ÇANKAYA UNIVERSITY

Department of Mathematics and Computer Science

MATH 156 Calculus for Engineering II

Practice Problems

2$^{\text{nd}}$ Midterm
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1. Domain, Range, and Level Curves

(p. 1060) In Exercises 1-4, find the domain and the range of the given function and identify its level curves.

1. $f(x,y) = 9x^2 + y^2$
Solution:
Domain: All points in the xy-plane
Range: $z \geq 0$
Level curves are ellipses with major axis along the y-axis and minor axis along the x-axis.

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2. $f(x,y) = e^{x+y}$
Solution:
Domain: All points in the xy-plane
Range: $0 < z < \infty$
Level curves are the straight lines $x + y = \ln z$ with slope -1, and $z > 0$.

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3. $g(x,y) = \dfrac{1}{xy}$

Solution:
Domain: All (x,y) such that $x \neq 0$ and $y \neq 0$
Range: $z \neq 0$
Level curves are hyperbolas with the x and y-axis as asymptotes.

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4. $g(x,y) = \sqrt{x^2 - y}$
Solution:
Domain: All (x,y) so that $x^2 - y \geq 0$
Range: $z \geq 0$
Level curves are parabolas $y = x^2 - c, c \geq 0$

In Exercises 5-8, find the domain and the range of the given function and identify its level surfaces.

5. $f(x,y,z) = x^2 + y^2 - z$
Solution:
Domain: All points (x,y,z) in space
Range: All real numbers
Level surfaces are paraboloidss of revolution with the z-axis as axis

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6. $g(x,y,z) = x^2 + 4y^2 + 9z^2$
Solution:

Domain: All points (x,y,z) in space
Rane: Nonnegative real numbers
Level surfaces are ellipsoids with center $(0,0,0)$

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7. $h(x,y,z) = \dfrac{1}{x^2 + y^2 + z^2}$
Solution:

Domain: All (x,y,z) such that $(x,y,z) \neq (0,0,0)$
Range: Positive real numbers
Level surfaces are spheres with center $(0,0,0)$ and radius $r > 0$.

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8. $h(x,y,z) = \dfrac{1}{x^2 + y^2 + z^2 + 1}$
Solution:

Domain: All points (x,y,z) in space
Range: $(0,1]$
Level surfaces are spheres with center $(0,0,0)$ and radius$r > 0$.

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2. Evaluating Limits

Find the limits in Exercises 9-14.
9. $\lim\limits_{(x,y)\to(\pi,\ln 2)} e^y \cos x$
Solution:

$$\lim_{(x,y)\to(\pi,\ln 2)} e^y \cos x = e^{\ln 2} \cos \pi = (2)(-1) = -2$$

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10. $\lim\limits_{(x,y)\to(0,0)} \dfrac{x+y}{x+\cos y}$
Solution:
$$\lim_{(x,y)\to(0,0)} \frac{x+y}{x+\cos y} = \frac{2+0}{2+\cos 0} = 2$$

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11. $\lim\limits_{(x,y)\to(1,1)} \dfrac{x-y}{x^2-y^2}$

Solution:

$$\lim_{\substack{(x,y)\to(1,1)\\ x\neq \pm y}} \frac{x-y}{x^2-y^2} = \lim_{\substack{(x,y)\to(1,1)\\ x\neq \pm y}} \frac{x-y}{(x-y)(x+y)}$$

$$= \lim_{\substack{(x,y)\to(1,1)\\ x\neq y}} \frac{1}{x+y} = \frac{1}{1+1} = \frac{1}{2}.$$

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12. $\lim\limits_{(x,y)\to(1,1)} \dfrac{x^3y^3-1}{xy-1}$

