ÇANKAYA UNIVERSITY
Department of Mathematics and Computer Science
MATH 156 Calculus for Engineering II
Practice Problems -b

Final Exam

May 26, 2008

9:00

1. 15.3 Double Integrals in Polar Form

(p. 1097) In Exercises 1-16, change the cartesian integral into an equivalent polar in tegral. Then evaluate the polar integral.

1. $\displaystyle\int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}} dy\ dx$

Solution:

$$\int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}} dy\ dx = \int_{0}^{\pi}\int_{0}^{1} r\ dr\ d\theta = \cdots = \frac{\pi}{2}.$$

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2. $\displaystyle\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dy\ dx$

Answer: $\displaystyle\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dy\ dx = \int_{0}^{2\pi}\int_{0}^{1} r\ dr\ d\theta = \cdots = \pi$

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3. $\displaystyle\int_{0}^{1}\int_{0}^{\sqrt{1-y^2}} \left(x^2+y^2\right)\ dx\ dy$

Answer: $\displaystyle\int_{0}^{1}\int_{0}^{\sqrt{1-y^2}} \left(x^2+y^2\right)\ dx\ dy = \int_{0}^{\pi/2}\int_{0}^{1} r^3\ dr\ d\theta = \cdots = \frac{\pi}{8}.$

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4. $\displaystyle\int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \left(x^2+y^2\right)\ dx\ dy$

Answer:

$$\int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \left(x^2+y^2\right)\ dx\ dy = \int_{0}^{2\pi}\int_{0}^{1} r^3\ dr\ d\theta = \cdots = \frac{\pi}{2}$$

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5. $\displaystyle\int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dy\ dx$

Solution:

$$\int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dy\ dx = \int_0^{2\pi}\int_0^a r\ dr\ d\theta = \cdots = \pi a^2$$

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6. $\int_0^6 \int_0^y x\ dx\ dy$

Answer: $\int_0^6 \int_0^y x\ dx\ dy = \frac{\pi}{4}$

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7. $\int_0^6 \int_0^{\ln y} e^{x+y}\ dxdy$

Answer: $8\ln 8 - 16 + e$.

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8. $\int_1^2 \int_y^{y^2} dxdy$

Answer: $\frac{5}{6}$.

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9. $\int_0^1 \int_0^{y^2} 3y^3 e^{xy}\ dxdy$

Answer: $e - 2$

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10. $\int_1^4 \int_0^{\sqrt{x}} \frac{3}{2} e^{y/\sqrt{x}}\ dydx$

Answer: $7(e-1)$

In Exercises 11-16, integrate f over the given region.

11. Quadrilateral $f(x,y) = x/y$ over the region in the first quadrant bounded by the lines $y = x, y = 2x, x = 1, x = 2$

Solution:

$$\int_1^2 \int_x^{2x} \frac{x}{y}\ dydx = \frac{3}{2}\ln 2.$$

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12. Square $f(x,y) = \frac{1}{xy}$ over the square $1 \le x \le 2, 1 \le y \le 2$

Solution:

$$\int_1^2 \int_1^2 \frac{1}{xy}\ dydx = (\ln 2)^2.$$

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13. Triangle $f(x,y) = x^2 + y^2$ over the triangular region with vertices $(0,0), (1,0), (0,1)$

Solution:

$$\int_0^1 \int_0^{1-x} \left(x^2 + y^2\right)\ dydx = \frac{1}{6}.$$

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14. Rectangle $f(x, y) = y \cos xy$ over the recrangle $0 \leq x \leq \pi, 0 \leq y \leq 1$.
Solution:
$$\int_0^1 \int_0^{\pi} y \cos xy\ dxdy = \frac{2}{\pi}.$$

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15. Triangle $f(u, v) = v - \sqrt{u}$ over the triangular region cut from the first quadrant of the uv-plane by the line $u + v = 1$.
Solution:
$$\int_0^1 \int_0^{1-u} \left(v - \sqrt{u}\right)\ dvdu = -\frac{1}{10}.$$

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16. Curved region $f(s, t) = e^s \ln t$ over the region in the first quadrant of the st-plane that lies above the curve $s = \ln t$ from $t = 1$ to $t = 2$.
Solution:
$$\int_1^2 \int_0^{\ln t} e^s \ln t = \frac{1}{4}.$$

Each of the exercises 17-20 gives an integral over a region in a cartesian coordinate plane. Sketch the region and evaluate the integral.

