

ÇANKAYA UNIVERSITY
Department of Mathematics and Computer Science
MATH 156 Calculus for Engineering II
Practice Problems

2nd Midterm
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17:40

1. 15.3 DOUBLE INTEGRALS IN POLAR FORM

(p. 1097) In Exercises 1-16, change the cartesian integral into an equivalent polar integral. Then evaluate the polar integral.

1. $\int_{-1}^1 \int_0^{\sqrt{1-x^2}} dy \, dx$

Solution:

$$\int_{-1}^1 \int_0^{\sqrt{1-x^2}} dy \, dx = \int_0^{\pi} \int_0^1 r \, dr \, d\theta = \cdots = \frac{\pi}{2}.$$

2. $\int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dy \, dx$

Answer: $\int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dy \, dx = \int_0^{2\pi} \int_0^1 r \, dr \, d\theta = \cdots = \pi$

3. $\int_0^1 \int_0^{\sqrt{1-y^2}} (x^2 + y^2) \, dx \, dy$

Answer: $\int_0^1 \int_0^{\sqrt{1-y^2}} (x^2 + y^2) \, dx \, dy = \int_0^{\pi/2} \int_0^1 r^3 \, dr \, d\theta = \cdots = \frac{\pi}{8}.$

4. $\int_{-1}^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} (x^2 + y^2) \, dx \, dy$

Answer:

$$\int_{-1}^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} (x^2 + y^2) \, dx \, dy = \int_0^{2\pi} \int_0^1 r^3 \, dr \, d\theta = \cdots = \frac{\pi}{2}$$

5. $\int_{-a}^a \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dy \, dx$

Solution:

$$\int_{-a}^a \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dy \, dx = \int_0^{2\pi} \int_0^a r \, dr \, d\theta = \cdots = \pi a^2$$

6. $\int_0^6 \int_0^y x \, dx \, dy$

Answer: $\int_0^6 \int_0^y x \, dx \, dy = \frac{\pi}{4}$

7. $\int_0^6 \int_0^{\ln y} e^{x+y} \, dx \, dy$

Answer: $8 \ln 8 - 16 + e$.

8. $\int_1^2 \int_y^{y^2} dx \, dy$

Answer: $\frac{5}{6}$.

9. $\int_0^1 \int_0^{y^2} 3y^3 e^{xy} \, dx \, dy$

Answer: $e - 2$

10. $\int_1^4 \int_0^{\sqrt{x}} \frac{3}{2} e^{y/\sqrt{x}} \, dy \, dx$

Answer: $7(e - 1)$

In Exercises 11-16, integrate f over the given region.

11. Quadrilateral $f(x, y) = x/y$ over the region in the first quadrant bounded by the lines $y = x, y = 2x, x = 1, x = 2$

Solution:

$$\int_1^2 \int_x^{2x} \frac{x}{y} \, dy \, dx = \frac{3}{2} \ln 2.$$

12. Square $f(x, y) = \frac{1}{xy}$ over the square $1 \leq x \leq 2, 1 \leq y \leq 2$

Solution:

$$\int_1^2 \int_1^2 \frac{1}{xy} \, dy \, dx = (\ln 2)^2.$$

13. Triangle $f(x, y) = x^2 + y^2$ over the triangular region with vertices $(0, 0), (1, 0), (0, 1)$

Solution:

$$\int_0^1 \int_0^{1-x} (x^2 + y^2) \, dy dx = \frac{1}{6}.$$

14. Rectangle $f(x, y) = y \cos xy$ over the rectangle $0 \leq x \leq \pi, 0 \leq y \leq 1$.

Solution:

$$\int_0^1 \int_0^\pi y \cos xy \, dx dy = \frac{2}{\pi}.$$

15. Triangle $f(u, v) = v - \sqrt{u}$ over the triangular region cut from the first quadrant of the uv -plane by the line $u + v = 1$.

Solution:

$$\int_0^1 \int_0^{1-u} (v - \sqrt{u}) \, dv du = -\frac{1}{10}.$$

16. Curved region $f(s, t) = e^s \ln t$ over the region in the first quadrant of the st -plane that lies above the curve $s = \ln t$ from $t = 1$ to $t = 2$.

Solution:

$$\int_1^2 \int_0^{\ln t} e^s \ln t \, ds = \frac{1}{4}.$$

Each of the exercises 17-20 gives an integral over a region in a cartesian coordinate plane. Sketch the region and evaluate the integral.

17. $\int_{-2}^0 \int_v^{-v} 2 \, dp dv$ (the pv -plane).

Answer: 8.

18. $\int_0^1 \int_0^{\sqrt{1-x^2}} 8t \, dt ds$ (the st -plane).

Answer: $\frac{8}{3}$.

19. $\int_{-\pi/3}^{\pi/3} \int_0^{\sec t} 3 \cos t \, du dt$ (the tu -plane)

Answer: 2π .

20. $\int_1^3 \int_1^{4-2x} \frac{4-2u}{v^2} \, dv du$ (the uv -plane)

Answer: 0.

2. REVERSING THE ORDER OF INTEGRATION

In Exercises 21-30, sketch the region of integration and write an equivalent double integral with the order of integration reversed.

