer:$I = \dfrac{1}{2}\left(e^{-y}\sin y - e^{-y}\cos y\right) + C$

---

23. $\displaystyle\int e^{2x} \cos 3x \; dx$

Answer:$I = \dfrac{e^{2x}}{13}\left(3\sin 3x + 2\cos 3x\right) + C$

24. $\int e^{-2x} \sin 2x \ dx$

Answer: $I = -\dfrac{e^{-2x}}{4} (\sin 2x + \cos 2x) + C$

---

25. $\int e^{\sqrt{3s+9}} \ ds$

Solution:

$\int e^{\sqrt{3s+9}} \ ds; \begin{bmatrix} 3s+9 = x^2 \\ ds = \frac{2}{3}x \ dx \end{bmatrix} \longrightarrow \int e^x \cdot \frac{2}{3}x \ dx = \frac{2}{3} \int xe^x \ dx;$

$[u = x, du = dx; dv = e^x \ dx, v = e^x];$

$\frac{2}{3} \int xe^x \ dx = \frac{2}{3} \left(xe^x - \int e^x \ dx \right) = \frac{2}{3} (xe^x - e^x) + C = \frac{2}{3} \left(\sqrt{3s+9} e^{\sqrt{3s+9}} - e^{\sqrt{3s+9}} \right) + C.$

---

26. $\int_0^1 x\sqrt{1-x} \ dx$

Solution:

$u = x, du = dx; dv = \sqrt{1-x} \ dx, v = -\frac{2}{3}\sqrt{(1-x)^3};$

$\int_0^1 x\sqrt{1-x} \ dx = \left[-\frac{2}{3}\sqrt{(1-x)^3} x \right]_0^1 + \frac{2}{3} \int_0^1 \sqrt{(1-x)^3} \ dx = \frac{2}{3} \left[-\frac{2}{5} (1-x)^{5/2} \right]_0^1 = \frac{4}{15}.$

---

27. $\int_0^{\pi/3} x \tan^2 x \ dx$

Solution:

$u = x, du = dx; dv = \tan^2 x, v = \int \tan^2 x \ dx = \int \left(\sec^2 x - 1 \right) \ dx = \tan x - x$

$\int_0^{\pi/3} x \tan^2 x \ dx = [x (\tan x - x)]_0^{\pi/3} - \int_0^{\pi/3} (\tan x - x) \ dx = \frac{\pi}{3} \left(\sqrt{3} - \frac{\pi}{3} \right) + \left[\ln|\cos x| + \frac{x^2}{2} \right]_0^{\pi/3}$

$= \frac{\pi}{3} \left(\sqrt{3} - \frac{\pi}{3} \right) + \ln\frac{1}{2} + \frac{\pi^2}{18} = \frac{\pi\sqrt{3}}{3} - \ln 2 - \frac{\pi^2}{18}.$

---

28. $\int \ln \left(x + x^2 \right) \ dx$

Solution:

$u = \ln \left(x + x^2 \right), du = \dfrac{(2x+1) \ dx}{x + x^2}; dv = dx, v = x;$

$\int \ln \left(x + x^2 \right) \ dx = x \ln \left(x + x^2 \right) - \int \frac{2x+1}{x(x+1)} \cdot x \ dx$

$= x \ln \left(x + x^2 \right) - \int \frac{(2x+1) \ dx}{x+1} = x \ln \left(x + x^2 \right) - \int \frac{2(x+1) - 1}{x+1} \ dx = x \ln \left(x + x^2 \right) - 2x +$
$\ln|x+1| + C.$

---

29. $\int \sin(\ln x) \ dx$

Solution:

$\int \sin(\ln x) \ dx; \begin{bmatrix} u = \ln x \\ du = \frac{1}{x} \ dx \\ dx = e^u \ du \end{bmatrix} \longrightarrow \int (\sin u) e^u \ du.$ From Exercise 21, $\int (\sin u) e^u \ du =$
$e^u \left(\dfrac{\sin u - \cos u}{2} \right) + C = \dfrac{1}{2} [-x \cos(\ln x) + x \sin(\ln x)] + C.$

---

30. $\int z(\ln z)^2 \; dz$

Solution:

$$\int z(\ln z)^2 \; dz; \begin{bmatrix} u = \ln z \\ du = \frac{1}{z} \; dz \\ dz = e^u \; du \end{bmatrix} \longrightarrow \int e^u u^2 e^u \; du = \int e^{2u} u^2 \; du$$

		e^{2u}
u^2	$\underrightarrow{(+)}$	$\frac{1}{2}e^{2u}$
$2u$	$\underrightarrow{(-)}$	$\frac{1}{4}e^{x}$
2	$\underrightarrow{(+)}$	$\frac{1}{8}e^{x}$
0		

