ÇANKAYA UNIVERSITY

Department of Mathematics and Computer Science

MATH 156 Calculus for Engineering II

Practice Problems

Final Exam

May 26, 2008

9:00

1. 15.3 Double Integrals in Polar Form

(p. 1097) In Exercises 1-16, change the cartesian integral into an equivalent polar in tegral. Then evaluate the polar integral.

1. $\displaystyle\int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}} dy\ dx$

Solution:

$$\int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}} dy\ dx = \int_{0}^{\pi}\int_{0}^{1} r\ dr\ d\theta = \cdots = \frac{\pi}{2}.$$

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2. $\displaystyle\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dy\ dx$

Answer: $\displaystyle\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dy\ dx = \int_{0}^{2\pi}\int_{0}^{1} r\ dr\ d\theta = \cdots = \pi$

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3. $\displaystyle\int_{0}^{1}\int_{0}^{\sqrt{1-y^2}} \left(x^2+y^2\right)\ dx\ dy$

Answer: $\displaystyle\int_{0}^{1}\int_{0}^{\sqrt{1-y^2}} \left(x^2+y^2\right)\ dx\ dy = \int_{0}^{\pi/2}\int_{0}^{1} r^3\ dr\ d\theta = \cdots = \frac{\pi}{8}.$

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4. $\displaystyle\int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \left(x^2+y^2\right)\ dx\ dy$

Answer:

$$\int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \left(x^2+y^2\right)\ dx\ dy = \int_{0}^{2\pi}\int_{0}^{1} r^3\ dr\ d\theta = \cdots = \frac{\pi}{2}$$

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5. $\displaystyle\int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dy\ dx$

Solution:

$$\int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dy\ dx = \int_{0}^{2\pi}\int_{0}^{a} r\ dr\ d\theta = \cdots = \pi a^2$$

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6. $\displaystyle\int_{0}^{2}\int_{0}^{\sqrt{4-y^2}} \left(x^2+y^2\right)\ dx\ dy$

Answer: $\displaystyle\int_{0}^{2}\int_{0}^{\sqrt{4-y^2}} \left(x^2+y^2\right)\ dx\ dy = \int_{0}^{\pi/2}\int_{0}^{2} r^3\ dr\ d\theta = \cdots = 2\pi$

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7. $\displaystyle\int_{0}^{6}\int_{0}^{y} x\ dx\ dy$

Answer: $\displaystyle\int_{0}^{6}\int_{0}^{y} x\ dx\ dy = .\int_{\pi/4}^{\pi/2}\int_{0}^{6\csc\theta} r^2\cos\theta\ dr\ d\theta = \cdots = 36$

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8. $\displaystyle\int_{0}^{2}\int_{0}^{x} y\ dy\ dx$

Answer: $\displaystyle\int_{0}^{2}\int_{0}^{x} y\ dy\ dx = .\int_{0}^{\pi/4}\int_{0}^{2\sec\theta} r^2\sin\theta\ dr\ d\theta = \frac{8}{3}\int_{0}^{\pi/4}\tan\theta\sec^2\theta\ d\theta = \frac{4}{3}$

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9. $\displaystyle\int_{-1}^{0}\int_{-\sqrt{1-x^2}}^{0} \frac{2}{1+\sqrt{x^2+y^2}}\ dy\ dx$

Answer: $\displaystyle\int_{-1}^{0}\int_{-\sqrt{1-x^2}}^{0} \frac{2}{1+\sqrt{x^2+y^2}}\ dy\ dx = \int_{\pi}^{3\pi/2}\int_{0}^{1}\tfrac{2r}{1+r}\ dr\ d\theta = \cdots = (1-\ln 2)\,\pi$

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10. $\displaystyle\int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{0} \frac{4\sqrt{x^2+y^2}}{1+x^2+y^2}\ dx\ dy$

Answer: $\displaystyle\int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{0} \frac{4\sqrt{x^2+y^2}}{1+x^2+y^2}\ dx\ dy = \int_{\pi/2}^{3\pi/2}\int_{0}^{1}\frac{4r^2}{1+r^2}\ dr\ d\theta = \cdots = 4\pi-\pi^2$

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11. $\displaystyle\int_{0}^{\ln 2}\int_{0}^{\sqrt{(\ln 2)^2-y^2}} e^{\sqrt{x^2+y^2}}\ dx\ dy$

Solution:

$$\int_{0}^{\ln 2}\int_{0}^{\sqrt{(\ln 2)^2-y^2}} e^{\sqrt{x^2+y^2}}\ dx\ dy = \int_{0}^{\pi/2}\int_{0}^{\ln 2} re^{r}\ dr\ d\theta = \cdots = \frac{\pi}{2}\left(2\ln 2-1\right)$$

