ÇANKAYA UNIVERSITY

Department of Mathematics and Computer Science

MATH 156 Calculus for Engineering II

Practice Problems

1^{st} Midterm

March 20, 2008

17:40

1. Convergent Series

(p.840) Find the sum of the series in Exercises 19-24.

19. $\displaystyle\sum_{n=3}^{\infty} \frac{1}{(2n-3)(2n-1)}$

Solution:

$$\frac{1}{(2n-3)(2n-1)} = \frac{\left(\frac{1}{2}\right)}{2n-3} - \frac{\left(\frac{1}{2}\right)}{(2n-1)}$$

$$\Longrightarrow s_n = \left[\frac{\left(\frac{1}{2}\right)}{3} - \frac{\left(\frac{1}{2}\right)}{5}\right] + \left[\frac{\left(\frac{1}{2}\right)}{5} - \frac{\left(\frac{1}{2}\right)}{7}\right] + \cdots + \left[\frac{\left(\frac{1}{2}\right)}{2n-3} - \frac{\left(\frac{1}{2}\right)}{2n-1}\right]$$

$$= \frac{\left(\frac{1}{2}\right)}{3} - \frac{\left(\frac{1}{2}\right)}{2n-1} \Longrightarrow \lim_{n\to\infty} s_n = \lim_{n\to\infty} \left[\frac{1}{6} - \frac{\left(\frac{1}{2}\right)}{2n-1}\right] = \frac{1}{6}$$

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20. $\displaystyle\sum_{n=2}^{\infty} \frac{-2}{n(n+1)}$

Solution:

$$\frac{-2}{n(n+1)} = \frac{-2}{n} + \frac{2}{n+1}$$

$$\Longrightarrow s_n = \left(\frac{-2}{2} + \frac{2}{3}\right) + \left(\frac{-2}{3} + \frac{2}{4}\right) + \cdots + \left(\frac{-2}{n} + \frac{2}{n+1}\right) = -\frac{2}{2} + \frac{2}{n+1}$$

$$\Longrightarrow \lim_{n\to\infty} s_n = \lim_{n\to\infty} \left(-1 + \frac{2}{n+1}\right) = -1$$

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21. $\displaystyle\sum_{n=1}^{\infty} \frac{9}{(3n-1)(3n+21)}$

Solution:

$$\frac{9}{(3n-1)(3n+21)} = \frac{3}{3n-1} - \frac{3}{3n+2}$$

$$\Longrightarrow s_n = \left(\frac{3}{2} - \frac{3}{5}\right) + \left(\frac{3}{5} - \frac{3}{8}\right) + \left(\frac{3}{8} - \frac{3}{11}\right) + \cdots + \left(\frac{3}{3n-1} - \frac{3}{3n+2}\right)$$

$= \dfrac{3}{2} - \dfrac{3}{3n+2} \Longrightarrow \lim_{n\to\infty} s_n = \lim_{n\to\infty}\left(\dfrac{3}{2} - \dfrac{3}{3n+2}\right) = \dfrac{3}{2}$

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22. $\displaystyle\sum_{n=3}^{\infty} \frac{-8}{(4n-3)(4n+1)}$

Solution:

$\dfrac{-8}{(4n-3)(4n+1)} = \dfrac{-2}{4n-3} + \dfrac{2}{4n+1}$

$\Longrightarrow s_n = \left(\dfrac{-2}{9} + \dfrac{2}{13}\right) + \left(\dfrac{-2}{13} + \dfrac{2}{17}\right) + \left(\dfrac{-2}{17} + \dfrac{2}{21}\right) + \cdots + \left(\dfrac{-2}{4n-3} + \dfrac{2}{4n+1}\right)$

$s_n = -\dfrac{2}{9} + \dfrac{2}{4n+1} \Longrightarrow \lim_{n\to\infty} s_n = \lim_{n\to\infty}\left(-\dfrac{2}{9} + \dfrac{2}{4n+1}\right) = -\dfrac{2}{9}$

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23. $\displaystyle\sum_{n=0}^{\infty} e^{-n}$

Solution:

$\displaystyle\sum_{n=0}^{\infty} e^{-n} = \sum_{n=0}^{\infty} \frac{1}{e^n}$, a convergent geometric series with $r = \dfrac{1}{e}$ and $a = 1 \Longrightarrow$ the sum is $\dfrac{1}{1-\frac{1}{e}} = \dfrac{e}{e-1}$.

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24. $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{3}{4^n}$

Solution:

$\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{3}{4^n} = \sum_{n=1}^{\infty} \left(-\frac{3}{4}\right)\left(-\frac{1}{4}\right)^n$ a convergent geometric series with $r = -\dfrac{1}{4}$ and $r = \dfrac{-3}{4}$

$\Longrightarrow$ the sum is $\dfrac{\left(-\frac{3}{4}\right)}{1-\left(-\frac{1}{4}\right)} = -\dfrac{3}{5}$

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2. Convergent or Divergent Series

Which of the series in Exercises 25-40 converge absolutely, which converge conditionally, and which diverge? Give reasons for your answers.

25. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$

Solution:

diverges, a p-series with $p = \dfrac{1}{2}$.

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26. $\displaystyle\sum_{n=3}^{\infty} \frac{-5}{n}$

Solution:

$\sum_{n=3}^{\infty} \frac{-5}{n} = -5 \sum_{n=3}^{\infty} \frac{1}{n}$, diverges since it is a nonzero multiple of the divergent harmonic series.