$\int e^{2u}u^2 \; du = \frac{u^2}{2}e^{2u} - \frac{u}{2}e^{2u} + \frac{1}{4}e^{2u} + C = \frac{e^{2u}}{4}\left[2u^2 - 2u + 1\right] + C = \frac{z^2}{4}\left[2(\ln z)^2 - 2(\ln z) + 1\right] + C.$

---

3. Integration of Rational Functions by Partial Fractions

10. $\int \frac{dx}{x^2 + 2x}$

Solution:

$\frac{1}{x^2 + 2x} = \frac{A}{x} + \frac{B}{x+2} \Longrightarrow 1 = A(x+2) + Bx; x = 0 \Longrightarrow A = \frac{1}{2}; x = -2 \Longrightarrow B = -\frac{1}{2};$

$\int \frac{dx}{x^2 + 2x} = \frac{1}{2}\int \frac{dx}{x} - \frac{1}{2}\int \frac{dx}{x+2} = \frac{1}{2}\left[\ln|x| - \ln|x+2|\right] + C$

---

11. $\int \frac{x+4}{x^2 + 5x - 6} \; dx$

Solution:

$\frac{x+4}{x^2+5x-6} = \frac{A}{x+6} + \frac{B}{x-1} \Longrightarrow x + 4 = A(x-1) + B(x+6); x = 1 \Longrightarrow B = \frac{5}{7}; x = -6 \Longrightarrow A = \frac{-2}{-7} = \frac{2}{7};$

$$\int \frac{x+4}{x^2+5x-6} \; dx = \frac{2}{7}\int \frac{dx}{x+6} + \frac{5}{7}\int \frac{dx}{x-1} = \frac{2}{7}\ln|x+6| + \frac{5}{7}\ln|x-1| + C$$

$$= \frac{1}{7}\ln\left|(x+6)^2(x-1)^5\right| + C$$

---

12. $\int \frac{2x+1}{x^2 - 7x + 12} \; dx$

Answer:

$\ln\left|\frac{(x-4)^9}{(x-3)^7}\right| + C$

---

13. $\int_4^8 \frac{y \; dy}{y^2 - 2y - 3}$

Answer:

$\frac{\ln 15}{2}$

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14. $\int_{1/2}^1 \frac{y+4}{y^2+y} \; dy$

Answer:
$\ln \frac{27}{4}$

---

15. $\int \frac{1}{t^3+t^2-2t}\ dt$
Solution:
$\frac{1}{t^3+t^2-t} = \frac{A}{t} + \frac{B}{t+2} + \frac{C}{t-1} \Longrightarrow 1 = A\left(t+2\right)\left(t-1\right) + Bt\left(t-1\right) + Ct\left(t+2\right); t=0 \Longrightarrow$
$A = -\frac{1}{2}; t=-2 \Longrightarrow B = \frac{1}{6};$

$t=1 \Longrightarrow C = \frac{1}{3}$

$$\int \frac{1}{t^3+t^2-t}\ dt = -\frac{1}{2}\int \frac{dt}{t} + \frac{1}{6}\int \frac{dt}{t+2} + \frac{1}{3}\int \frac{dt}{t-1} = -\frac{1}{2}\ln|t| + \frac{1}{6}\ln|t+2| + \frac{1}{3}\ln|t-1| + C$$

---

17. $\int_0^1 \frac{x^3}{x^2+2x+1}\ dx$
Solution:
$\frac{x^3}{x^2+2x+1} = (x-2) + \frac{3x+2}{\left(x+1\right)^2}$ (after long division);

$\Longrightarrow \frac{3x+2}{\left(x+1\right)^2} = \frac{A}{x+1} + \frac{B}{\left(x+1\right)^2} \Longrightarrow 3x+2 = A\left(x+1\right) + B; A=3, A+B=2; B=-1.$

$$\int_0^1 (x-2)\ dx + 3\int_0^1 \frac{dx}{x+1} + \int_0^1 \frac{dx}{(x+1)^2} = 3\ln 2 - 2.$$

---

19. $\int \frac{dx}{\left(x^2-1\right)^2}$
Solution:
$\frac{dx}{\left(x^2-1\right)^2} = \frac{A}{x+1} + \frac{B}{x-1} + \frac{C}{\left(x+1\right)^2} + \frac{D}{\left(x-1\right)^2} \Longrightarrow 1 = A\left(x+1\right)\left(x-1\right)^2 + B\left(x-1\right)\left(x+1\right)^2 +$
$C\left(x-1\right)^2 + D\left(x+1\right)^2;$