Solution:

$$\lim_{(x,y)\to(1,1)} \frac{x^3y^3-1}{xy-1} = \lim_{(x,y)\to(1,1)} \frac{(xy-1)(x^2y^2+xy+1)}{xy-1}$$

$$= \lim_{(x,y)\to(1,1)} \left(x^2y^2+xy+1\right) = 1^21^2+(1)(1)+1 = 3$$

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13. $\lim\limits_{P\to(1,-1,e)} \ln|x+y+z|$

Solution:

$$\lim_{P\to(1,-1,e)} \ln|x+y+z| = \lim_{P\to(1,-1,e)} \ln|1+(-1)+e| = \ln e = 1$$

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14. $\lim\limits_{P\to(1,-1,-1)} \tan^{-1}(x+y+z)$

Solution:

$$\lim_{P\to(1,-1,-1)} \tan^{-1}(x+y+z) = \tan^{-1}(x+(-1)+(-1)) = \tan^{-1}(-1) = -\frac{\pi}{4}$$

By considering different paths of approach, show that the limits in Exercises 15 and 16 do not exist.

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15. $\lim\limits_{\substack{(x,y)\to(0,0)\\ y\neq x^2}} \dfrac{y}{x^2-y}$

Solution:

Let $y = kx, k \neq 0$. Then

$$\lim_{\substack{(x,y)\to(0,0)\\ y\neq x^2}} \frac{y}{x^2-y} = \lim_{(x,kx^2)\to(0,0)} \frac{kx^2}{x^2-kx^2} = \frac{k}{1-k^2}$$ which gives different limits for different values of $k \implies$ limit does not exist.

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16. $\displaystyle\lim_{\substack{(x,y)\to(0,0)\\ xy\neq 0}} \frac{x^2+y^2}{xy}$

Solution:
Let $y = kx, k \neq 0$. Then

$\displaystyle\lim_{\substack{(x,y)\to(0,0)\\ xy\neq 0}} \frac{x^2+y^2}{xy} = \lim_{(x,kx)\to(0,0)} \frac{x^2+(kx)^2}{x(kx)} = \frac{1+k^2}{k}$ which gives different limits for different values of $k \Longrightarrow$ limit does not exist.

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17. Let $f(x,y) = \dfrac{x^2-y^2}{x^2+y^2}$ for $(x,y)\neq(0,0)$. Is it possible to define $f(0,0)$ in a way that makes f continuous at the origin? Why?

Solution:
Let $y = kx$. Then

$$\lim_{\substack{(x,y)\to(0,0)\\ xy\neq 0}} \frac{x^2-y^2}{x^2+y^2} = \lim_{(x,kx)\to(0,0)} \frac{x^2-(kx)^2}{x^2+(kx)^2} = \frac{1-k^2}{1+k^2}$$

which gives different limits for different values of $k \Longrightarrow$ limit does not exist so $f(0,0)$ cannot be defined in a way that makes f continuous at the origin.

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18. Let

$$h(x) = \begin{cases} \dfrac{\sin(x-y)}{|x|+|y|} & if \quad |x|+|y|\neq 0 \\ 0 & if \quad (x,y)=(0,0) \end{cases}$$

Is f continuous at the origin? Why?

Solution:
Along the x-axis, $y=0$ and $\displaystyle\lim_{(x,y)\to(0,0)} \frac{\sin(x-y)}{|x|+|y|} = \lim_{x\to 0} \frac{\sin x}{|x|} = \begin{cases} 1 & if \quad x>0 \\ -1 & if \quad x<0 \end{cases}$, so the limit fails to exist $\Longrightarrow f$ is not continuous at $(0,0)$.

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3. Partial Derivatives

In Exercises 19-24, find the partial derivative of the function with respect to each variable.