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17. $\displaystyle\int_{-2}^0 \int_v^{-v} 2\ dpdv$ (the pv-plane).
Answer: 8.

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18. $\displaystyle\int_0^1 \int_0^{\sqrt{1-x^2}} 8t\ dt\ ds$ (the st-plane).
Answer: $\frac{8}{3}$.

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19. $\displaystyle\int_{-\pi/3}^{\pi/3} \int_0^{\sec t} 3 \cos t\ du\ dt$ (the tu-plane)
Answer: 2π.

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20. $\displaystyle\int_1^3 \int_1^{4-2x} \frac{4 - 2u}{v^2}\ dv\ du$ (the uv-plane)
Answer: 0.

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2. Reversing the order of integration

In Exercises 21-30, sketch the region of integration and write an equivalent double integral with the order of integration reversed.

21. $\int_0^1 \int_2^{4-2x} dy\ dx$

Solution:

$$\int_0^1 \int_2^{4-2x} dy\ dx = \int_2^4 \int_0^{(4-y)/2} dx\ dy$$

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22. $\int_0^2 \int_{y-2}^0 dx\ dy$

Solution:

$$\int_0^2 \int_{y-2}^0 dx\ dy = \int_{-2}^0 \int_0^{x+2} dy\ dx$$

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23. $\int_0^1 \int_y^{\sqrt{y}} dx\ dy$

Solution:

$$\int_0^1 \int_y^{\sqrt{y}} dx\ dy = \int_0^1 \int_x^{x^2} dy\ dx$$

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24. $\int_0^1 \int_{1-x}^{1-x^2} dy\ dx$

Solution:

$$\int_0^1 \int_{1-x}^{1-x^2} dy\ dx = \int_0^1 \int_{1-y}^{\sqrt{1-y}} dx\ dy$$

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25. $\int_0^1 \int_1^{e^x} dy\ dx$

Solution:

$$\int_0^1 \int_1^{e^x} dy\ dx = \int_1^e \int_{\ln y}^1 dx\ dy$$

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26. $\int_0^{\ln 2} \int_{e^x}^2 dy\ dx$

Solution:

$$\int_0^{\ln 2} \int_{e^y}^2 dx\ dy = \int_1^2 \int_0^{\ln x} dy\ dx$$

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27. $\int_0^{3/2} \int_0^{9-4x^2} 16x\ dy\ dx$

Solution:

$$\int_0^{3/2}\int_0^{9-4x^2} 16x\ dy\ dx = \int_0^9\int_0^{\frac{1}{2}\sqrt{9-y}} 16x\ dx\ dy.$$

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28. $\int_0^2\int_0^{4-y^2} y\ dx\ dy$

Solution:
$$\int_0^2\int_0^{4-y^2} y\ dx\ dy = \int_0^4\int_0^{\sqrt{4-x}} y\ dy\ dx.$$

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29. $\int_0^1\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} 3y\ dx\ dy.$

Solution:
$$\int_0^1\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} 3y\ dx\ dy = \int_{-1}^1\int_0^{\sqrt{1-x^2}} 3y\ dy\ dx.$$

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30. $\int_0^2\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} 6x\ dy\ dx.$

Solution:
$$\int_0^2\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} 6x\ dy\ dx = \int_{-2}^2\int_0^{\sqrt{4-y^2}} 6x\ dx\ dy.$$

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3. Evaluating Double Integrals

In Exercises 31-40, sketch the region of integration, reverse the order of integration, and evaluate the integral.