21. $\int_0^1 \int_2^{4-2x} dy \, dx$

Solution:

$$\int_0^1 \int_2^{4-2x} dy \, dx = \int_2^4 \int_0^{(4-y)/2} dx \, dy$$

22. $\int_0^2 \int_{y-2}^0 dx \, dy$

Solution:

$$\int_0^2 \int_{y-2}^0 dx \, dy = \int_{-2}^0 \int_0^{x+2} dy \, dx$$

23. $\int_0^1 \int_y^{\sqrt{y}} dx \, dy$

Solution:

$$\int_0^1 \int_y^{\sqrt{y}} dx \, dy = \int_0^1 \int_x^{x^2} dy \, dx$$

24. $\int_0^1 \int_{1-x}^{1-x^2} dy \, dx$

Solution:

$$\int_0^1 \int_{1-x}^{1-x^2} dy \, dx = \int_0^1 \int_{1-y}^{\sqrt{1-y}} dx \, dy$$

25. $\int_0^1 \int_1^{e^x} dy \, dx$

Solution:

$$\int_0^1 \int_1^{e^x} dy \, dx = \int_1^e \int_{\ln y}^1 dx \, dy$$

26. $\int_0^{\ln 2} \int_{e^x}^2 dy \, dx$

Solution:

$$\int_0^{\ln 2} \int_{e^x}^2 dx \, dy = \int_1^2 \int_0^{\ln x} dy \, dx$$

27. $\int_0^{3/2} \int_0^{9-4x^2} 16x \, dy \, dx$

Solution:

$$\int_0^{3/2} \int_0^{9-4x^2} 16x \, dy \, dx = \int_0^9 \int_0^{\frac{1}{2}\sqrt{9-y}} 16x \, dx \, dy.$$

28. $\int_0^2 \int_0^{4-y^2} y \, dx \, dy$

Solution:

$$\int_0^2 \int_0^{4-y^2} y \, dx \, dy = \int_0^4 \int_0^{\sqrt{4-x}} y \, dy \, dx.$$

29. $\int_0^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} 3y \, dx \, dy.$

Solution:

$$\int_0^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} 3y \, dx \, dy = \int_{-1}^1 \int_0^{\sqrt{1-x^2}} 3y \, dy \, dx.$$

30. $\int_0^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} 6x \, dy \, dx.$

Solution:

$$\int_0^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} 6x \, dy \, dx = \int_{-2}^2 \int_0^{\sqrt{4-y^2}} 6x \, dx \, dy.$$

3. EVALUATING DOUBLE INTEGRALS

In Exercises 31-40, sketch the region of integration, reverse the order of integration, and evaluate the integral.

31. $\int_0^\pi \int_x^\pi \frac{\sin y}{y} \, dy \, dx.$

Solution:

$$\int_0^\pi \int_x^\pi \frac{\sin y}{y} \, dy \, dx = \int_0^\pi \int_0^y \frac{\sin y}{y} \, dx \, dy = \int_0^\pi \sin y \, dy = 2.$$

32. $\int_0^2 \int_x^2 2y^2 \sin xy \, dy \, dx.$

Solution:

$$\begin{aligned} \int_0^2 \int_x^2 2y^2 \sin xy \, dy \, dx &= \int_0^2 \int_0^y 2y^2 \sin xy \, dx \, dy \\ &= \int_0^2 [-2y \cos xy]_0^y \, dy = \int_0^2 (-2y \cos y^2 + 2y) \, dy = [-\sin y^2 + y^2]_0^2 = 4 - \sin 4. \end{aligned}$$

33. $\int_0^1 \int_y^1 x^2 e^{xy} \, dx \, dy.$

Solution:

$$\int_0^1 \int_y^1 x^2 e^{xy} \, dx \, dy = \int_0^1 \int_0^x x^2 e^{xy} \, dy \, dx = \int_0^1 [xe^{xy}]_0^x \, dx$$

$$= \int_0^1 (xe^{x^2} - x) \, dx = \left[\frac{1}{2}e^{x^2} - \frac{x^2}{2} \right]_0^1 = \frac{e-2}{2}.$$

34. $\int_0^2 \int_0^{4-x^2} \frac{xe^{2y}}{4-y} \, dy \, dx.$

Solution:

$$\begin{aligned} \int_0^2 \int_0^{4-x^2} \frac{xe^{2y}}{4-y} \, dy \, dx &= \int_0^4 \int_0^{\sqrt{4-y}} \frac{xe^{2y}}{4-y} \, dx \, dy \\ &= \int_0^4 \left[\frac{x^2 e^{2y}}{2(4-y)} \right]_0^{\sqrt{4-y}} dy = \int_0^4 \frac{e^{2y}}{2} \, dy = \left[\frac{e^{2y}}{4} \right]_0^4 = \frac{e^8-1}{4}. \end{aligned}$$

