ÇANKAYA UNIVERSITY
Department of Mathematics and Computer Science
MATH 156 Calculus for Engineering II
Practice Problems Final Exam

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09:00

1. 16.1 Line Integrals

(p. 1149)

9. Evaluate $\int_C (x+y)\ ds$ where C is the straight-line segment $x=t, y=(1-t), z=0$, from $(0,1,0)$ to $(1,0,0)$.
Solution:
$\mathbf{r}(t)=t\mathbf{i}+(1-t)\mathbf{j}, 0\le t\le 1 \Longrightarrow \frac{d\mathbf{r}}{dt}=\mathbf{i}-\mathbf{j}\Longrightarrow \left|\frac{d\mathbf{r}}{dt}\right|=\sqrt{2};$
$x=t$ and $y=1-t\Longrightarrow x+y=t+(1-t)=1$
$\Longrightarrow \int_C f(x,y,z)\ ds=\int_0^1 f(t,1-t,0)\ \left|\frac{d\mathbf{r}}{dt}\right|\ dt=\int_0^1 (1)\left(\sqrt{2}\right)\ dt=\left[\sqrt{2}t\right]_0^1=\sqrt{2}.$

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10. Evaluate $\int_C (x-y+z-2)\ ds$ where C is the straight-line segment $x=t, y=(1-t), z=1$, from $(0,1,1)$ to $(1,0,1)$.
Solution:
$\mathbf{r}(t)=t\mathbf{i}+(1-t)\mathbf{j}+\mathbf{k}, 0\le t\le 1 \Longrightarrow \frac{d\mathbf{r}}{dt}=\mathbf{i}-\mathbf{j}\Longrightarrow \left|\frac{d\mathbf{r}}{dt}\right|=\sqrt{2};$
$x=t, y=1-t$ and $z=1\Longrightarrow x-y+z-2=t-(1-t)+1-2=2t-2$
$\Longrightarrow \int_C f(x,y,z)\ ds=\int_0^1 (2t-2)\ \sqrt{2}\ dt=\sqrt{2}\left[t^2-2t\right]_0^1=-\sqrt{2}.$

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11. Evaluate $\int_C (x-y+z-2)\ ds$ where C is the straight-line segment $x=t, y=(1-t), z=1$, from $(0,1,1)$ to $(1,0,1)$.
Solution:
$\mathbf{r}(t)=t\mathbf{i}+(1-t)\mathbf{j}+\mathbf{k}, 0\le t\le 1 \Longrightarrow \frac{d\mathbf{r}}{dt}=\mathbf{i}-\mathbf{j}\Longrightarrow \left|\frac{d\mathbf{r}}{dt}\right|=\sqrt{2};$
$x=t, y=1-t$ and $z=1\Longrightarrow x-y+z-2=t-(1-t)+1-2=2t-2$
$\Longrightarrow \int_C f(x,y,z)\ ds=\int_0^1 (2t-2)\ \sqrt{2}\ dt=\sqrt{2}\left[t^2-2t\right]_0^1=-\sqrt{2}.$

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12. Evaluate $\int_C \sqrt{x^2+y^2}\, ds$ along the curve $\mathbf{r}(t)=(4-\cos t)\,\mathbf{i}+(4\sin t)\,\mathbf{j}+3t\mathbf{k}, -2\pi\le t\le 2\pi$.
Solution:

$\mathbf{r}(t) = (4\cos t)\,\mathbf{i} + (4\sin t)\,\mathbf{j} + 3t\mathbf{k}, -2\pi \le t \le 2\pi \implies \dfrac{d\mathbf{r}}{dt} = (-4\sin t)\,\mathbf{i} + (4\cos t)\,\mathbf{j} + 3\mathbf{k} \implies$
$\left|\dfrac{d\mathbf{r}}{dt}\right| = \sqrt{16\sin^2 t + 16\cos^2 t + 9} = 5;$
$\sqrt{x^2+y^2} = \sqrt{16\cos^2 t + 16\sin^2 t} = 4$
$\implies \displaystyle\int_C f(x,y,z)\ ds = \int_{-2\pi}^{2\pi} (4)(5)\ dt = [20t]_{-2\pi}^{2\pi} = 80\pi.$

