ÇANKAYA UNIVERSITY
Department of Mathematics and Computer Science

MATH 156 Calculus for Engineering II

2^{nd} Midterm
SOLUTIONS
April 24, 2008
17:40-19:30

Surname : ______________________
Name : ______________________
ID # : ______________________
Department : ______________________
Section : ______________________
Instructor : ______________________
Signature : ______________________

- The exam consists of 5 questions.
- Please read the questions carefully and write your answers under the corresponding questions. Be neat.
- Show all your work. Correct answers without sufficient explanation might not get full credit.
- Calculators are not allowed.

GOOD LUCK!

Please do not write below this line.

Q1	Q2	Q3	Q4	Q5	TOTAL
20	20	20	20	20	100

1. Let $\mathbf{u} = 2\mathbf{i} + 2\mathbf{j} + \mathbf{k}, \mathbf{v} = 3\mathbf{i} - 3\mathbf{j} + \sqrt{7}\mathbf{k}$.
a) (5 pts.) Find $\cos\theta$ where θ is the angle between u and v.
b) (5 pts.) Find a vector $\mathbf{w}$ such that $\mathbf{w} \cdot \mathbf{u} = 0$ and $\mathbf{w} \cdot \mathbf{v} = 0$.
c) (5 pts.) Find the parametric equations for the line L through $P(1,1,1)$ and parallel to $\mathbf{v}$.
d) (5 pts.) Find the distance between the point $S(2,2,0)$ and the line L above.
Solution:
(a)

$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}| \cdot |\mathbf{v}|} = \frac{(2)(3) + (2)(-3) + (1)\left(\sqrt{7}\right)}{\sqrt{(2)^2 + (2)^2 + (1)^2} \cdot \sqrt{(3)^2 + (-3)^2 + \left(\sqrt{7}\right)^2}} = \frac{6 - 6 + \sqrt{7}}{3 \cdot 5} = \frac{\sqrt{7}}{15}$$

(b)

$$\mathbf{w} = \mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 2 & 1 \\ 3 & -3 & \sqrt{7} \end{vmatrix} = \begin{vmatrix} 2 & 1 \\ -3 & \sqrt{7} \end{vmatrix} \mathbf{i} - \begin{vmatrix} 2 & 1 \\ 3 & \sqrt{7} \end{vmatrix} \mathbf{j} + \begin{vmatrix} 2 & 2 \\ 3 & -3 \end{vmatrix} \mathbf{k}$$

$$= \left(2\sqrt{7} + 3\right)\mathbf{i} - \left(2\sqrt{7} - 3\right)\mathbf{j} - 12\mathbf{k}$$

$$(c) L : \begin{cases} x = 1 + 3t \\ y = 1 - 3t \\ z = 1 + \sqrt{7}t \end{cases} \quad -\infty < t < \infty$$

(d) $S(2,2,0), P(1,1,1)$ and $\mathbf{v} = 3\mathbf{i} - 3\mathbf{j} + \sqrt{7}\mathbf{k} \Longrightarrow \overrightarrow{PS} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & -1 \\ 3 & -3 & \sqrt{7} \end{vmatrix}$

$$= \left(\sqrt{7} - 3\right)\mathbf{i} + \left(-\sqrt{7} - 3\right)\mathbf{j} + (-3 - 3)\mathbf{k}$$

$$= \left(\sqrt{7} - 3\right)\mathbf{i} + \left(-3 - \sqrt{7}\right)\mathbf{j} + 6\mathbf{k}$$

$$\Longrightarrow d = \frac{\left|\overrightarrow{PS} \times \mathbf{v}\right|}{|\mathbf{v}|} = \frac{\sqrt{\left(\sqrt{7} - 3\right)^2 + \left(-3 - \sqrt{7}\right)^2 + (6)^2}}{\sqrt{(3)^2 + (-3)^2 + \left(\sqrt{7}\right)^2}}$$

$$= \frac{\sqrt{7 - 6\sqrt{7} + 9 + 9 + +6\sqrt{7} + 7 + 36}}{5} = \frac{\sqrt{68}}{5}$$

2.

a) (10 pts.) Find the equation of the plane $\mathcal{P}$ passing through $P(1,1,-1), Q(2,0,2)$, and $R(0,-2,1)$.

b) (10 pts.) Find the parametric equations for the line of intersection of the above plane $\mathcal{P}$ and the plane $3x-6y-2z=10$.