$x=-1 \Longrightarrow C = \frac{1}{4}; x=1 \Longrightarrow D = \frac{1}{4};$ coefficient of $x^3 = A+B \Longrightarrow A+B=0;$
constant $= A-B+C+D \Longrightarrow A-B+C+D=1 \Longrightarrow A-B = \frac{1}{2};$ thus $A = \frac{1}{4} \Longrightarrow B = -\frac{1}{4};$

$$\int \frac{dx}{\left(x^2-1\right)^2} = \frac{1}{4}\int \frac{dx}{x+1} - \frac{1}{4}\int \frac{dx}{x-1} + \frac{1}{4}\int \frac{dx}{\left(x+1\right)^2} + \frac{1}{4}\int \frac{dx}{\left(x-1\right)^2} = \frac{1}{4}\ln\left|\frac{x+1}{x-1}\right| - \frac{x}{2\left(x^2-1\right)} + C$$

---

22. $\int_1^{\sqrt{3}} \frac{3t^2+t+4}{t^3+t}\ dt$
Solution:

$\frac{3t^2+t+4}{t^3+t} = \frac{A}{t} + \frac{Bt+C}{t^2+1} \Longrightarrow 3t^2+t+4 = A\left(t^2+1\right) + \left(Bt+C\right)t; t=0 \Longrightarrow A=4;$
coefficient of $t^2 = A+B \Longrightarrow A+B=3 \Longrightarrow B=-1;$
coefficient of$t = C \Longrightarrow C=1;$

$$\int_1^{\sqrt{3}} \frac{3t^2+t+4}{t^3+t}\,dt = 4\int_1^{\sqrt{3}} \frac{dt}{t} + \int_1^{\sqrt{3}} \frac{(-t+1)}{t^2+1}\,dt = \left[4\ln|t| - \frac{1}{2}\ln\left(t^2+1\right) + \tan^{-1} t\right]_1^{\sqrt{3}}$$

$$= \left(4\ln\sqrt{3} - \frac{1}{2}\ln 4 + \tan^{-1}\sqrt{3}\right) - \left(4\ln 1 - \frac{1}{2}\ln 2 + \tan^{-1} 1\right) = \ln\left(\frac{9}{\sqrt{2}}\right) + \frac{\pi}{12}$$

---

25. $\displaystyle\int \frac{2s+2}{\left(s^2+1\right)(s-1)^3}\,ds$

Solution:

$$\frac{2s+2}{\left(s^2+1\right)(s-1)^3} = \frac{As+B}{s^2+1} + \frac{C}{s-1} + \frac{D}{(s-1)^2} + \frac{E}{(s-1)^3}$$

$$\Longrightarrow 2s+2 = (As+B)(s-1)^3 + C\left(s^2+1\right)(s-1)^2 + D\left(s^2+1\right)(s-1) + E\left(s^2+1\right)$$

$$= \left[As^4 + (-3A+B)s^3 + (3A-3B)s^2 + (-A+3B)s - B\right] + C\left(s^4 - 2s^3 + 2s^2 - 2s + 1\right)$$

$$+D\left(s^3 - s^2 + s - 1\right) + E\left(s^2+1\right)$$

$= (A+C)s^4 + (-3A+B-2C+D)s^3 + (3A-3B+2C-D+E)s^2 + (-A+3B-2C+D)s + (-B+C-D+E)$

$$\Longrightarrow \left\{\begin{array}{c} A+C=0 \\ -3A+B-2C+D=0 \\ 3A-3B+2C-D+E=0 \\ -A+3B-2C+D=2 \\ -B+C-D+E=2 \end{array}\right\} \text{summing all equations} \Longrightarrow 2E=4 \Longrightarrow E=2;$$

summing eqs (2) and (3) $\Longrightarrow -2B+2=0 \Longrightarrow B=1$;summing eqs (3) and (4) $\Longrightarrow 2A+2=2 \Longrightarrow A=0$;

$C=0$ from eq (1); then $-1+0-D+2=2$ from eq (5) $\Longrightarrow D=-1$;

$\displaystyle\int \frac{2s+2}{\left(s^2+1\right)(s-1)^3}\,ds = \int \frac{ds}{s^2+1} - \int \frac{ds}{(s-1)^2} + 2\int \frac{ds}{(s-1)^3} = -(s-1)^{-2} + (s-1)^{-1} + 2\tan^{-1} s + C.$

