ÇANKAYA UNIVERSITY

Department of Mathematics and Computer Science

MATH 156 Calculus for Engineering II

Practice Problems-f

Final Exam
May 26, 2008
09:00

1. Substitutions in Multiple Integrals

(p. 1135) In Exercises 1-10, sketch the region of integration and evaluate the integral.

1.

a. Solve the system

$$u = x - y \ , \qquad v = 2x + y$$

for x and y. Then find the value of the Jacobian $\dfrac{\partial (x, y)}{\partial (u, v)}$.

b.Find the image under the transformation $u = x - y,\ v = 2x + y$ of the triangular region with vertices $(0, 0), (1, 1), (1, -2)$ in the xy-plane. Sketch the transformed region in the uv-plane.

Solution:

(a) $x - y = u$ and $2x + y = v \Longrightarrow 3x = u + v$ and $y = x - u \Longrightarrow x = \dfrac{1}{3}(u + v)$ and $y = \dfrac{1}{3}(-2u + v)$;

$$\frac{\partial (x, y)}{\partial (u, v)} = \begin{vmatrix} \frac{1}{3} & \frac{1}{3} \\ -\frac{2}{3} & \frac{1}{3} \end{vmatrix} = \frac{1}{3}$$

(b) The line segment $y = x$ from $(0, 0)$ to $(1, 1)$ is $x - y = 0 \Longrightarrow u = 0$;
the line segment $y = -2x$ from $(0, 0)$ to $(1, -2)$ is $2x + y = 0 \Longrightarrow v = 0$;
the line segment $x = 1$ from $(1, 1)$ to $(1, 2)$ is $(x - y) + (2x + y) = 3 \Longrightarrow u + v = 3$.

2.

a.Solve the system

$$u = x + 2y \ , \qquad v = x - y$$

for x and y in terms of u and v. Then find the value of the Jacobian $\dfrac{\partial (x, y)}{\partial (u, v)}$.

b. Find the image under the transformation $u = x + 2y,\ v = x - y$ of the triangular region in the xy-plane bounded by the lines $y = 0, y = x$, and $x + 2y = 2$. Sketch the transformed region in the uv-plane.

Solution:

(a) $x + 2y = u$ and $x - y = v \Longrightarrow 3y = u - v$ and $x = v + y \Longrightarrow y = \dfrac{1}{3}(u - v)$ and $x = \dfrac{1}{3}(u + 2v)$;

$$\frac{\partial (x, y)}{\partial (u, v)} = \begin{vmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & -\frac{1}{3} \end{vmatrix} = -\frac{1}{3}$$

(b) The triangular region in the xy-plane has vertices $(0, 0), (2, 0), \left(\dfrac{2}{3}, \dfrac{2}{3}\right)$.

The line segment $y=x$ from $(0,0)$ to $\left(\frac{2}{3},\frac{2}{3}\right)$ is $x-y=0 \Longrightarrow v=0$;
the line segment $y=0$ from $(0,0)$ to $(2,0) \Longrightarrow u=v$;
the line segment segment $x+2y=2$ from $\left(\frac{2}{3},\frac{2}{3}\right)$ to $(2,0) \Longrightarrow u=2$.

3.
a. Solve the system

$$u = 3x+2y\ , \qquad v = x+4y$$

for x and y in terms of u and v. Then find the value of the Jacobian $\dfrac{\partial(x,y)}{\partial(u,v)}$.

b. Find the image under the transformation $u=3x+2y,\ v=x+4y$ of the triangular region in the xy-plane bounded by the x-axis, the y-axis, and the line $x+y=1$. plane. Sketch the transformed region in the uv-plane.
Solution:
(a) $3x+2y=u$ and $x+4y=v \Longrightarrow -5x=-2u+v$ and $y=\frac{1}{2}(u-3x) \Longrightarrow x=\frac{1}{5}(2u-v)$ and $y=\frac{1}{10}(3v-u)$;
$\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{2}{5} & -\frac{1}{5} \\ -\frac{1}{10} & \frac{3}{10} \end{vmatrix} = \dfrac{1}{10}.$
(b) The x-axis $y=0 \Longrightarrow u=3v$;

the y-axis $x=0 \Longrightarrow v=2u$;
the line $x+y=1$ $\frac{1}{5}(2u-v)+\frac{1}{10}(3v-u)=1$
$\Longrightarrow 2(2u-v)+(3v-u)=10 \Longrightarrow 3u+v=10.$