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27. $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$

Solution:

Since $f(x) = \frac{1}{x^{1/2}} \Longrightarrow f'(x) = -\frac{1}{2x^{3/2}} < 0 \Longrightarrow f(x)$ is decreasing $\Longrightarrow a_{n+1} < a_n$, and

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$, the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ converges by the Alternating Series Test. Since

$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges, the given series converges conditionally.

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28. $\sum_{n=1}^{\infty} \frac{1}{2n^3}$

Solution:

converges by the Direct Comparison Test since $\frac{1}{2n^3} < \frac{1}{n^3}$ for $n \geq 1$, which is the nth term of a convergent p-series.

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29. $\sum_{n=1}^{\infty} \frac{(-1)^n}{\ln(n+1)}$

Solution:

The given series does not converge absolutely by the Direct Comparison Test since

$\frac{1}{\ln(n+1)} < \frac{1}{n+1}$, which is the nth term of a divergent series. Since

$f(x) = \frac{1}{\ln(x+1)} \Longrightarrow f'(x) = -\frac{1}{(\ln(x+1))^2 (x+1)} < 0 \Longrightarrow f(x)$ is decreasing

$a_{n+1} < a_n$, and $\lim_{n\to\infty} \frac{1}{\ln(x+1)} = 0$, the given series converges conditionally by the Alternating Series Test.

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30. $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$

Solution:

$$\int_2^{\infty} \frac{1}{x(\ln x)^2} dx = \lim_{b\to\infty} \int_2^b \frac{1}{x(\ln x)^2} dx = \lim_{b\to\infty} \left[-(\ln x)^{-1}\right]_2^b = -\lim_{b\to\infty} \left(\frac{1}{\ln b} - \frac{1}{\ln 2}\right) = \frac{1}{\ln 2} \Longrightarrow$$
the series converges absolutely by the Integral Test.

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31. $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n^3}$

Solution:

converges absolutely by the Direct Comparison Test since

$\dfrac{\ln n}{n^3} < \dfrac{n}{n^3} = \dfrac{1}{n^2}$, the nth term of a convergent p-series.

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32. $\displaystyle\sum_{n=3}^{\infty} \frac{\ln n}{\ln(\ln n)}$

Solution:

diverges by the Direct Comparison Test for $e^{n^n} > n \Longrightarrow \ln\left(e^{n^n}\right) > \ln n \Longrightarrow n^n > \ln n \Longrightarrow$ $\ln n^n > \ln(\ln n)$

$\Longrightarrow n \ln n > \ln(\ln n) \Longrightarrow \dfrac{\ln n}{\ln(\ln n)} > \dfrac{1}{n}$, the nth term of the divergent harmonic series.

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33. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n\sqrt{n^2+1}}$

Solution:

$\displaystyle\lim_{n\to\infty} \frac{\left(\frac{1}{n\sqrt{n^2+1}}\right)}{\left(\frac{1}{n^2}\right)} = \sqrt{\lim_{n\to\infty} \frac{n^2}{n^2+1}} = \sqrt{1} = 1 \Longrightarrow$ the series converges absolutely by the Limit Comparison Test.

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34. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n 3n^2}{n^3+1}$

Solution:

Since $f(x) = \dfrac{3x^2}{x^3+1} \Longrightarrow f'(x) = \dfrac{3x(2-x^3)}{(x^3+1)^2} < 0$ when $x \geq 2 \Longrightarrow a_{n+1} < a_n$ for $n \geq 2$ and $\displaystyle\lim_{n\to\infty} \frac{3n^2}{n^3+1} = 0$, the series converges by the Alternating Series Test. The series does not converge absolutely: By the Limit

Comparison Test, $\displaystyle\lim_{n\to\infty} \frac{\left(\frac{3n^2}{n^3+1}\right)}{\left(\frac{1}{n}\right)} = \lim_{n\to\infty} \frac{3n^3}{n^3+1} = 3$. Therefore the convergence is conditional.

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35. $\displaystyle\sum_{n=1}^{\infty} \frac{n+1}{n!}$

Solution:

converges absolutely by the Ratio Test since $\displaystyle\lim_{n\to\infty}\left[\frac{n+2}{(n+1)!}\cdot\frac{n!}{n+1}\right] = \lim_{n\to\infty}\frac{n+2}{(n+1)^2} = 0 < 1$

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36. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n\left(n^2+1\right)}{2n^2+n-1}$

Solution:

diverges since $\displaystyle\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{(-1)^n\left(n^2+1\right)}{2n^2+n+1}$ does not exist

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37. $\displaystyle\sum_{n=1}^{\infty} \frac{(-3)^n}{n!}$

Solution:

converges absolutely by the Ratio Test since $\displaystyle\lim_{n\to\infty}\left[\frac{3^{n+1}}{(n+1)!}\cdot\frac{n!}{3^n}\right] = \lim_{n\to\infty}\frac{3}{n+1} = 0 < 1$

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38. $\displaystyle\sum_{n=1}^{\infty} \frac{2^n 3^n}{n^n}$

Solution:

converges absolutely by the Root Test since $\displaystyle\lim_{n\to\infty}\sqrt[n]{a_n} = \lim_{n\to\infty}\sqrt[n]{\frac{2^n3^n}{n^n}} = \lim_{n\to\infty}\frac{6}{n} = 0 < 1$