---

4. Trigonometric Substitutions

10. $\displaystyle\int \frac{5\,dx}{\sqrt{25x^2-9}}$

Solution:

$$x = \frac{3}{5}\sec\theta, 0<\theta<\frac{\pi}{2}, dx = \frac{3}{5}\sec\theta\tan\theta,$$

$$\sqrt{25x^2-9} = \sqrt{9\sec^2\theta - 9} = 3\tan\theta;$$

$$\int \frac{5\,dx}{\sqrt{25x^2-9}} = \int \frac{5\left(\frac{3}{5}\sec\theta\tan\theta\right)\,d\theta}{3\tan\theta} = \int \sec\theta\,d\theta = \ln|\sec\theta+\tan\theta| + C$$

$$= \ln\left|\frac{5x}{3} + \frac{\sqrt{25x^2-9}}{3}\right| + C.$$

---

15. $\int \frac{x^3\ dx}{\sqrt{x^2+4}}$

Solution:

$x = 2\tan\theta, -\frac{\pi}{2} < \theta < \frac{\pi}{2}, dx = 2\sec^2\theta\ d\theta; \sqrt{x^2+4} = 2\sec\theta;$

$$\int \frac{x^3\ dx}{\sqrt{x^2+4}} = \int \frac{(8\tan^3\theta)(\cos\theta)\ d\theta}{\cos^2\theta} = 8\int \frac{\sin^3\theta\ d\theta}{\cos^4\theta} = 8\int \frac{(\cos^2\theta - 1)(-\sin\theta)\ d\theta}{\cos^4\theta};$$

$[t = \cos\theta] \longrightarrow 8\int \frac{t^2-1}{t^4}\ dt = 8\int \left(\frac{1}{t^2} - \frac{1}{t^4}\right)\ dt = 8\left(-\frac{1}{t} + \frac{1}{3t^3}\right) + C = 8\left(-\sec\theta + \frac{\sec^3\theta}{3}\right) + C$

$$= 8\left(-\frac{\sqrt{x^2+4}}{2} + \frac{(x^2+4)^{3/2}}{8\cdot 3}\right) + C = \frac{1}{3}\left(x^2+4\right)^{3/2} - 4\sqrt{x^2+4} + C.$$

---

18. $\int \frac{\sqrt{9-w^2}}{w^2}\ dw$

Solution:

$w = 3\sin\theta, -\frac{\pi}{2} < \theta < \frac{\pi}{2}, dw = 3\cos\theta\ d\theta, \sqrt{9-w^2} = 3\cos\theta;$

$$\int \frac{\sqrt{9-w^2}}{w^2}\ dw = \int \frac{3\cos\theta\ 3\cos\theta\ d\theta}{9\sin^2\theta} = \int \cot^2\theta\ d\theta = \int \left(\csc^2\theta - 1\right)\ d\theta$$

$$= -\cot\theta - \theta + C = -\frac{\sqrt{9-w^2}}{w} - \sin^{-1}\left(\frac{w}{3}\right) + C$$

---

22. $\int \frac{x^2\ dx}{(x^2-1)^{5/2}}$

Solution:

$x = \sec\theta, 0 < \theta < \frac{\pi}{2}, dx = \sec\theta\tan\theta\ d\theta, \left(x^2-1\right)^{5/2} = 3\cos\theta;$

$$\int \frac{x^2\ dx}{(x^2-1)^{5/2}} = \int \frac{\sec^2\theta\sec\theta\tan\theta\ d\theta}{\tan^5\theta} = \int \frac{\cos\theta}{\sin^4\theta}\ d\theta = -\frac{1}{3\sin^3\theta} + C = -\frac{x^3}{3\left(x^2-1\right)^{3/2}} + C$$

---

30. $\int_{\ln(3/4)}^{\ln(4/3)} \frac{e^t\ dt}{(1+e^{2t})^{3/2}}$

Solution:

Let $e^t = \tan\theta, t = \ln(\tan\theta), \tan^{-1}\left(\frac{3}{4}\right) \le \theta \le \tan^{-1}\left(\frac{4}{3}\right), dt = \frac{\sec^2\theta}{\tan\theta}\ d\theta,$

$1 + e^{2t} = 1 + \tan^2\theta = \sec^2\theta;$

$$\int_{\ln(3/4)}^{\ln(4/3)} \frac{e^t\ dt}{(1+e^{2t})^{3/2}} = \int_{\tan^{-1}(3/4)}^{\tan^{-1}(4/3)} \frac{(\tan\theta)\left(\frac{\sec^2\theta}{\tan\theta}\right)\ d\theta}{\sec^3\theta} = \int_{\tan^{-1}(3/4)}^{\tan^{-1}(4/3)} \cos\theta\ d\theta$$

$$= [\sin\theta]_{\tan^{-1}(3/4)}^{\tan^{-1}(4/3)} = \frac{4}{5} - \frac{3}{5} = \frac{1}{5}.$$