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19. $g(r,\theta) = r\cos\theta + r\sin\theta$

Solution:
$\dfrac{\partial g}{\partial r} = \cos\theta + \sin\theta, \dfrac{\partial g}{\partial \theta} = -r\sin\theta + r\cos\theta$

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20. $f(x,y) = \dfrac{1}{2}\ln\left(x^2+y^2\right) + \tan^{-1}\dfrac{y}{x}$

Solution:

$$\frac{\partial f}{\partial x} = \frac{1}{2}\left(\frac{2x}{x^2+y^2}\right) + \frac{\left(-\frac{y}{x^2}\right)}{1+\left(\frac{y}{x}\right)^2} = \frac{x}{x^2+y^2} + \frac{y}{x^2+y^2} = \frac{x-y}{x^2+y^2}$$
$$\frac{\partial f}{\partial y} = \frac{1}{2}\left(\frac{2y}{x^2+y^2}\right) + \frac{\left(\frac{1}{x}\right)}{1+\left(\frac{y}{x}\right)^2} = \frac{y}{x^2+y^2} + \frac{x}{x^2+y^2} = \frac{x+y}{x^2+y^2}$$

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21. $f(R_1, R_2, R_3) = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}$

Solution:
$\frac{\partial f}{\partial R_1} = -\frac{1}{R_1^2}, \frac{\partial f}{\partial R_2} = -\frac{1}{R_2^2}, \frac{\partial f}{\partial R_3} = -\frac{1}{R_3^2}$

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22. $h(x, y, z) = \sin(2\pi x + y - 3z)$

Solution:
$h_x(x, y, z) = 2\pi\cos(2\pi x + y - 3z), h_y(x, y, z) = \cos(2\pi x + y - 3z),$
$h_z(x, y, z) = -3\cos(2\pi x + y - 3z)$

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23. $P(n, R, T, V) = \frac{nRT}{V}$

Solution:
$\frac{\partial P}{\partial n} = \frac{RT}{V}, \frac{\partial P}{\partial R} = \frac{nT}{V}, \frac{\partial P}{\partial T} = \frac{nR}{V}, \frac{\partial P}{\partial V} = -\frac{nRT}{V^2}$

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24. $f(r, l, T, w) = \frac{1}{2rl}\sqrt{\frac{T}{\pi w}}$

Solution:

$f_r(r, l, T, w) = -\frac{1}{2r^2 l}\sqrt{\frac{T}{\pi w}},$

$f_l(r, l, T, w) = -\frac{1}{2rl^2}\sqrt{\frac{T}{\pi w}},$

$f_T(r, l, T, w) = \left(\frac{1}{2rl}\right)\left(\frac{1}{\sqrt{\pi w}}\right)\left(\frac{1}{2\sqrt{T}}\right) = \frac{1}{4rl}\sqrt{\frac{1}{T\pi w}} = \frac{1}{4rlT}\sqrt{\frac{T}{\pi w}},$

$f_w(r, l, T, w) = \left(\frac{1}{2rl}\right)\sqrt{\frac{T}{\pi}}(-\frac{1}{2}w^{-3/2}) = -\frac{1}{4rlw}\sqrt{\frac{T}{\pi w}}$

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4. Second-Order Partials

Find the second-order partial derivatives of the functions in Exercises 25-28.

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25. $g(x, y) = y + \frac{x}{y}$

Solution:
$\frac{\partial g}{\partial x} = \frac{1}{y},$

$$\frac{\partial g}{\partial y} = 1 - \frac{x}{y^2},$$
$$\frac{\partial^2 g}{\partial x^2} = 0,$$
$$\frac{\partial^2 g}{\partial y^2} = \frac{2x}{y^3}$$

$$\frac{\partial^2 g}{\partial x \partial y} = -\frac{1}{y^2}$$

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26. $g(x, y) = e^x + y \sin x$
Solution:
$g_x(x, y) = e^x + y \cos x,$
$g_y(x, y) = e^x + \sin x,$
$g_{xx}(x, y) = e^x - y \sin x,$
$g_{yy}(x, y) = 0,$
$g_{xy}(x, y) = g_{yx}(x, y) = \cos x$