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4. a. Solve the system

$$u = 2x-3y\ , \qquad v = -x+y$$

for x and y in terms of u and v. Then find the value of the Jacobian $\dfrac{\partial(x,y)}{\partial(u,v)}$.

b. Find the image under the transformation $u=2x-3y,\ v=-x+y$ of the parallelogram with boundaries $x=-3, x=0, y=x$, and $y=x+1$. Sketch the transformed region in the uv-plane.

Solution:
(a) $2x-3y=u$ and $-x+y=v \Longrightarrow -x=u+3v$ and $y=v+x \Longrightarrow x=-u-3v$ and $y=-u-2v$;

$\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} -1 & -3 \\ -1 & -2 \end{vmatrix} = -1.$
(b) The line $x=-3 \Longrightarrow -u-3v=-3$ or $u+3v=3$;
$x=0 \Longrightarrow u+3v=0$;
$y=x+1 \Longrightarrow v=1$.

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5. Evaluate the integral

$$\int_0^4 \int_{x=y/2}^{x=(y/2)+1} \frac{2x-y}{2}\ dx\ dy$$

directly by integration with respect to x and y
Solution:

$$\int_0^4 \int_{x=y/2}^{x=(y/2)+1} \left(x-\frac{y}{2}\right)\ dx\ dy = \int_0^4 \left[\frac{x^2}{2}-\frac{xy}{2}\right]_{\frac{y}{2}}^{\frac{y}{2}+1}\ dy$$
$$= \frac{1}{2}\int_0^4 \left[\left(\frac{y}{2}+1\right)^2 - \left(\frac{y}{2}\right)^2 - \left(\frac{y}{2}+1\right)y + \left(\frac{y}{2}\right)y\right]\ dy = \frac{1}{2}\int_0^4 (y+1-y)\ dy$$
$$= \frac{1}{2}\int_0^4\ dy = 2$$

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6. Evaluate

$$\int\int_R \left(2x^2-xy-y^2\right)\ dx\ dy$$

for the region R in the first quadrant bounded by the lines
$y=-2x+4, y=-2x+7, y=x-2, y=x+1$
Solution:

$$\int\int_R \left(2x^2-xy-y^2\right)\ dx\ dy = \int\int_R (x-y)(2x+y)\ dx\ dy$$
$$= \int\int_G uv\left|\frac{\partial(x,y)}{\partial(u,v)}\right|\ du\ dv = \frac{1}{3}\int\int_G uv\ du\ dv\ ;$$

We find the boundaries of G from the boundaries of R

$$\Longrightarrow \frac{1}{3}\int\int_G uv\ du\ dv = \frac{1}{3}\int_{-1}^{2}\int_4^7 uv\ dv\ du = \frac{1}{3}\int_{-1}^{2} u\left[\frac{v^2}{2}\right]_4^7\ du$$
$$= \left(\frac{11}{2}\right)\left[\frac{u^2}{2}\right]_{-1}^{2} = \left(\frac{11}{4}\right)(4-1) = \frac{33}{4}$$

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7. Evaluate

$$\int\int_R \left(3x^2+14xy+8y^2\right)\ dx\ dy$$

for the region R in the first quadrant bounded by the lines
$y=-(3/2)x+1, y=-(3/2)x+3, y=-(1/4)x-2, y=(-1/4)x+1$
Solution:

$$\int\int_R \left(3x^2+14xy+8y^2\right)\ dx\ dy = \int\int_R (3x+2y)(x+4y)\ dx\ dy$$
$$= \int\int_G uv\left|\frac{\partial(x,y)}{\partial(u,v)}\right|\ du\ dv = \frac{1}{10}\int\int_G uv\ du\ dv\ ;$$

