ÇANKAYA UNIVERSITY
Department of Mathematics and Computer Science

MATH 156 Calculus for Engineering II

1st Midterm
SOLUTIONS
March 20, 2008
17:40-19:30

Surname : ____________________

Name : ____________________

ID # : ____________________

Department : ____________________

Section : ____________________

Instructor : ____________________

Signature : ____________________

- The exam consists of 5 questions.
- Please read the questions carefully and write your answers under the corresponding questions. Be neat.
- Show all your work. Correct answers without sufficient explanation might not get full credit.
- Calculators are not allowed.

GOOD LUCK!

Please do not write below this line.

Q1	Q2	Q3	Q4	Q5	TOTAL
18	28	18	16	20	100

1. Calculate (if possible) the sum of each of the following series:

a) $\sum_{n=0}^{\infty} \frac{n+1}{e^n}$, b) $\sum_{n=0}^{\infty} \frac{1}{(n+4)(n+5)}$

Solution:

a) We know by differentiating the geometric series that

$$\sum_{n=1}^{\infty} n x^{n-1} x = \frac{1}{(1-x)^2} \text{ for } -1 < x < 1.$$

Then

$$\sum_{n=0}^{\infty} \frac{n+1}{e^n} = \sum_{k=1}^{\infty} k \left(\frac{1}{e}\right)^{k-1} = \frac{1}{\left(1-\frac{1}{e}\right)^2} = \frac{e^2}{(e-1)^2}.$$

b) $\frac{1}{(n+4)(n+5)} = \frac{1}{n+4} - \frac{1}{n+5}$

$$\Longrightarrow s_n = \left[\frac{1}{4} - \frac{1}{5}\right] + \left[\frac{1}{5} - \frac{1}{6}\right] + \cdots + \left[\frac{1}{n+4} - \frac{1}{n+5}\right]$$

$$\Longrightarrow s_n = \frac{1}{4} - \frac{1}{n+5} \Longrightarrow \lim_{n\to\infty} s_n = \lim_{n\to\infty} \left[\frac{1}{4} - \frac{1}{n+5}\right] = \frac{1}{4}$$

Thus, we get

$$\sum_{n=0}^{\infty} \frac{1}{(n+4)(n+5)} = \frac{1}{4}$$

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2. In each part, determine whether the series is convergent or divergent. Show your work and name the test used.

a) $\displaystyle\sum_{n=0}^{\infty}\left(\frac{n+2}{2n+1}\right)^n$, b) $\displaystyle\sum_{n=1}^{\infty}\frac{e^{1/n}}{n^2}$, c) $\displaystyle\sum_{n=1}^{\infty}\frac{n^n}{4^n n!}$, d) $\displaystyle\sum_{n=1}^{\infty}\frac{n+3}{\sqrt[3]{8n^4-2}}$

Solution:

a) converges by the Root Test:

$$\lim_{n\to\infty}\sqrt[n]{a_n}=\lim_{n\to\infty}\sqrt[n]{\left(\frac{n+2}{2n+1}\right)^n}=\lim_{n\to\infty}\frac{n+2}{2n+1}=\frac{1}{2}<1$$

b) $\displaystyle\sum_{n=1}^{\infty}\frac{e^{1/n}}{n^2}$ converges by the Limit Comparison Test when compared with $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}$, a convergent p-series:

$$\lim_{n\to\infty}\frac{a_n}{b_n}=\lim_{n\to\infty}\frac{\frac{e^{1/n}}{n^2}}{\frac{1}{n^2}}=\lim_{n\to\infty}e^{1/n}=1$$

c) $\displaystyle\sum_{n=1}^{\infty}\frac{n^n}{4^n n!}$ converges by the Ratio Test:

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\frac{(n+)^{n+1}}{4^{n+1}(n+1)!}\cdot\frac{4^n n!}{n^n}=\lim_{n\to\infty}\frac{1}{4}\left(\frac{n+1}{n}\right)^n=\frac{e}{4}<1.$$

d) $\displaystyle\sum_{n=1}^{\infty}\frac{n+3}{\sqrt[3]{8n^4-2}}$ diverges by the Limit Comparison Test with $\dfrac{1}{n^{1/3}}$, the nth term of a divergent p-series:

$$\lim_{n\to\infty}\frac{a_n}{b_n}=\lim_{n\to\infty}\frac{\frac{n+3}{\sqrt[3]{8n^4-2}}}{\frac{1}{n^{1/3}}}=\lim_{n\to\infty}\frac{n^{4/3}+3n^{1/3}}{\sqrt[3]{8n^4-2}}=\frac{1}{2}$$

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3. In each part, determine whether the series is absolutely convergent, conditionally convergent, or divergent. Show your work and name the test used.

a) $\sum_{n=2}^{\infty} (-1)^n \frac{1}{\sqrt[5]{n^3} - 2}$, b) $\sum_{n=2}^{\infty} (-1)^n \frac{1}{n (\ln n)^{10}}$

Solution:

a) $\sum_{n=2}^{\infty} (-1)^n \frac{1}{\sqrt[5]{n^3} - 2}$ converges conditionally since

$$u_n = \frac{1}{\sqrt[5]{n^3} - 2} > u_{n+1} = \frac{1}{\sqrt[5]{(n+1)^3} - 2} > 0, \forall n \geq 4, \text{ so } u_n \text{ is decreasing and}$$

$$\lim_{n \to \infty} u_n = \lim_{n \to \infty} \frac{1}{\sqrt[5]{n^3} - 2} = 0 \Longrightarrow \text{convergence by the Alternating Series Test; but}$$

$$\sum_{n=2}^{\infty} |a_n| = \sum_{n=2}^{\infty} \frac{1}{\sqrt[5]{n^3} - 2} \text{ diverges by the Direct Comparison Test, because}$$

$$\frac{1}{\sqrt[5]{n^3} - 2} > \frac{1}{\sqrt[5]{n^3}} = \frac{1}{n^{3/5}} \text{ and } \sum_{n=2}^{\infty} \frac{1}{n^{3/5}} \text{ is a divergent } p\text{-series.}$$

b) $\sum_{n=2}^{\infty} (-1)^n \frac{1}{n (\ln n)^{10}}$ converges absolutely by the Integral Test,

$$f(x) = \frac{1}{x (\ln x)^{10}} \text{ is continuous and positive on } [2, \infty), \text{ and}$$

$$f'(x) = -\frac{10 + \ln x}{x^2 (\ln x)^{11}} < 0 \text{ if } x > e^{-10}, \text{ so that}$$

f is eventually decreasing and we can use the Integral Test.

$$\int_2^{\infty} \frac{1}{x (\ln x)^{10}} dx = \lim_{b \to \infty} \left[\frac{(\ln x)^{1-10}}{1 - 10} \right]_2^b = \lim_{b \to \infty} \left[\frac{(\ln b)^{-9}}{-9} \right] - \frac{(\ln 2)^{-9}}{-9} = \frac{1}{9 (\ln 2)^9}$$

Thus the improper integral converges and has value $\frac{1}{9 (\ln 2)^9}$. Therefore

$$\sum_{n=2}^{\infty} \frac{1}{n (\ln n)^{10}} \text{ converges by the Integral Test and hence the given series converges absolutely.}$$

4. Find the radius and the interval of convergence of the power series $\displaystyle\sum_{n=0}^{\infty} \frac{(2x+3)^n}{\sqrt{n+1}}$.

Solution:

If $a_n = \dfrac{(2x+3)^n}{\sqrt{n+1}}$, then

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(2x+3)^{n+1}}{\sqrt{(n+1)+1}} \cdot \frac{\sqrt{n+1}}{(2x+3)^n} \right| = \lim_{n\to\infty} \left(|2x+3| \frac{\sqrt{n+1}}{\sqrt{n+2}} \right) = |2x+3|,$$

so by the Ratio Test the series converges when

$|2x+3| < 1 \iff -1 < 2x+3 < 1 \iff -2 < x < -1.$

So $R = \dfrac{1}{2}$.