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27. $f(x, y) = x + xy - 5x^3 + \ln\left(x^2 + 1\right)$
Solution:
$$\frac{\partial f}{\partial x} = 1 + y - 15x^2 + \frac{2x}{x^2+1},$$
$$\frac{\partial f}{\partial y} = x,$$
$$\frac{\partial^2 f}{\partial x^2} = -30x + \frac{2 - 2x^2}{(x^2+1)^2},$$
$$\frac{\partial^2 f}{\partial y^2} = 0,$$
$$\frac{\partial^2 f}{\partial y \partial x} = 1$$

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28. $f(x, y) = y^2 - 3xy + \cos y + 7e^y$
Solution:
$f_x(x, y) = -3y,$
$f_y(x, y) = 2y - 3x - \sin y + 7e^y$
$f_{xx}(x, y) = 0,$
$f_{yy}(x, y) = 2 - \cos y + 7e^y,$
$f_{xy}(x, y) = f_{yx}(x, y) = -3.$

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5. Chain Rule calculations

29. Find dw/dt at $t = 0$ if $w = \sin(xy + \pi)$, $x = e^t$, and $y = \ln(t + 1)$.
Solution:
$$\frac{\partial w}{\partial x} = y \cos(xy + \pi),$$
$$\frac{\partial w}{\partial y} = x \cos(xy + \pi),$$

$\dfrac{dx}{dt} = e^t,$
$\dfrac{dy}{dt} = \dfrac{1}{t+1},$
$\Longrightarrow \dfrac{dw}{dt} = [y\cos(xy+\pi)]\,e^t + [x\cos(xy+\pi)]\left(\dfrac{1}{t+1}\right); t=0 \Longrightarrow x=1, y=0$
$\dfrac{dw}{dt}\,|_{t=0} = (0)(1) + [(1)(-1)]\left(\dfrac{1}{0+1}\right) = -1.$

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30. Find dw/dt at $t = 1$ if $w = xe^y + y\sin z - \cos z,\ \ x = 2\sqrt{t},\ y = t - 1 + \ln t,\ z = \pi t.$
Solution:
$\dfrac{\partial w}{\partial x} = e^y$
$\dfrac{\partial w}{\partial y} = xe^y + \sin z,$
$\dfrac{\partial w}{\partial z} = y\cos z + \sin z,$
$\dfrac{dw}{\partial t} = t^{-1/2}$
$\dfrac{dy}{dt} = 1 + \dfrac{1}{t}$
$\dfrac{dz}{\partial t} = \pi$
$\Longrightarrow \dfrac{dw}{dt} = e^y t^{-1/2} + (xe^y + \sin z)\left(1 + \dfrac{1}{t}\right) + (y + \cos z + \sin z)\,\pi; t = 1 \Longrightarrow x = 2, y = 0,$ and $z = \pi$
$\Longrightarrow \dfrac{dw}{dt}\,|_{t=1} = (1)(1) + ((2)(1) - 0)(2) + (0+0)\,\pi = 5.$

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31. Find $\partial w/\partial r$ and $\partial w/\partial s$ when $r = \pi$ and $s = 0$ if $w = \sin(2x - y),\ x = r + \sin s,\ y = rs.$
Solution:
$\dfrac{\partial w}{\partial x} = 2\cos(2x - y),$
$\dfrac{\partial w}{\partial y} = -\cos(2x - y),$
$\dfrac{\partial x}{\partial r} = 1,$
$\dfrac{\partial x}{\partial s} = \cos s,$
$\dfrac{\partial y}{\partial r} = s,$
$\dfrac{\partial y}{\partial s} = r$
$\Longrightarrow \dfrac{\partial w}{\partial r} = [2\cos(2x - y)](1) + [-\cos(2x - y)](s); r = \pi$ and $s = 0$ implies $x = \pi$ and $y = 0$
$\Longrightarrow \dfrac{\partial w}{\partial r}\,|_{(\pi,0)} = (2\cos 2\pi) - (\cos 2\pi)(0) = 2;$
$\Longrightarrow \dfrac{\partial w}{\partial s} = [12\cos(2x - y)](\cos s) + [-\cos(2x - y)](r)$
$\Longrightarrow \dfrac{\partial w}{\partial s}\,|_{(\pi,0)} = (2\cos 2\pi)(\cos 0) - (\cos 2\pi)(\pi) = 2 - \pi$