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13. Find the integral of $f(x,y,z) = x+y+z$ over the straight line segment from $(1,2,3)$ to $(0,-1,1)$.
Solution:
$\mathbf{r}(t) = (\mathbf{i}+2\mathbf{j}+3\mathbf{k}) + t(-\mathbf{i}-3\mathbf{j}-2\mathbf{k}) = (1-t)\,\mathbf{i} + (2-3t)\,\mathbf{j} + (3-2t)\,\mathbf{k}, 0 \le t \le 1$
$\implies \dfrac{d\mathbf{r}}{dt} = -\mathbf{i}-3\mathbf{j}-2\mathbf{k} \implies \left|\dfrac{d\mathbf{r}}{dt}\right| = \sqrt{1+9+4} = \sqrt{14};$
$\implies x+y+z = (1-t)+(2-3t)+(3-2t) = 6-6t$
$\implies \displaystyle\int_C f(x,y,z)\ ds = \int_0^1 (6-6t)\ \sqrt{14}\ dt = 6\sqrt{14}\left[t - \frac{t^2}{2}\right]_0^1 = \left(6\sqrt{14}\right)\left(\frac{1}{2}\right) = 3\sqrt{14}.$

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14. Find the line integral of $f(x,y,z) = \dfrac{\sqrt{3}}{x^2+y^2+z^2}$ over the curve $\mathbf{r}(t) = t\mathbf{i}+t\mathbf{j}+t\mathbf{k}, 1 \le t \le \infty$
Solution:
$\mathbf{r}(t) = t\mathbf{i}+t\mathbf{j}+t\mathbf{k}, 1 \le t \le \infty$
$\implies \dfrac{d\mathbf{r}}{dt} = \mathbf{i}+\mathbf{j}+\mathbf{k} \implies \left|\dfrac{d\mathbf{r}}{dt}\right| = \sqrt{3};$
$\implies \dfrac{\sqrt{3}}{x^2+y^2+z^2} = \dfrac{\sqrt{3}}{t^2+t^2+t^2} = \dfrac{\sqrt{3}}{3t^2}$
$\implies \displaystyle\int_C f(x,y,z)\ ds = \int_0^\infty \left(\frac{\sqrt{3}}{3t^2}\right)\ \sqrt{3}\ dt = \left[-\frac{1}{t}\right]_0^\infty = \lim_{b\to\infty}\left(-\frac{1}{b}+1\right) = 1.$

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15. Integrate $f(x,y,z) = x + \sqrt{y} - z^2$ over the path from $(0,0,0)$ to $(1,1,1)$ given by

$$C_1 : \mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j}, 0 \le t \le 1$$
$$C_2 : \mathbf{r}(t) = \mathbf{i}+\mathbf{j}+t\mathbf{k}, 0 \le t \le 1$$

Solution:
C_1: $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j}, 0 \le t \le 1.$
$\implies \dfrac{d\mathbf{r}}{dt} = \mathbf{i} + 2t\mathbf{j} \implies \left|\dfrac{d\mathbf{r}}{dt}\right| = \sqrt{1+4t^2} = \sqrt{14};$
$\implies x + \sqrt{y} - z^2 = t + \sqrt{t^2} - 0 = t + |t| = 2t$ since $t \ge 0$
$\implies \displaystyle\int_{C_1} f(x,y,z)\ ds = \int_0^1 2t\sqrt{1+4t^2}\ dt = \left[\frac{1}{6}\left(1+4t^2\right)^{3/2}\right]_0^1 = \frac{1}{6}(5)^{3/2} - \frac{1}{6} = \frac{1}{6}\left(5\sqrt{5}-1\right).$
C_2: $\mathbf{r}(t) = \mathbf{i}+\mathbf{j}+t\mathbf{k}, 0 \le t \le 1.$
$\implies \dfrac{d\mathbf{r}}{dt} = \mathbf{k} \implies \left|\dfrac{d\mathbf{r}}{dt}\right| = 1;$
$\implies x + \sqrt{y} - z^2 = 1 + \sqrt{1} - t^2 = 2 - t^2$
$\implies \displaystyle\int_{C_2} f(x,y,z)\ ds = \int_0^1 \left(2-t^2\right)(1)\ dt = \left[2t - \frac{1}{3}t^3\right]_0^1 = 2 - \frac{1}{3} = \frac{5}{3};$

therefore

$$\int_C f(x,y,z)\ ds = \int_{C_1} f(x,y,z)\ ds + \int_{C_2} f(x,y,z)\ ds = \frac{5}{6}\sqrt{5} + \frac{3}{2}$$

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16. Integrate $f(x,y,z) = x + \sqrt{y} - z^2$ over the path from $(0,0,0)$ to $(1,1,1)$ given by

$$\begin{aligned} C_1 &: \mathbf{r}(t) = t\mathbf{k}, 0 \le t \le 1 \\ C_2 &: \mathbf{r}(t) = t\mathbf{j} + \mathbf{k}, 0 \le t \le 1 \\ C_3 &: \mathbf{r}(t) = t\mathbf{i} + \mathbf{j} + \mathbf{k}, 0 \le t \le 1 \end{aligned}$$