When $x = -2$, the series is $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}$, a conditionally convergent series by the Alternating series Test.

When $x = -1$, we have the series $\displaystyle\sum_{n=0}^{\infty} \frac{1}{\sqrt{n+1}}$, a divergent series by the Limit Comparison Test.

(a) the radius is $R = \dfrac{1}{2}$; the interval of convergence is $-2 \le x < -1$, i.e., $I = [-2, -1)$

(b) the interval of absolute convergence is $-2 < x < -1$

(c) the series converges conditionally at $x = -2$

5.

a) Estimate $\int_0^{1/2} \frac{t^2}{1+t^4} dt$ correct to an error less than 10^{-4}.

b) Use series to evaluate $\lim_{x\to 0} \frac{(x-\tan^{-1}x)(e^{2x}-1)}{2x^2-1+\cos(2x)}$

Solution:

a) $$\int_0^{1/2} \frac{t^2}{1+t^4} dt = \int_0^{1/2} t^2 \frac{1}{1-(-t^4)} dt$$

$$= \int_0^{1/2} \left(t^2 - t^6 + t^{10} - t^{14} + \cdots\right) dt = \left[\frac{t^3}{3} - \frac{t^7}{7} + \frac{t^{11}}{11} - \frac{t^{15}}{15} + \cdots\right]_0^{1/2} = \left[\sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+3}}{4n+3}\right]_0^{1/2}$$

$$= \left[\frac{1}{2^3\cdot 3} - \frac{1}{2^7\cdot 7} + \frac{1}{2^{11}\cdot 11} - \cdots\right] \approx \frac{1}{24} - \frac{1}{896} \pm \frac{1}{2048\cdot 11} = \frac{109}{2688} \pm \frac{1}{22528}$$

$$|\text{error}| < \frac{1}{2^{11}\cdot 11} \approx 0.4438\times 10^{-4}.$$

b) $$\lim_{x\to 0} \frac{(x-\tan^{-1}x)(e^{2x}-1)}{2x^2-1+\cos(2x)}$$

$$\frac{(x-\tan^{-1}x)(e^{2x}-1)}{2x^2-1+\cos(2x)} = \frac{\left(x-\left(x-\frac{x^3}{3}+\frac{x^5}{5}-\cdots\right)\right)\left(1+2x+\frac{4x^2}{2'!}+\frac{8x^3}{3!}+\cdots-1\right)}{2x^2-1+\left(1-\frac{4x^2}{2!}+\frac{2^4x^4}{4!}-\cdots\right)}$$

$$= \frac{\left(\frac{x^3}{3}-\frac{x^5}{5}+\cdots\right)\left(2x+\frac{4x^2}{2'!}+\frac{8x^3}{3!}+\cdots\right)}{\frac{2^4x^4}{4!}-\cdots}$$

$$= \frac{\left(\frac{1}{3}-\frac{x^2}{5}+\cdots\right)\left(2+\frac{4x}{2'!}+\frac{8x^2}{3!}+\cdots\right)}{\frac{2^4}{4!}-\frac{2^6x^2}{6!}+\cdots}$$

Thus, we get

$$\lim_{x\to 0} \frac{(x-\tan^{-1}x)(e^{2x}-1)}{2x^2-1+\cos(2x)} = \lim_{x\to 0} \frac{\left(\frac{1}{3}-\frac{x^2}{5}+\cdots\right)\left(2+\frac{4x}{2'!}+\frac{8x^2}{3!}+\cdots\right)}{\frac{2^4}{4!}-\frac{2^6x^2}{6!}+\cdots} = \frac{\frac{2}{3}}{\frac{16}{24}} = 1$$