We find the boundaries of G from the boundaries of R

$$\Longrightarrow \frac{1}{10}\int\int_G uv\ du\ dv = \frac{1}{10}\int_2^6\int_0^4 uv\ dv\ du = \frac{1}{10}\int_2^6 u\left[\frac{v^2}{2}\right]_0^4\ du = \frac{4}{5}\int_2^6 u\ du$$
$$= \left(\frac{4}{5}\right)\left[\frac{u^2}{2}\right]_2^6 = \left(\frac{4}{5}\right)(18-2) = \frac{64}{5}.$$

8. Use the transformation and parallelogram R in Exercise 4 to evaluate the integral

$$\int\int_R 2(x-y)\ dx\ dy$$

Solution:

$$\int\int_R 2(x-y)\ dx\ dy = \int\int_G -2v\left|\frac{\partial(x,y)}{\partial(u,v)}\right|\ du\ dv = \int\int_G -2v\ du\ dv$$

$$\Longrightarrow \int\int_G -2v\ du\ dv = \int_0^1\int_{-3v}^{3-3v} -2v\ du\ dv$$

$$= \int_0^1 -2v(3-3v+3v)\ dv = \int_0^1 -6v\ dv = \left[-3v^2\right]_0^1 = -3.$$

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9. Let R be the region in the first quadrant of the xy-plane bounded by the hyperbolas $xy=1, xy=9$ and the lines $y=x, y=4x$. Use $x=u/v, y=uv$ with $u>0$ and $v>0$ to evaluate

$$\int\int_R\left(\sqrt{\frac{y}{x}}+\sqrt{xy}\right)\ dx\ dy$$

Solution:

$x=u/v$ and $y=uv \Longrightarrow \dfrac{y}{x}=v^2$ and $xy=u^2$;

$\dfrac{\partial(x,y)}{\partial(u,v)} = J(u,v) = \begin{vmatrix} v^{-1} & -uv^{-2} \\ v & u \end{vmatrix} = \dfrac{2u}{v}$;

$y=x \Longrightarrow uv=\dfrac{u}{v} \Longrightarrow v=1$, and

$y=4x \Longrightarrow v=2$;

$xy=1 \Longrightarrow u=1$, and

$xy=9 \Longrightarrow u=3$, thus

$$\int\int_R\left(\sqrt{\frac{y}{x}}+\sqrt{xy}\right)\ dx\ dy = \int_1^3\int_1^2 (v+u)\left(\frac{2u}{v}\right)\ dv\ du$$

$$= \int_1^3\int_1^2\left(2u+\frac{2u^2}{v}\right)\ dv\ du = \int_1^3\left[2uv+2u^2\ln v\right]_1^2\ du$$

$$= \int_1^3\left(2u+2u^2\ln 2\right)\ du = 8+\frac{2}{3}(26)(\ln 2).$$

10.

a. Find the Jacobian of the transformation $x=u, y=uv$, and sketch the region $u=G: 1\le u\le 2, 1\le uv\le 2$ in the uv-plane.

b. Evaluate $\displaystyle\int_1^2\int_1^2 \frac{y}{x}\ dy\ dx$

Solution:

(a)

$\dfrac{\partial(x,y)}{\partial(u,v)} = J(u,v) = \begin{vmatrix} 1 & 0 \\ u & v \end{vmatrix} = u$

(b)

$x=1 \Longrightarrow u=1$, and $x=2 \Longrightarrow u=2$;