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39. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n(n+1)(n+2)}}$

Solution:

converges absolutely by the Limit Comparison Test since $\displaystyle\lim_{n\to\infty}\frac{\left(\frac{1}{n^{3/2}}\right)}{\frac{1}{\sqrt{n(n+1)(n+2)}}} = \sqrt{\lim_{n\to\infty}\frac{n(n+1)(n+2)}{n^3}} =$
1

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40. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\sqrt{n^2-1}}$

Solution:

converges absolutely by the Limit Comparison Test since $\displaystyle\lim_{n\to\infty}\frac{\left(\frac{1}{n^2}\right)}{\frac{1}{n\sqrt{n^2-1}}} = \sqrt{\lim_{n\to\infty}\frac{n^2\left(n^2-1\right)}{n^4}} = 1$

In Exercises 41-50, (a) find the series' radius and interval of convergence. Then identify the values of x for which the series converges (b) absolutely and (c) conditionally.

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41. $\sum_{n=1}^{\infty} \frac{(x+4)^n}{n3^n}$

Solution:

$$\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| < 1 \Longrightarrow \lim_{n\to\infty} \left|\frac{(x+4)^{n+1}}{(n+1)\,3^{n+1}} \cdot \frac{n3^n}{(x+4)^n}\right| < 1 \Longrightarrow \frac{|x+4|}{3} \lim_{n\to\infty} \left(\frac{n}{n+1}\right) < 1 \Longrightarrow \frac{|x+4|}{3} < 1$$

$\Longrightarrow -3 < x+4 < 3 \Longrightarrow -7 < x < -1$; at $x = -7$ we have

$\sum_{n=1}^{\infty} \frac{(-1)^n 3^n}{n3^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$, the alternating harmonic series, which converges conditionally; at $x = -1$ we have

$\sum_{n=1}^{\infty} \frac{3^n}{n3^n} = \sum_{n=1}^{\infty} \frac{1}{n}$, the divergent harmonic series

(a) the radius is 3; the interval of convergence is $-7 \le x < -1$
(b) the interval of absolute convergence is $-7 < x < -1$
(c) the series converges conditionally at $x = -7$.

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42. $\sum_{n=1}^{\infty} \frac{(x-1)^{2n-2}}{(2n-1)!}$

Solution:

$$\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| < 1 \Longrightarrow \lim_{n\to\infty} \left|\frac{(x-1)^{2n}}{(2n+1)!} \cdot \frac{(2n-1)!}{(x-1)^{2n-2}}\right| < 1 \Longrightarrow (x-1)^2 \lim_{n\to\infty} \frac{1}{2n\,(2n+1)} = 0 < 1$$

which holds for all x

(a) the radius is ∞; the interval of convergence is $-\infty < x < \infty$
(b) the interval of absolute convergence is $-\infty < x < \infty$
(c) there are no values for which the series converges conditionally.

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43. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}(3x-1)^n}{n^2}$

Solution:

$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| < 1 \Longrightarrow \lim_{n\to\infty}\left|\frac{(3x-1)^{n+1}}{(n+1)^2}\cdot\frac{n^2}{(3x-1)^n}\right| < 1 \Longrightarrow |3x-1|\lim_{n\to\infty}\frac{n^2}{(n+1)^2} < 1 \Longrightarrow$
$|3x-1|(1) < 1$

$|3x-1| < 1 \Longrightarrow -1 < 3x-1 < 1 \Longrightarrow 0 < 3x < 2 \Longrightarrow 0 < x < \frac{2}{3}$; at $x = 0$ we have $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}(-1)^n}{n^2} = \sum_{n=1}^{\infty}\frac{(-1)^{2n-1}}{n^2} = -\sum_{n=1}^{\infty}\frac{1}{n^2}$, a nonzero constant multiple of a convergent p-series, which is absolutely convergent;
at $x = \frac{2}{3}$ we have $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}(1)^n}{n^2} = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2}$, which converges absolutely

(a) the radius is $\frac{1}{3}$; the interval of convergence is $0 \le x \le \frac{2}{3}$
(b) the interval of absolute convergence is $0 \le x \le \frac{2}{3}$
(c) there are no values for which the series converges conditionally.

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44. $\sum_{n=0}^{\infty}\frac{(n+1)(2x+1)^n}{(2n+1)2^n}$
Solution:

$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| < 1 \Longrightarrow \lim_{n\to\infty}\left|\frac{n+2}{2n+3}\cdot\frac{(2x+1)^{n+1}}{2^{n+1}}\cdot\frac{2n+1}{n+1}\cdot\frac{2^n}{(2x+1)^n}\right| < 1 \Longrightarrow \frac{|2x+1|}{2}\lim_{n\to\infty}\left|\frac{n+2}{2n+3}\cdot\frac{2n}{n}\right.$
$1 \Longrightarrow \frac{|2x+1|}{2}(1) < 1$