31. $\int_0^\pi\int_x^\pi \frac{\sin y}{y}\ dy\ dx.$

Solution:
$$\int_0^\pi\int_x^\pi \frac{\sin y}{y}\ dy\ dx = \int_0^\pi\int_0^y \frac{\sin y}{y}\ dx\ dy = \int_0^\pi \sin y\ dy = 2.$$

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32. $\int_0^2\int_x^2 2y^2\sin xy\ dy\ dx.$

Solution:
$$\int_0^2\int_x^2 2y^2\sin xy\ dy\ dx = \int_0^2\int_0^y 2y^2\sin xy\ dx\ d$$
$$= \int_0^2 [-2y\cos xy]_0^y\ dy = \int_0^2 \left(-2y\cos y^2 + 2y\right)\ dy = \left[-\sin y^2 + y^2\right]_0^2 = 4 - \sin 4.$$

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33. $\int_0^1\int_y^1 x^2 e^{xy}\ dx\ dy.$

Solution:
$$\int_0^1\int_y^1 x^2 e^{xy}\ dx\ dy = \int_0^1\int_0^x x^2 e^{xy}\ dy\ dx = \int_0^1 [xe^{xy}]_0^x\ dx$$

$$= \int_0^1 \left(xe^{x^2} - x \right) \ dx = \left[\frac{1}{2} e^{x^2} - \frac{x^2}{2} \right]_0^1 = \frac{e-2}{2}.$$

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34. $\displaystyle\int_0^2 \int_0^{4-x^2} \frac{xe^{2y}}{4-y} \ dy \ dx.$

Solution:

$$\int_0^2 \int_0^{4-x^2} \frac{xe^{2y}}{4-y} \ dy \ dx = \int_0^4 \int_0^{\sqrt{4-y}} \frac{xe^{2y}}{4-y} \ dx \ dy$$

$$= \int_0^4 \left[\frac{x^2 e^{2y}}{2(4-y)} \right]_0^{\sqrt{4-y}} \ dy = \int_0^4 \frac{e^{2y}}{2} \ dy = \left[\frac{e^{2y}}{4} \right]_0^4 = \frac{e^8 - 1}{4}.$$

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35. $\displaystyle\int_0^{2\sqrt{\ln 3}} \int_{y/2}^{\sqrt{\ln 3}} e^{x^2} \ dx \ dy$

Solution:

$$\int_0^{2\sqrt{\ln 3}} \int_{y/2}^{\sqrt{\ln 3}} e^{x^2} \ dx \ dy = \int_0^{\sqrt{\ln 3}} \int_0^{2x} e^{x^2} \ dy \ dx$$

$$= \int_0^{\sqrt{\ln 3}} 2xe^{x^2} = \left[e^{x^2} \right]_0^{\sqrt{\ln 3}} = e^{\ln 3} - 1 = 2.$$

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36. $\displaystyle\int_0^3 \int_{\sqrt{x/3}}^1 e^{y^3} \ dy \ dx$

Solution:

$$\int_0^3 \int_{\sqrt{x/3}}^1 e^{y^3} \ dy \ dx = \int_0^3 \int_0^{3y^2} e^{y^3} \ dx \ dy$$

$$\int_0^1 3y^2 e^{y^3} \ dy = \left[e^{y^3} \right]_0^1 = e - 1$$

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37. $\displaystyle\int_0^{1/16} \int_{y^{1/4}}^{1/2} \cos\left(16\pi x^5\right) \ dx \ dy$