Solution:

C_1: $\mathbf{r}(t) = t\mathbf{k}, 0 \le t \le 1$

$\Longrightarrow \dfrac{d\mathbf{r}}{dt} = \mathbf{k} \Longrightarrow \left|\dfrac{d\mathbf{r}}{dt}\right| = 1;$

$\Longrightarrow x + \sqrt{y} - z^2 = 0 + \sqrt{0} - t^2 = -t^2$

$\Longrightarrow \displaystyle\int_{C_1} f(x,y,z)\ ds = \int_0^1 \left(-t^2\right)(1)\ dt = \left[-\frac{t^3}{3}\right]_0^1 = -\frac{1}{3};$

C_2: $\mathbf{r}(t) = t\mathbf{j} + \mathbf{k}, 0 \le t \le 1.$

$\Longrightarrow \dfrac{d\mathbf{r}}{dt} = \mathbf{j} \Longrightarrow \left|\dfrac{d\mathbf{r}}{dt}\right| = 1;$

$\Longrightarrow x + \sqrt{y} - z^2 = 0 + \sqrt{t} - 1 = \sqrt{t} - 1$

$\Longrightarrow \displaystyle\int_{C_2} f(x,y,z)\ ds = \int_0^1 \left(\sqrt{t} - 1\right)(1)\ dt = \left[\frac{2}{3}t^{3/2} - t\right]_0^1 = \frac{2}{3} - 1 = -\frac{1}{3};$

C_3: $\mathbf{r}(t) = t\mathbf{i} + \mathbf{j} + \mathbf{k}, 0 \le t \le 1.$

$\Longrightarrow \dfrac{d\mathbf{r}}{dt} = \mathbf{i} \Longrightarrow \left|\dfrac{d\mathbf{r}}{dt}\right| = 1;$

$\Longrightarrow x + \sqrt{y} - z^2 = t + \sqrt{1} - 1 = t + \sqrt{1} - 1 = t$

$\Longrightarrow \displaystyle\int_{C_3} f(x,y,z)\ ds = \int_0^1 (t)(1)\ dt = \left[\frac{t^2}{2}\right]_0^1 = \frac{1}{2};$

$\Longrightarrow \displaystyle\int_C f(x,y,z)\ ds = \int_{C_1} f\ ds + \int_{C_2} f\ ds + \int_{C_3} f\ ds = -\frac{1}{3} + \left(-\frac{1}{3}\right) + \frac{1}{2} = -\frac{1}{6}.$

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17. Integrate $f(x,y,z) = \dfrac{x+y+z}{x^2+y^2+z^2}$ over the path $\mathbf{r}(t) = t\mathbf{i} + t\mathbf{j} + t\mathbf{k}, 0 < a \le t \le b.$

Solution:

$\mathbf{r}(t) = t\mathbf{i} + t\mathbf{j} + t\mathbf{k}, 0 < a \le t \le b.$

$\Longrightarrow \dfrac{d\mathbf{r}}{dt} = \mathbf{i} + \mathbf{j} + \mathbf{k} \Longrightarrow \left|\dfrac{d\mathbf{r}}{dt}\right| = \sqrt{3};$

$\Longrightarrow \dfrac{x+y+z}{x^2+y^2+z^2} = \dfrac{t+t+t}{t^2+t^2+t^2} = \dfrac{1}{t}$

$\Longrightarrow \displaystyle\int_C f(x,y,z)\ ds = \int_a^b \left(\frac{1}{t}\right)\sqrt{3}\ dt = \left[\sqrt{3}\ln t\right]_a^b = \sqrt{3}\ln\left(\frac{b}{a}\right)$ since $0 < a \le b.$

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18. Integrate $f(x,y,z) = -\sqrt{x^2+z^2}$ over the circle $\mathbf{r}(t) = (a\cos t)\mathbf{j} + (a\sin t)\mathbf{k}, 0 \le t \le 2\pi.$

Solution:

$\mathbf{r}(t) = (a\cos t)\mathbf{j} + (a\sin t)\mathbf{k}, 0 \le t \le 2\pi.$

$\Longrightarrow \dfrac{d\mathbf{r}}{dt} = (-a\sin t)\mathbf{j} + (a\cos t)\mathbf{k} \Longrightarrow \left|\dfrac{d\mathbf{r}}{dt}\right| = \sqrt{a^2\sin^2 t + a^2\cos^2 t} = |a|;$

$$-\sqrt{x^2+z^2} = -\sqrt{0+a^2\sin^2 t} = \begin{cases} -|a|\sin t & 0 \le t \le \pi \\ |a|\sin t & \pi \le t \le 2\pi \end{cases}$$

$$\Longrightarrow \int_C f(x,y,z)\ ds = \int_0^{\pi} -|a|^2 \sin t\ dt + \int_{\pi}^{2\pi} |a|^2 \sin t\ dt = \left[a^2\cos t\right]_0^{\pi} - \left[a^2 \cos t\right]_{\pi}^{2\pi} = -4a^2.$$