$|2x+1| < 2 \Longrightarrow -2 < 2x+1 < 2 \Longrightarrow -3 < 2x < 1 \Longrightarrow -\frac{3}{2} < x < \frac{1}{2}$; at $x = -\frac{3}{2}$ we have $\sum_{n=1}^{\infty}\frac{n+1}{2n+1}\cdot\frac{(-2)^n}{2^n} = \sum_{n=1}^{\infty}\frac{(-1)^n(n+1)}{2n+1}$, which diverges by the nth Term Test for Divergence since
$\lim_{n\to\infty}\left(\frac{n+1}{2n+1}\right) = \frac{1}{2} \ne 0$; at $x = \frac{1}{2}$ we have $\sum_{n=1}^{\infty}\frac{n+1}{2n+1}\cdot\frac{2^n}{2^n} = \sum_{n=1}^{\infty}\frac{n+1}{2n+1}$, which diverges by the nth
Term Test for Divergence

(a) the radius is 1; the interval of convergence is $-\frac{3}{2} < x < \frac{1}{2}$
(b) the interval of absolute convergence is $-\frac{3}{2} < x < \frac{1}{2}$
(c) there are no values for which the series converges conditionally.

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45. $\sum_{n=1}^{\infty}\frac{x^n}{n^n}$
Solution:

$\lim\limits_{n\to\infty}\left|\dfrac{u_{n+1}}{u_n}\right| < 1 \Longrightarrow \lim\limits_{n\to\infty}\left|\dfrac{x^{n+1}}{(n+1)^{n+1}}\cdot\dfrac{n^n}{x^n}\right| < 1 \Longrightarrow |x|\lim\limits_{n\to\infty}\left(\dfrac{n}{n+1}\right)^n\left(\dfrac{1}{n+1}\right) < 1 \Longrightarrow$
$\dfrac{|x|}{e}\lim\limits_{n\to\infty}\left(\dfrac{1}{n+1}\right) < 1$

$\dfrac{|x|}{e}\cdot 0 < 1$; whic holds for all x

(a) the radius is ∞; the series converges for all x
(b) the series converges absolutely for all x
(c) there are no values for which the series converges conditionally.

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46. $\sum\limits_{n=1}^{\infty}\dfrac{x^n}{\sqrt{n}}$
Solution:

$\lim\limits_{n\to\infty}\left|\dfrac{u_{n+1}}{u_n}\right| < 1 \Longrightarrow \lim\limits_{n\to\infty}\left|\dfrac{x^{n+1}}{\sqrt{n+1}}\cdot\dfrac{\sqrt{n}^n}{x^n}\right| < 1 \Longrightarrow |x|\lim\limits_{n\to\infty}\sqrt{\dfrac{n}{n+1}} < 1 \Longrightarrow |x| < 1$; when $x=-1$ we have
$\sum\limits_{n=1}^{\infty}\dfrac{(-1)^n}{\sqrt{n}}$, which converges by the Alternating Series Test; we have $\sum\limits_{n=1}^{\infty}\dfrac{1}{\sqrt{n}}$, a divergent p-series

(a) the radius is 1; the interval of convergence is $-1 \le x < 1$
(b) the interval of absolute convergence is $-1 < x < 1$
(c) the series converges conditionally at $x=-1$.

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47. $\sum\limits_{n=1}^{\infty}\dfrac{(n+1)\,x^{2n-1}}{3^n}$
Solution:

$\lim\limits_{n\to\infty}\left|\dfrac{u_{n+1}}{u_n}\right| < 1 \Longrightarrow \lim\limits_{n\to\infty}\left|\dfrac{(n+2)\,x^{2n+1}}{3^{n+1}}\cdot\dfrac{3^n}{(n+1)\,x^{2n-1}}\right| < 1 \Longrightarrow \dfrac{x^2}{3}\lim\limits_{n\to\infty}\left(\dfrac{n+2}{n+1}\right) < 1$

$-\sqrt{3} < x < \sqrt{3}$; the series $\sum\limits_{n=1}^{\infty}-\dfrac{(n+1)}{\sqrt{3}}$ and $\sum\limits_{n=1}^{\infty}\dfrac{(n+1)}{\sqrt{3}}$, obtained with $x=\pm\sqrt{3}$, both diverge

(a) the radius is $\sqrt{3}$; the interval of convergence is $-\sqrt{3} < x < \sqrt{3}$
(b) the interval of absolute convergence is $-\sqrt{3} < x < \sqrt{3}$
(c) there are no values for which the series converges conditionally

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48. $\sum\limits_{n=0}^{\infty}\dfrac{(-1)^n\,(x-1)^{2n+1}}{2n+1}$
Solution:

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right|<1 \Longrightarrow \lim_{n\to\infty}\left|\frac{(x-1)\,x^{2n+3}}{2n+3}\cdot\frac{2n+1}{(x-1)^{2n+1}}\right|<1 \Longrightarrow (x-1)^2\lim_{n\to\infty}\frac{2n+3}{2n+1}<1$$

$(x-1)^2(1)<1 \Longrightarrow |x-1|<1 \Longrightarrow -1<x-1<1 \Longrightarrow 0<x<2$; at $x=0$ we have

$\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n(-1)^{2n+1}}{2n+1}=\sum_{n=0}^{\infty}\frac{(-1)^{3n+1}}{2n+1}$ which converges conditionally by the Alternating Series Test and the fact that

$\displaystyle\sum_{n=0}^{\infty}\frac{1}{2n+1}$ diverges; at $x=0$ we have

$\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n(1)^{2n+1}}{2n+1}=\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}$ which also converges conditionally

(a) the radius is 1; the interval of convergence is $0\le x\le 2$
(b) the interval of absolute convergence is $0<x<2$
(c) the series converges conditionally at $x=0$ and $x=2$

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49. $\displaystyle\sum_{n=1}^{\infty}(\operatorname{csc} hn)\,x^n$
Solution: Omitted

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50. $\displaystyle\sum_{n=1}^{\infty}(\coth n)\,x^n$

Solution: Omitted

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4. Maclaurin Series

Each of the series in Exercises 51-56 is the value of the Taylor series at $x=0$ of a function $f(x)$ at a particular point. What function and what point? What is the sum of the series?