$y=1 \Longrightarrow uv=1 \Longrightarrow v=\dfrac{1}{u}$, and $y=2 \Longrightarrow uv=2 \Longrightarrow v=\dfrac{2}{u}$;

thus,

$$\int_1^2 \int_1^2 \frac{y}{x}\, dy\ dx = \int_1^2 \int_{1/u}^{2/u} \left(\frac{uv}{u}\right)\ u\ dv\ du = \int_1^u \int_{1/u}^{2/u} uv\ dv\ du$$
$$= \int_1^2 u \left[\frac{v^2}{2}\right]_{1/u}^{2/u} du = \int_1^2 u\left(\frac{2}{u^2} - \frac{1}{2u^2}\right)\ du = \frac{3}{2}\int_1^2 u\left(\frac{1}{u^2}\right)\ du$$
$$= \frac{3}{2}\ln 2.$$

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13. Evaluate the integral

$$\int_0^{2/3} \int_y^{2-2y} (x+2y)\, e^{(y-x)}\ dx\ dy$$

Solution:
Let $u = x + 2y$ and $v = x - y \Longrightarrow 2x - y = (2u + v) - v = 2u$ and
$\left|\frac{\partial (x, y)}{\partial (u, v)}\right| = J(u, v) = -\frac{1}{3};$
next, $u = x - \frac{v}{2} = x - \frac{y}{2}$ and $v = y$, so the boundaries of R are transformed to the boundaries of G:

boundaries of R	Substitute	Simplify
$x = y$	$\frac{1}{3}(u+2v) = \frac{1}{3}(u-v)$	$v = 0$
$x = 2 - 2y$	$\frac{1}{3}(u+2v) = 2 - \frac{2}{3}(u-v)$	$u = 2$
$y = 0$	$0 = \frac{1}{3}(u-v)$	$v = u$

$$\Longrightarrow \int_0^{2/3} \int_y^{2-2y} (x+2y)\, e^{(y-x)}\ dx\ dy = \int_0^2 \int_0^u ue^{-v}\left|-\frac{1}{3}\right|\ dv\ du$$
$$= \frac{1}{3}\int_0^2 u\left[-e^{-v}\right]_0^u\ du = \frac{1}{3}\int_0^2 u\left(1 - e^{-u}\right)\ du = \frac{1}{3}\left[u\left(u + e^{-v}\right) - \frac{u^2}{2} + e^{-u}\right]_0^2 = \frac{1}{3}\left(3e^{-2} + 1\right) \approx 0.4687.$$

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14. Evaluate the integral

$$\int_0^2 \int_{y/2}^{(y+4)/2} y^3\,(2x-y)\, e^{(2x-y)^2}\ dx\ dy$$

Solution:
Let $x = u + \frac{v}{2}$ and $= y = v \Longrightarrow 2x - y = (2u + v) - v = 2u$ and
$\left|\frac{\partial (x, y)}{\partial (u, v)}\right| = J(u, v) = \begin{vmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{vmatrix} = 1;$
next, $u = x - \frac{v}{2} = x - \frac{y}{2}$ and $v = y$, so the boundaries of R are transformed to the boundaries of G:

boundaries of R	Substitute	Simplify
$x = \frac{y}{2}$	$u + \frac{v}{2} = \frac{v}{2}$	$u = 0$
$x = \frac{y}{2} + 2$	$u + \frac{v}{2} = \frac{v}{2} + 2$	$u = 2$
$y = 0$	$v = 0$	$v = 0$
$y = 2$	$v = 2$	$v = 2$

$$\Longrightarrow \int_0^2 \int_{y/2}^{(y/2)+2} y^3\,(2x-y)\, e^{(2x-y)^2}\ dx\ dy = \int_0^2 \int_0^2 v^3\,(2u)\, e^{4u^2}\ du\ dv$$
$$= \int_0^2 v^3\left[\frac{1}{4}e^{4u^2}\right]_0^2\ dv = \frac{1}{4}\int_0^2 v^3\left(e^{16} - 1\right)\ dv = \frac{1}{4}\left(e^{16} - 1\right)\left[\frac{v^4}{4}\right]_0^2 = e^{16} - 1.$$