Solution:

(a)

$\overrightarrow{PQ}=\mathbf{i}-\mathbf{j}+3\mathbf{k}, \overrightarrow{PR}=-\mathbf{i}-3\mathbf{j}+2\mathbf{k} \Longrightarrow \overrightarrow{PQ}\times\overrightarrow{PR}=\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 3 \\ -1 & -3 & 2 \end{vmatrix}=7\mathbf{i}-5\mathbf{j}-4\mathbf{k}$ is normal to the plane

$$\Longrightarrow (7)(x-1)+(-5)(y-1)+(-4)(z+1)=0 \Longrightarrow 7x-5y-4z=6.$$

(b)

The line of intersection is parallel to $\mathbf{v}=\mathbf{n}_1\times\mathbf{n}_2=\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 7 & -5 & -4 \\ 3 & -6 & -2 \end{vmatrix}=-14\mathbf{i}+2\mathbf{j}-27\mathbf{k}.$

Now to find a point.in the intersection, solve

$$\left\{\begin{array}{c} 7x-5y-4z=6 \\ 3x-6y-2z=10 \end{array}\right. \Longrightarrow \left\{\begin{array}{c} 7x-5y-4z=6 \\ -6x+12y+4z=-20 \end{array}\right. \Longrightarrow x+7y=-14 \Longrightarrow x=0 \text{ and}$$

$y=-2 \Longrightarrow (0,-2,1)$

is a point on the line we seek. Therefore the line $x=-14t, y=-2+2t, z=1-27t$

3.

a) (10 pts.) Given $z = e^{3xy^2} + 4x^3 - y^3 \ln x$. Find $\dfrac{\partial^2 z}{\partial y \partial x}$.

b) (10 pts.) Discuss the continuity of the following function at the origin.

$$f(x,y) = \begin{cases} \dfrac{-3xy}{x^2+y^2} & \text{if} \quad (x,y) \neq (0,0) \\ 0 & \text{if} \quad (x,y) = (0,0) \end{cases}$$

Solution:

(a)

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)$$

$$= \frac{\partial}{\partial y}\left(3y^2 e^{3xy^2} + 12x^2 - y^3 \frac{1}{x}\right)$$

$$= 6ye^{3xy^2} + 3y^2 6xye^{3xy^2} - \frac{3y^2}{x}$$

$$= 6ye^{3xy^2}\left(1 + 3xy^2\right) - \frac{3y^2}{x}$$

(b)

$\displaystyle \lim_{(x,y)\to(0,0)} \frac{-3xy}{x^2+y^2} = \lim_{(x,kx)\to(0,0)} \frac{-3x(kx)}{x^2+(kx)^2} = \lim_{x\to 0} \frac{-3kx^2}{x^2(1+k^2)} = \frac{-3k}{1+k^2}$ different limits for different values of

$k \Longrightarrow$ the limit does not exist at the origin and so f is not continuous at the origin.

---

4. Suppose $F(x, y, z) = x^2y + y^2z + \cos(xz)$.
a) (7 pts.) Find the directional derivative of F at the point $P(0, 2, 1)$ in the direction of $\mathbf{v} = \mathbf{i} - \mathbf{j} + 2\mathbf{k}$.
b) (7 pts.) For the level surface $F(x, y, z) = 5$, find the equation of the tangent plane at the point $P(0, 2, 1)$.
c) (6 pts.) On the level surface $F(x, y, z) = 5$, find $\dfrac{\partial z}{\partial y}$.

Solution:
(a)

$$\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{\mathbf{i} - \mathbf{j} + 2\mathbf{k}}{\sqrt{6}}$$

$$\nabla F = (2xy - z\sin(xz))\,\mathbf{i} + \left(x^2 + 2yz\right)\mathbf{j} + \left(y^2 - x\sin(xz)\right)\mathbf{k} \Longrightarrow \nabla F(0, 2, 1) = 4\mathbf{j} + 4\mathbf{k}$$

$$D_{\mathbf{u}}F(0, 2, 1) = \nabla F(0, 2, 1) \cdot \mathbf{u} = (4\mathbf{j} + 4\mathbf{k}) \cdot \left(\frac{1}{\sqrt{6}}\mathbf{i} - \frac{1}{\sqrt{6}}\mathbf{j} + \frac{2}{\sqrt{6}}\mathbf{k}\right) = -\frac{4}{\sqrt{6}} + \frac{8}{\sqrt{6}} = \frac{4}{\sqrt{6}}$$