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51. $1-\frac{1}{4}+\frac{1}{16}-\cdots+(-1)^n\frac{1}{4^n}+\cdots$
Solution:

The given series has the form

$1-x+x^2-x^3+\cdots+(-x)^n+\cdots=\dfrac{1}{1+x}$, where $x=\dfrac{1}{4}$; the sum is $\dfrac{1}{1+\frac{1}{4}}=\dfrac{4}{5}$.

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52. $\frac{2}{3} - \frac{4}{18} + \frac{8}{81} - \cdots + (-1)^{n-1}\frac{2^n}{n3^n} + \cdots$
Solution:

The given series has the form

$$x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^{n-1}\frac{x^n}{n} + \cdots = \ln(1+x), \text{ where } x = \frac{2}{3}\text{; the sum is } \ln\left(\frac{5}{3}\right) \approx 0,510825624.$$

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53. $\pi - \frac{\pi^3}{3!} + \frac{\pi^5}{5!} - \cdots + (-1)^n \frac{\pi^{2n+1}}{(2n+1)!} + \cdots$
Solution:

The given series has the form

$$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \cdots = \sin x, \text{ where } x = \pi\text{; the sum is } \sin\pi = 0.$$

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54. $1 - \frac{\pi^2}{9.2!} + \frac{\pi^4}{81.4!} - \cdots + (-1)^n \frac{\pi^{2n}}{3^{2n}(2n)!} + \cdots$
Solution:

The given series has the form

$$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots = \cos x, \text{ where } x = \frac{\pi}{3}\text{; the sum is } \cos\frac{\pi}{3} = \frac{1}{2}.$$

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55. $1 + \ln 2 + \frac{(\ln 2)^2}{2!} + \cdots + \frac{(\ln 2)^n}{n!} + \cdots$
Solution:

The given series has the form

$$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots = e^x, \text{ where } x = \ln 2\text{; the sum is } e^{\ln 2} = 2.$$

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56. $\frac{1}{\sqrt{3}} - \frac{1}{9\sqrt{3}} + \frac{1}{45\sqrt{3}} - \cdots + (-1)^{n-1}\frac{1}{(2n-1)\left(\sqrt{3}\right)^{2n-1}} + \cdots$
Solution:

The given series has the form

$$x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots + (-1)^n \frac{x^{2n-1}}{(2n-1)} + \cdots = \tan^{-1} x, \text{ where } x = \frac{1}{\sqrt{3}}\text{; the sum is } \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}.$$

Find the Taylor series at $x = 0$ for the functions in Exercises 57-64.

57. $\dfrac{1}{1-2x}$
Solution:

Consider $\dfrac{1}{1-2x}$ as the sum of a convergent geometric series with $a=1$ and $r=2x$

$$\Longrightarrow \frac{1}{1-2x} = 1+(2x)+(2x)^2+(2x)^3+\cdots = \sum_{n=0}^{\infty} 2^n x^n \text{ where } |2x|<1$$

$$\Longrightarrow |x| < \frac{1}{2}.$$

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58. $\dfrac{1}{1+x^3}$
Solution:

Consider $\dfrac{1}{1+x^3}$ as the sum of a convergent geometric series with $a=1$ and $r=-x^3$

$$\Longrightarrow \frac{1}{1+x^3} = \frac{1}{1-(-x^3)} = 1+\left(-x^3\right)+\left(-x^3\right)^2+\left(-x^3\right)^3+\cdots = \sum_{n=0}^{\infty} (-1)^n x^{3n} \text{ where } \left|-x^3\right|<1$$

$$\Longrightarrow \left|x^3\right| < 1 \Longrightarrow |x| < 1.$$

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59. $\sin \pi x$
Solution:

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \Longrightarrow \sin(\pi x) = \sum_{n=0}^{\infty} \frac{(-1)^n (\pi x)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n+1} x^{2n+1}}{(2n+1)!}$$

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60. $\sin \dfrac{2x}{3}$
Solution:

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \Longrightarrow \sin \frac{2x}{3} = \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{2x}{3}\right)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n+1} x^{2n+1}}{3^{2n+1}(2n+1)!}$$

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61. $\cos\left(x^{5/2}\right)$
Solution:

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \Longrightarrow \cos\left(x^{5/2}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n \left(x^{5/2}\right)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{5n}}{(2n)!}$$

---

62. $\cos\sqrt{5x}$
Solution:

$$\cos x=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}\Longrightarrow \cos\left(\sqrt{5x}\right)=\cos(5x)^{1/2}=\sum_{n=0}^{\infty}\frac{(-1)^n\left((5x)^{1/2}\right)^{2n}}{(2n)!}=\sum_{n=0}^{\infty}\frac{(-1)^n 5^n x^n}{(2n)!}$$

---

63. $e^{(\pi x/2)}$
Solution:

$$e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!}\Longrightarrow e^{(\pi x/2)}=\sum_{n=0}^{\infty}\frac{\left(\frac{\pi x}{2}\right)^n}{n!}=\sum_{n=0}^{\infty}\frac{\pi^n x^n}{2^n n!}$$

---

64. e^{-x^2}
Solution:

$$e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!}\Longrightarrow e^{-x^2}=\sum_{n=0}^{\infty}\frac{(-x^2)^n}{n!}=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{n!}$$

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5. Taylor Series

In Exercises 65-68, find the first four nonzero terms of the Taylor series generated by f at $x=a$.