35. $\int_0^{2\sqrt{\ln 3}} \int_{y/2}^{\sqrt{\ln 3}} e^{x^2} \, dx \, dy$

Solution:

$$\begin{aligned} \int_0^{2\sqrt{\ln 3}} \int_{y/2}^{\sqrt{\ln 3}} e^{x^2} \, dx \, dy &= \int_0^{\sqrt{\ln 3}} \int_0^{2x} e^{x^2} \, dy \, dx \\ &= \int_0^{\sqrt{\ln 3}} 2xe^{x^2} \, dx = \left[e^{x^2} \right]_0^{\sqrt{\ln 3}} = e^{\ln 3} - 1 = 2. \end{aligned}$$

36. $\int_0^3 \int_{\sqrt{x/3}}^1 e^{y^3} \, dy \, dx$

Solution:

$$\begin{aligned} \int_0^3 \int_{\sqrt{x/3}}^1 e^{y^3} \, dy \, dx &= \int_0^3 \int_0^{3y^2} e^{y^3} \, dx \, dy \\ &= \int_0^1 3y^2 e^{y^3} \, dy = \left[e^{y^3} \right]_0^1 = e - 1 \end{aligned}$$

37. $\int_0^{1/16} \int_{y^{1/4}}^{1/2} \cos(16\pi x^5) \, dx \, dy$

Solution:

$$\begin{aligned} \int_0^{1/16} \int_{y^{1/4}}^{1/2} \cos(16\pi x^5) \, dx \, dy &= \int_0^{1/2} \int_0^{x^4} \cos(16\pi x^5) \, dy \, dx \\ &= \int_0^{1/2} x^4 \cos(16\pi x^5) \, dx = \left[\frac{\sin(16\pi x^5)}{80\pi} \right]_0^{1/2} = \frac{1}{80\pi}. \end{aligned}$$

38. $\int_0^8 \int_{\sqrt[3]{x}}^2 \frac{dy \, dx}{y^4+1}$

Solution:

$$\begin{aligned} \int_0^8 \int_{\sqrt[3]{x}}^2 \frac{dy \, dx}{y^4+1} &= \int_0^2 \int_0^{y^3} \frac{1}{y^4+1} \, dx \, dy \\ &= \int_0^8 \frac{y^3}{y^4+1} \, dy = \frac{1}{4} [\ln(y^4+1)]_0^2 = \frac{\ln 17}{4}. \end{aligned}$$

39. Square region $\int \int_R (y - 2x^2) \, dA$ where R is the region bounded by the square $|x| + |y| = 1$.

Solution:

$$\begin{aligned} & \int \int_R (y - 2x^2) \, dA \\ &= \int_{-1}^0 \int_{-x-1}^{x+1} (y - 2x^2) \, dy \, dx + \int_0^1 \int_{x-1}^{1-x} (y - 2x^2) \, dy \, dx \\ &= \int_{-1}^0 \left[\frac{1}{2}y^2 - 2x^2y \right]_{-x-1}^{x+1} dx + \int_0^1 \left[\frac{1}{2}y^2 - 2x^2y \right]_{x-1}^{1-x} dx \\ &= \int_{-1}^0 \left[\frac{1}{2}(x+1)^2 - 2x^2(x+1) - \frac{1}{2}(-x-1)^2 + 2x^2(-x-1) \right] dx \\ &+ \int_{-1}^0 \left[\frac{1}{2}(1-x)^2 - 2x^2(1-x) - \frac{1}{2}(x-1)^2 + 2x^2(x-1) \right] dx \\ &= \frac{1}{2}. \end{aligned}$$

40. Triangular region $\int \int_R xy \, dA$ where R is the region bounded by the lines $y = x, y = 2x$, and $x + y = 2$

Solution:

$$\begin{aligned} & \int \int_R xy \, dA = \int_0^{2/3} \int_x^{2x} xy \, dy \, dx + \int_{2/3}^1 \int_x^{2-x} xy \, dy \, dx \\ &= \int_0^{2/3} \left[\frac{1}{2}xy^2 \right]_x^{2x} dx + \int_{2/3}^1 \left[\frac{1}{2}xy^2 \right]_x^{2-x} dx \\ &= \frac{13}{81}. \end{aligned}$$

4. VOLUME BENEATH A SURFACE $z = f(x, y)$

41. Find the volume of the region bounded by the paraboloid $z = x^2 + y^2$ and below by the triangle enclosed by the lines $y = x, x = 0$, and $x + y = 2$ in the plane.

Solution:

$$V = \int_0^1 \int_x^{2-x} (x^2 + y^2) \, dy \, dx = \cdots = \frac{4}{3}.$$

42. Find the volume of the solid that is bounded above by the cylinder $z = x^2$ and below by the region enclosed by the parabola $y = 2 - x^2$ and the line $y = x$ in the xy -plane.

Solution:

$$V = \int_{-2}^1 \int_x^{2-x^2} x^2 \, dy \, dx = \cdots = \frac{63}{20}.$$

43. Find the volume of the solid whose base is the region in the xy -plane that is bounded by the parabola $y = 4 - x^2$ and the line $y = 3x$, while the top of the solid is bounded by the plane $z = x + 4$.

Solution:

$$V = \int_{-4}^1 \int_{3x}^{4-x^2} (x + 4) \, dy \, dx = \cdots = \frac{625}{12}.$$