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12. $\displaystyle\int_{0}^{1}\int_{0}^{\sqrt{1-x^2}} e^{-\left(x^2+y^2\right)}\ dy\ dx$

Solution:

$$\int_{0}^{1}\int_{0}^{\sqrt{1-x^2}} e^{-\left(x^2+y^2\right)}\ dy\ dx = \int_{0}^{\pi/2}\int_{0}^{1} re^{-r^2}\ dr\ d\theta = \cdots = \frac{\pi\,(e-1)}{4e}$$

13. $\displaystyle\int_0^2 \int_0^{\sqrt{1-(x-1)^2}} \frac{x+y}{x^2+y^2}\, dy\, dx$

Solution:

$$\int_0^2 \int_0^{\sqrt{1-(x-1)^2}} \frac{x+y}{x^2+y^2}\, dy\, dx = \int_0^{\pi/2} \int_0^{2\cos\theta} \frac{r(\cos\theta+\sin\theta)}{r^2}\, r\, dr\, d\theta = \cdots = \frac{\pi}{2}+1$$

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14. $\displaystyle\int_0^2 \int_{-\sqrt{1-(y-1)^2}}^0 xy^2\, dx\, dy.$

Solution:

$$\int_0^2 \int_{-\sqrt{1-(y-1)^2}}^0 xy^2\, dx\, dy = \int_{\pi/2}^{\pi} \int_0^{2\sin\theta} \sin^2\theta\cos\theta\, r^4\, dr\, d\theta = \cdots = -\frac{4}{5}.$$

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15. $\displaystyle\int_{-1}^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \ln\left(x^2+y^2+1\right)\, dx\, dy.$

Solution:

$$\int_{-1}^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \ln\left(x^2+y^2+1\right)\, dx\, dy = 4\int_0^{\pi/2} \int_0^1 \ln\left(r^2+1\right) r\, dr\, d\theta = \cdots = \pi(\ln 4 - 1).$$

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16. $\displaystyle\int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{2}{\left(1+x^2+y^2\right)^2}\, dy\, dx$

Solution:

$$\int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{2}{\left(1+x^2+y^2\right)^2}\, dy\, dx = 4\int_0^{\pi/2} \int_0^1 \frac{2r}{\left(1+r^2\right)^2}\, dr\, d\theta = \cdots = \pi.$$

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2. Finding Area in Polar Coordinates

17. Find the area of the region cut from the first quadrant by the curve $r = 2(2-\sin 2\theta)^{1/2}$.

Answer: $\displaystyle\int_0^{\pi/2} \int_0^{2\sqrt{2-\sin 2\theta}} r\, dr\, d\theta = 2\int_0^{\pi/2} (2-\sin 2\theta)\, d\theta = 2(\pi-1).$

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18. Find the area of the region that lies inside the cardioid $r = 1+\cos\theta$ and outside the circle $r = 1$.

Answer: $\displaystyle A = 2\int_0^{\pi/2} \int_1^{1+\cos\theta} r\, dr\, d\theta = \int_0^{\pi/2} \left(2\cos\theta + \cos^2\theta\right)\, d\theta = \frac{8+\pi}{4}.$

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19. Find the area enclosed by one leaf of the rose $r = 12\cos 3\theta$.

Answer: $\displaystyle A = 2\int_0^{\pi/6} \int_0^{12\cos 3\theta} r\, dr\, d\theta = \cdots = 12\pi.$

22. Find the area of the region common to the interiors of the cardioids $r = 1 + \cos\theta$ and $r = 1 - \cos\theta$.

Answer: $A = 4\int_0^{\pi/2}\int_0^{1-\cos\theta} r\ dr\ d\theta = \cdots = \dfrac{3\pi}{2} - 4$

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3. Theory and Examples

33. Integrate $f(x,y) = \dfrac{\ln(x^2+y^2)}{\sqrt{x^2+y^2}}$ over the region $1 \le x^2 + y^2 \le e$

Solution:

$$\int_0^{2\pi}\int_1^{\sqrt{e}}\left(\frac{\ln r^2}{r}\right) r\ dr\ d\theta = \int_0^{2\pi}\int_1^{\sqrt{e}} 2\ln r\ dr\ d\theta = \int_0^{2\pi}\int_1^{\sqrt{e}} 2\ln r\ dr\ d\theta = \cdots = 2\pi\left(2 - \sqrt{e}\right)$$

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34. Integrate $f(x,y) = \dfrac{\ln(x^2+y^2)}{x^2+y^2}$ over the region $1 \le x^2 + y^2 \le e$

Solution:

$$\int_0^{2\pi}\int_1^{\sqrt{e}}\left(\frac{\ln r^2}{r^2}\right) r\ dr\ d\theta = \int_0^{2\pi}\int_1^{\sqrt{e}}\left(\frac{2\ln r}{r}\right) dr\ d\theta = \cdots = 2\pi$$