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65. $f(x)=\sqrt{3+x^2}$ at $x=-1$
Solution:

$$f(x)=\sqrt{3+x^2}=\left(3+x^2\right)^{1/2}\Rightarrow f'(x)=x\left(3+x^2\right)^{-1/2}\Rightarrow f''(x)=-x^2\left(3+x^2\right)^{-3/2}$$

$$f'''(x)=3x^3\left(3+x^2\right)^{-5/2}-3x\left(3+x^2\right)^{-3/2};f(-1)=2,f'(-1)=-\frac{1}{2},f''(-1)=-\frac{1}{8}+\frac{1}{2}=\frac{3}{8}$$

$$f'''(-1)=-\frac{3}{32}+\frac{3}{8}=\frac{9}{32}$$

$$f(x)=\sqrt{3+x^2}=2-\frac{(x+1)}{2.1!}+\frac{3(x+1)^2}{2^3.2!}+\frac{9(x+1)^3}{2^5.3!}+\cdots$$

---

66. $f(x)=\dfrac{1}{1-x}$ at $x=2$
Solution:

$$f(x) = \frac{1}{1-x} = (1-x)^{-1} \Rightarrow f'(x) = (1-x)^{-2} \Rightarrow f''(x) = 2(1-x)^{-3}$$

$$\Rightarrow f'''(x) = 6(1-x)^{-4}; f(2) = -1, f'(2) = 1, f''(2) = -2, f'''(2) = 6$$

$$\Rightarrow \frac{1}{1-x} = -1 + (x-2) - (x-2)^2 + (x-2)^3 - \cdots$$

---

67. $f(x) = \dfrac{1}{x+1}$ at $x = 3$
Solution:

$$f(x) = \frac{1}{x+1} = (x+1)^{-1} \Longrightarrow f'(x) = -(x+1)^{-2} \Longrightarrow f''(x) = 2(x+1)^{-3}$$

$$\Longrightarrow f'''(x) = -6(x+1)^{-4}; f(3) = \frac{1}{4}, f'(3) = -\frac{1}{4^2}, f''(3) = \frac{2}{4^3}, f'''(3) = \frac{-6}{4^4}$$

$$\Longrightarrow \frac{1}{x+1} = \frac{1}{4} - \frac{1}{4^2}(x-3) + \frac{1}{4^3}(x-3)^2 - \frac{1}{4^4}(x-3)^3 + \cdots$$

---

68. $f(x) = \dfrac{1}{x}$ at $x = a > 0$
Solution:

$$f(x) = \frac{1}{x} = x^{-1} \Rightarrow f'(x) = -x^{-2} \Rightarrow f''(x) = 2x^{-3} \Rightarrow f'''(x) = -6x^{-4}$$

$$\Rightarrow f(a) = \frac{1}{a}, f'(a) = -\frac{1}{a^2}, f''(a) = \frac{2}{a^3}, f'''(a) = \frac{-6}{a^4}$$

$$\Longrightarrow \frac{1}{x} = \frac{1}{a} - \frac{1}{a^2}(x-a) + \frac{1}{a^3}(x-a)^2 - \frac{1}{a^4}(x-a)^3 + \cdots$$

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6. Nonelementary Integrals

Use series to approximate the values of the integrals in Exercises 77-80 with an error of magnitude less than 10^{-8}. The answer section gives the integrals' values rounded to 10 decimal places.)

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77. $\displaystyle\int_0^{1/2} e^{-x^3}\,dx$
Solution:

$$\int_0^{1/2} e^{-x^3}\,dx = \int_0^{1/2} \left(1 - x^3 + \frac{x^6}{2!} - \frac{x^9}{3!} + \frac{x^{12}}{4!} - \cdots\right) dx$$

$$= \left[x - \frac{x^4}{4} + \frac{x^7}{7.2!} - \frac{x^{10}}{10.3!} + \frac{x^{13}}{13.4!} - \cdots \right]_0^{1/2}$$

$$\approx \frac{1}{2} - \frac{1}{2^4.4} + \frac{1}{2^7.7.2!} - \frac{1}{2^{10}.10.3!} + \frac{1}{2^{13}.13.4!} - \frac{1}{2^{16}.16.5!} \approx 0,484917143$$

---

78. $\int_0^1 x \sin\left(x^3\right) dx$

Solution:

$$\int_0^1 x \sin\left(x^3\right) dx = \int_0^1 x \left(x^3 - \frac{x^9}{3!} + \frac{x^{15}}{5!} - \frac{x^{21}}{7!} + \frac{x^{27}}{9!} - \cdots \right) dx$$

$$= \int_0^1 \left(x^4 - \frac{x^{10}}{3!} + \frac{x^{16}}{5!} - \frac{x^{22}}{7!} + \frac{x^{28}}{9!} - \cdots \right) dx$$

$$= \left[\frac{x^5}{5} - \frac{x^{11}}{11.3!} + \frac{x^{17}}{17.5!} - \frac{x^{23}}{23.7!} + \frac{x^{29}}{29.9!} - \cdots \right]_0^1 \approx 0,185330149$$