(b)
We know that $\nabla F(0, 2, 1) = 4\mathbf{j} + 4\mathbf{k}$ is normal to the surface at the point $P(0, 2, 1)$. Thus the equation of the required tangent plane is

$$(4\mathbf{j} + 4\mathbf{k}) \cdot ((x - 0)\,\mathbf{i} + (y - 2)\,\mathbf{j} + (z - 1)\,\mathbf{k}) = 0 \Longrightarrow 4y + 4z = 12 \Longrightarrow y + z = 3.$$

(c)

$$F(x, y, z) = x^2y + y^2z + \cos(xz) = 5 \Longrightarrow \frac{\partial}{\partial y}\left(x^2y + y^2z + \cos(xz)\right) = \frac{\partial}{\partial y}(5)$$

$$x^2 + 2yz + y^2z_y - xz_y\sin(xz) = 0$$

$$\left(y^2 - x\sin(xz)\right)z_y = -x^2 - 2yz$$

$$z_y = \frac{x^2 + 2yz}{x\sin(xz) - y^2}$$

---

5. Given $f(x,y) = 1 + 4x + 4y - x^2 - y^2$
a) (8 pts.) Find the local extrema and saddle points (if exist) for f.
b) (12 pts.) Find the absolute maximum and minimum of f on the triangular region in the first quadrant bounded by the lines $x = 0, y = 0$ and $y = 1 - x$

Solution:

(a)

$f_x(x,y) = 4 - 2x = 0$ and $f_y(x,y) = 4 - 2y = 0$

$\Longrightarrow x = 2$ and $y = 2 \Longrightarrow (2,2)$ is the critical point;

$f_{xx}(2,2) = -2,\ f_{yy}(2,2) = -2,\ \ f_{xy}(2,2) = 0$

$\Longrightarrow f_{xx}f_{yy} - f_{xy}^2 = 4 > 0$ and $f_{xx} < 0 \Longrightarrow$ local maximum value of $f(2,2) = 9$.

(b) $f(x,y) = 1 + 4x + 4y - x^2 - y^2$

Let O$(0,0)$,A$(0,1)$,B$(1,0)$.

(i) On OA, $f(x,y) = f(0,y) = -y^2 + 4y + 1$ for $0 \le y \le 1$

$\Longrightarrow f'(0,y) = -2y + 4 = 0 \Longrightarrow y = 2.$

But $(0,2)$ is not in the interior of OA.

Endpoints: $f(0,0) = 1$ and $f(0,1) = 4$.

(ii) On AB, $f(x,y) = f(x,1-x) = -2x^2 + 2x + 4$

for $0 \le x \le 1 \Longrightarrow f'(x,2) = -4x + 2 = 0$

$\Longrightarrow x = \dfrac{1}{2}, y = \dfrac{1}{2}.$

$\left(\dfrac{1}{2}, \dfrac{1}{2}\right)$ is an interior critical point of AB with $f\left(\dfrac{1}{2}, \dfrac{1}{2}\right) = \dfrac{9}{2}.$

Endpoints: $f(1,0) = 4$ and $f(0,1) = 4$.

(iv) On OB, $f(x,y) = f(x,0) = -x^2 + 4x + 1$ for $0 \le x \le 1 \Longrightarrow f'(x,0) = -2x + 4 = 0 \Longrightarrow$ $x = 2$ and

$y = 0 \Longrightarrow (2,0)$ is not in the interior of OB.

Endpoints: $f(0,0) = 1$ and $f(1,0) = 4$.

(v) For the interior of the rectangular region, $f_x(x,y) = 4 - 2x = 0$ and $f_y(x,y) = 4 - 2y = 0 \Longrightarrow x = 2$ and

$y = 2$ so $(2,2)$ is not in the interior of R.

Thus the absolute maximum is $\dfrac{9}{2}$ at $\left(\dfrac{1}{2}, \dfrac{1}{2}\right)$ and the absolute minimum is 1 at $(0,0)$.