Solution:

$$\int_0^{1/16} \int_{y^{1/4}}^{1/2} \cos\left(16\pi x^5\right) \ dx \ dy = \int_0^{1/2} \int_0^{x^4} \cos\left(16\pi x^5\right) \ dy \ dx$$

$$= \int_0^{1/2} x^4 \cos\left(16\pi x^5\right) \ dx = \left[\frac{\sin\left(16\pi x^5\right)}{80\pi} \right]_0^{1/2} = \frac{1}{80\pi}.$$

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38. $\displaystyle\int_0^8 \int_{\sqrt[3]{x}}^2 \frac{dy \ dx}{y^4 + 1}$

Solution:

$$\int_0^8 \int_{\sqrt[3]{x}}^2 \frac{dy \ dx}{y^4 + 1} = \int_0^2 \int_0^{y^3} \frac{1}{y^4 + 1} \ dx \ dy$$

$$= \int_0^8 \frac{y^3}{y^4 + 1} \ dy = \frac{1}{4} \left[\ln\left(y^4 + 1\right) \right]_0^2 = \frac{\ln 17}{4}.$$

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39. Square region $\int\int_R \left(y-2x^2\right)\, dA$ where R is the region bounded by the square $|x|+|x|=1$.
Solution:

$$\int\int_R \left(y-2x^2\right)\, dA$$
$$= \int_{-1}^{0}\int_{-x-1}^{x+1} \left(y-2x^2\right)\, dy\, dx + \int_{0}^{1}\int_{x-1}^{1-x} \left(y-2x^2\right)\, dy\, dx$$
$$= \int_{-1}^{0} \left[\frac{1}{2}y^2 - 2x^2 y\right]_{-x-1}^{x+1} dx + \int_{0}^{1} \left[\frac{1}{2}y^2 - 2x^2 y\right]_{x-1}^{1-x} dx$$
$$= \int_{-1}^{0} \left[\frac{1}{2}(x+1)^2 - 2x^2(x+1) - \frac{1}{2}(-x-1)^2 + 2x^2(-x-1)\right] dx$$
$$+ \int_{-1}^{0} \left[\frac{1}{2}(1-x)^2 - 2x^2(1-x) - \frac{1}{2}(x-1)^2 + 2x^2(x-1)\right]$$
$$= \frac{1}{2}.$$

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40. Triangular region $\int\int_R xy\, dA$ where R is the region bounded by the lines $y=x, y=2x$, and $x+y=2$
Solution:

$$\int\int_R xy\, dA = \int_{0}^{2/3}\int_{x}^{2x} xy\, dy\, dx + \int_{2/3}^{1}\int_{x}^{2-x} xy\, dy\, dx$$
$$= \int_{0}^{2/3} \left[\frac{1}{2}xy^2\right]_{x}^{2x} dx + \int_{2/3}^{1} \left[\frac{1}{2}xy^2\right]_{x}^{2-x} dx$$
$$= \frac{13}{81}.$$

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4. Volume Beneath a Surface $z = f(x,y)$

41. Find the volume of the region bounded by the paraboloid $z=x^2+y^2$ and below by the triangle enclosed by the lines $y=x, x=0$, and $x+y=2$ in the plane.
Solution:

$$V = \int_{0}^{1}\int_{x}^{2-x} \left(x^2+y^2\right)\, dy\, dx = \cdots = \frac{4}{3}.$$

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42. Find the volume of the solid that is bounded above by the cylinder $z=x^2$ and below by the region enclosed by the parabola $y=2-x^2$ and the line $y=x$ in the xy-plane.
Solution:

$$V = \int_{-2}^{1}\int_{x}^{2-x^2} x^2\, dy\, dx = \cdots = \frac{63}{20}.$$

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43. Find the volume of the solid whose base is the region in the xy-plane that is bounded by the parabola $y=4-x^2$ and the line $y=3x$, while the top of the solid is bounded by the plane $z=x+4$.
Solution:

$$V = \int_{-4}^{1}\int_{3x}^{4-x^2} (x+4)\, dy\, dx = \cdots = \frac{625}{12}.$$