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79. $\int_0^{1/2} \frac{\tan^{-1} x}{x} dx$

Solution:

$$\int_0^{1/2} \frac{\tan^{-1} x}{x} dx = \int_0^{1/2} \left(1 - \frac{x^2}{3} + \frac{x^4}{5} - \frac{x^6}{7} + \frac{x^8}{9} - \frac{x^{10}}{11} + \cdots \right) dx$$

$$= \left[x - \frac{x^3}{9} + \frac{x^5}{25} - \frac{x^7}{49} + \frac{x^9}{81} - \frac{x^{11}}{121} + \cdots \right]_0^{1/2}$$

$$\approx \frac{1}{2} - \frac{1}{9.2^3} + \frac{1}{25.2^5} - \frac{1}{49.2^7} + \frac{1}{81.2^9} - \frac{1}{121.2^{11}} + \frac{1}{13^2.2^{13}} - \frac{1}{15^2.2^{15}} + \frac{1}{17^2.2^{17}} - \frac{1}{19^2.2^{19}} + \frac{1}{21^2.2^{21}}$$

$$\approx 0,4872223583$$

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80. $\int_0^{1/64} \frac{\tan^{-1} x}{\sqrt{x}} dx$

Solution:

$$= \int_0^{1/64} \frac{\tan^{-1} x}{\sqrt{x}} dx = \int_0^{1/64} \frac{1}{\sqrt{x}} \left(x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \right) dx$$

$$= \int_0^{1/64} \left(x^{1/2} - \frac{1}{3} x^{5/2} + \frac{1}{5} x^{9/2} - \frac{1}{7} x^{13/2} + \cdots \right) dx$$

$$= \left[\frac{2}{3} x^{3/2} - \frac{2}{21} x^{7/2} + \frac{2}{55} x^{11/2} - \frac{2}{105} x^{15/2} + \cdots \right]_0^{1/64}$$

$$= \left[\frac{2}{3.8^3} - \frac{2}{21.8^7} + \frac{2}{55.8^{11}} - \frac{2}{105.8^{15}} + \cdots \right] \approx 0.0013020379$$

In Exercises 81-86 use power series to evaluate the limit.

81. $\lim\limits_{x\to 0}\dfrac{7\sin x}{e^{2x}-1}$
Solution:

$$\lim_{x\to 0}\frac{7\sin x}{e^{2x}-1}=\lim_{x\to 0}\frac{7\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-\cdots\right)}{\left(2x+\frac{2^2x^2}{2!}+\frac{2^3x^3}{3!}+\cdots\right)}=\lim_{x\to 0}\frac{7\left(1-\frac{x^2}{3!}+\frac{x^4}{5!}-\cdots\right)}{\left(2+\frac{2^2x}{2!}+\frac{2^3x^2}{3!}+\cdots\right)}=\frac{7}{2}$$

82. $\lim\limits_{\theta\to 0}\dfrac{e^{\theta}-e^{-\theta}-2\theta}{\theta-\sin\theta}$
Solution:

$$\lim_{\theta\to 0}\frac{e^{\theta}-e^{-\theta}-2\theta}{\theta-\sin\theta}=\lim_{\theta\to 0}\frac{\left(1+\theta+\frac{\theta^2}{2!}+\frac{\theta^3}{3!}+\cdots\right)-\left(1-\theta+\frac{\theta^2}{2!}-\frac{\theta^3}{3!}+\cdots\right)-2\theta}{\theta-\left(\theta-\frac{\theta^3}{3!}+\frac{\theta^5}{5!}-\cdots\right)}$$

$$=\lim_{\theta\to 0}\frac{2\left(\frac{\theta^3}{3!}+\frac{\theta^5}{5!}+\cdots\right)}{\left(\frac{\theta^3}{3!}-\frac{\theta^5}{5!}+\cdots\right)}=\lim_{\theta\to 0}\frac{2\left(\frac{1}{3!}+\frac{\theta^2}{5!}+\cdots\right)}{\left(\frac{1}{3!}-\frac{\theta^2}{5!}+\cdots\right)}=2$$

83. $\lim\limits_{t\to 0}\left(\dfrac{1}{2-2\cos t}-\dfrac{1}{t^2}\right)$
Solution:

$$\lim_{t\to 0}\left(\frac{1}{2-2\cos t}-\frac{1}{t^2}\right)=\lim_{t\to 0}\frac{t^2-2+2\cos t}{2t^2\left(1-\cos t\right)}$$

$$=\lim_{t\to 0}\frac{t^2-2+2\left(1-\frac{t^2}{2}+\frac{t^4}{4!}-\cdots\right)}{2t^2\left(1-1+\frac{t^2}{2}-\frac{t^4}{4!}+\cdots\right)}==\lim_{t\to 0}\frac{2\left(\frac{1}{4!}-\frac{t^2}{6!}+\cdots\right)}{\left(1-\frac{2t^2}{4!}+\cdots\right)}=\frac{1}{12}$$