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2. Line Integrals over Plane Curves

In Exercises 19-22, integrate f over the given region

19. $f(x,y) = \dfrac{x^3}{y} C : y = x^2/2, 0 \le x \le 2$

Solution:

$\mathbf{r}(x) = x\mathbf{j} + y\mathbf{j} = x\mathbf{j} + \dfrac{x^2}{2}\mathbf{j}, 0 \le x \le 2.$

$\Longrightarrow \dfrac{d\mathbf{r}}{dx} = \mathbf{i} + x\mathbf{j} \Longrightarrow \left|\dfrac{d\mathbf{r}}{dx}\right| = \sqrt{1+x^2};$

$f(x,y) = f\left(x, \dfrac{x^2}{2}\right) = \dfrac{x^3}{\frac{x^2}{2}} = 2x$

$\Longrightarrow \displaystyle\int_C f\ ds = \int_0^2 2x\sqrt{1+x^2}\ dx = \left[\frac{2}{3}\left(1+x^2\right)^{3/2}\right]_0^2 = \frac{10\sqrt{5}-2}{3}.$

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20. $f(x,y) = \left(x+y^2\right)/\sqrt{1+x^2}, C : y = x^2/2$ from $(1, 1/2)$ to $(0,0)$

Solution:

$\mathbf{r}(t) = (1-t)\,\mathbf{i} + \dfrac{1}{2}(1-t)^2\,\mathbf{j}, 0 \le t \le 1.$

$\Longrightarrow \left|\dfrac{d\mathbf{r}}{dt}\right| = \sqrt{1+(1-t)^2};$

$\Longrightarrow f(x,y) = f\left((1-t), \dfrac{1}{2}(1-t)^2\right) = \dfrac{(1-t)+\frac{1}{4}(1-t)^4}{\sqrt{1+(1-t)^2}}$

$\Longrightarrow \displaystyle\int_C f\ ds = \int_0^1 \frac{(1-t)+\frac{1}{4}(1-t)^4}{\sqrt{1+(1-t)^2}}\sqrt{1+(1-t)^2}\ dt = \int_0^1 \left((1-t)+\frac{1}{4}(1-t)^4\right)\ dt$

$= \left[-\dfrac{1}{2}(1-t)^2 - \dfrac{1}{20}(1-t)^5\right]_0^1 = \dfrac{11}{20}.$

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21. $f(x,y) = x+y\ C : x^2+y^2 = 4$ in the first quadrant from $(2,0)$ to $(0,2)$.

Solution:

$\mathbf{r}(t) = (2\cos t)\,\mathbf{i} + (2\sin t)\,\mathbf{j}, 0 \le t \le \dfrac{\pi}{2}.$

$\Longrightarrow \dfrac{d\mathbf{r}}{dx} = (-2\sin t)\,\mathbf{i} + (2\cos t)\,\mathbf{j}$

$\Longrightarrow \left|\dfrac{d\mathbf{r}}{dt}\right| = 2;$

$\Longrightarrow f(x,y) = f(2\cos t, 2\sin t) = 2\cos t + 2\sin t$

$$\implies \int_C f\ ds = \int_0^{\frac{\pi}{2}} (2\cos t + 2\sin t)(2)\ dt = [4\sin t - 4\cos t]_0^{\pi/2} = 4 - (-4) = 8.$$

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22. $f(x, y) = x^2 - y$, $C : x^2 + y^2 = 4$ in the first quadrant from $(0, 2)$ to $\left(\sqrt{2}, \sqrt{2}\right)$.

Solution:

$\mathbf{r}(t) = (2\sin t)\,\mathbf{i} + (2\cos t)\,\mathbf{j}, 0 \le t \le \dfrac{\pi}{4}.$

$\implies \dfrac{d\mathbf{r}}{dx} = (2\cos t)\,\mathbf{i} + (-2\sin t)\,\mathbf{j}$

$\implies \left|\dfrac{d\mathbf{r}}{dt}\right| = 2;$

$\implies f(x, y) = f(2\sin t, 2\cos t) = 4\sin^2 t - 2\cos t$

$$\implies \int_C f\ ds = \int_0^{\pi/4} \left(4\sin^2 t - 2\cos t\right)(2)\ dt = [4t - 2\sin 2t - 4\sin t]_0^{\pi/4} = \pi - 2\left(1 + \sqrt{2}\right).$$