84. $\lim\limits_{h\to 0}\dfrac{(\sinh)/h-\cosh}{h^2}$

Solution:

$$\lim_{h\to 0}\frac{(\sinh)/h-\cosh}{h^2}=\lim_{h\to 0}\frac{(\sinh)/h-\cosh}{h^2}=\lim_{h\to 0}\frac{\left(1-\frac{h^2}{3'}+\frac{h^4}{5!}-\cdots\right)-\left(1-\frac{h^2}{2!}+\frac{h^4}{4!}-\cdots\right)}{h^2}$$

$$\lim_{h\to 0}\frac{\left(\frac{h^2}{2!}-\frac{h^2}{3'}+\frac{h^4}{5!}-\frac{h^4}{4!}+\frac{h^6}{6!}-\frac{h^6}{7!}+\cdots\right)}{h^2}=\lim_{h\to 0}\left(\frac{1}{2!}-\frac{1}{3'}+\frac{h^2}{5!}-\frac{h^2}{4!}+\frac{h^4}{6!}-\frac{h^4}{7!}+\cdots\right)=\frac{1}{3}$$

85. $\lim\limits_{z\to 0}\dfrac{1-\cos^2 z}{\ln\left(1-z\right)+\sin z}$
Solution:

$$\lim_{z\to 0}\frac{1-\cos^2 z}{\ln(1-z)+\sin z}=\lim_{z\to 0}\frac{1-\left(1-z^2+\frac{z^4}{3}-\cdots\right)}{\left(-z-\frac{z^2}{3}-\frac{z^3}{3}-\cdots\right)+\left(z-\frac{z^3}{3'}+\frac{z^5}{5!}-\cdots\right)}$$

$$=\lim_{z\to 0}\frac{\left(z^2-\frac{z^4}{3}+\cdots\right)}{\left(-\frac{z^2}{2}-\frac{2z^3}{3}-\frac{z^4}{4}-\cdots\right)}=\lim_{z\to 0}\frac{\left(1-\frac{z^2}{3}+\cdots\right)}{\left(-\frac{1}{2}-\frac{2z}{3}-\frac{z^2}{4}-\cdots\right)}=-2$$

---

86. $\lim_{y\to 0}\frac{y^2}{\cos y-\cosh y}$
Solution: omitted

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87. Use a series representation of $\sin 3x$ to find values of r and s for which

$$\lim_{x\to 0}\left(\frac{\sin 3x}{x^3}+\frac{r}{x^2}+s\right)=0.$$

Solution:

$$\lim_{x\to 0}\left(\frac{\sin 3x}{x^3}+\frac{r}{x^2}+s\right)=\lim_{x\to 0}\left[\frac{\left(3x-\frac{(3x)^3}{6}+\frac{(3x)^5}{120}-\cdots\right)}{x^3}+\frac{r}{x^2}+s\right]$$

$$=\lim_{x\to 0}\left(\frac{3}{x^2}-\frac{9}{2}+\frac{81x^2}{40}+\cdots+\frac{r}{x^2}\right)=0$$

$$\Longrightarrow \frac{r}{x^2}+\frac{3}{x^2}=0 \text{ and } s-\frac{9}{2}\Rightarrow r=-3 \text{ and } s=\frac{9}{2}.$$

Problem: Find the sum of the series
$$\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}$$
correct to three decimal places.

Solution:

We first observe that the series is convergent by the Alternating Series Test because
(i)
$$\frac{1}{(n+1)!}<\frac{1}{n!\,(n+1)}<\frac{1}{n!}$$
(ii)
$$0<\frac{1}{n!}<\frac{1}{n}\to 0 \text{ as } n\to\infty$$

To get a feel for how many terms we need to use in our approximation, let's write out the first few terms of the series:

$$\begin{aligned} s &= \frac{1}{0!}-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}-\frac{1}{5!}+\frac{1}{6!}-\frac{1}{7!}+\cdots \\ &= 1-1+\frac{1}{2}-\frac{1}{6}+\frac{1}{24}-\frac{1}{120}+\frac{1}{720}-\frac{1}{5040}+\cdots \end{aligned}$$

Notice that

$$u_7 = \frac{1}{5040} < \frac{1}{5000} = 0.0002$$

and

$$s_6 = 1 - 1 + \frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \frac{1}{120} + \frac{1}{720} \approx 0.368056$$

By the Alternating Series Estimation Theorem we know that

$$|s - s_6| \leq u_7 < 0.0002$$

This error of less than 0.0002 does not affect the third decimal place, so we have

$$s \approx 0.368$$

correct to three decimal places.

NOTE: The rule that the error (in using s_n to approximate s) is smaller than the first neglected term is, in general, valid only for alternating series that satisfy the conditions of the Alternating Series Estimation Theorem. *The rule does not apply to other types of series.*

Problem: How many terms of the series do we need to add in order to find the sum to the indicated accuracy

$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^4}$ ($|\text{error}| < 0.001$)

Solution:

The series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^4}$$

satisfies (i) of the Alternating Series Test because $\frac{1}{(n+1)^4} < \frac{1}{n^4}$ and $\lim_{n\to\infty} \frac{1}{n^4} = 0$, so the series is convergent.

Now $u_5 = \frac{1}{5^4} = 0.0016 > 0.001$ and $u_6 = \frac{1}{6^4} \approx 0.00077 < 0.001$, so by the Alternating Series Estimation

Theorem, $n = 5$.