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32. Find $\partial w/\partial u$ and $\partial w/\partial v$ when $u = v = 0$ if $w = \ln\sqrt{1+x^2}\sin - \tan^{-1} x$ and $x = 2e^u \cos v$.
Solution:
$$\frac{\partial w}{\partial u} = \frac{dw}{dx}\frac{\partial x}{\partial u} = \left(\frac{x}{1+x^2} - \frac{1}{x^2+1}\right)(2e^u \cos v)\,; u = v = 0 \Longrightarrow x = 2$$
$$\Longrightarrow \frac{\partial w}{\partial u}\,|_{(0,0)} = \left(\frac{2}{5} - \frac{1}{5}\right)(2) = \frac{2}{5};$$
$$\frac{\partial w}{\partial v} = \frac{dw}{dx}\frac{\partial x}{\partial v} = \left(\frac{x}{1+x^2} - \frac{1}{x^2+1}\right)(-2e^u \sin v)$$
$$\Longrightarrow \frac{\partial w}{\partial v}\,|_{(0,0)} = \left(\frac{2}{5} - \frac{1}{5}\right)(0) = 0.$$

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33. Find the value of the derivative of $f(x,y,z) = xy + yz + zx$ with respect to t on the curve $x = \cos t, y = \sin t, z = \cos(2t)$ at $t = 1$.
Solution:
$$\frac{\partial f}{\partial x} = y+z, \frac{\partial f}{\partial y} = x+z, \frac{\partial f}{\partial z} = y+x, \frac{dx}{dt} = -\sin t, \frac{dy}{dt} = \cos t, \frac{dz}{dt} = -2\sin 2t$$
$\Longrightarrow \frac{df}{dt} = -(y+z)(\sin t) + (x+z)(\cos t) - 2(y+x)(\sin 2t)\,; t = 1 \Longrightarrow x = \cos 1, y = \sin 1,$ and $z = \cos 2$
$$\Longrightarrow \frac{df}{dt}\,|_{t=1} = -(\sin 1 + \cos 2)(\sin 1) + (\cos 1 + \cos 2)(\cos 1) - 2(\sin 1 + \cos 1)(\sin 2).$$

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34. Show that if $w = f(s)$ is any differentiable function of s and if $s = y + 5x$, then
$$\frac{\partial w}{\partial x} - 5\frac{\partial w}{\partial y} = 0$$

Solution:
$$\frac{\partial w}{\partial x} = \frac{dw}{ds}\frac{\partial s}{\partial x} = (5)\frac{dw}{ds}$$
and
$$\frac{\partial w}{\partial y} = \frac{dw}{ds}\frac{\partial s}{\partial y} = (1)\frac{dw}{ds} = \frac{dw}{ds}$$
$$\Longrightarrow \frac{\partial w}{\partial x} - 5\frac{\partial w}{\partial y} = (5)\frac{dw}{ds} - (5)\frac{dw}{ds} = 0.$$

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6. Implicit Differentiation

Assuming that the equations in Exercises 35 and 36 define y as a differentiable function of x, find the value of dy/dx at point P.

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35. $1 - x - y^2 - \sin xy = 0$, $P(0,1)$
Solution:
$F(x,y) = 1 - x - y^2 - \sin xy \Longrightarrow F_x = -1 - y\cos xy$ and $F_y = -2y - x\cos xy$
$$\Longrightarrow \frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{-1 - y\cos xy}{-2y - x\cos xy} = \frac{1 + y\cos xy}{-2y - x\cos xy}$$
$\Longrightarrow$ at $(x,y) = (0,1)$ we have
$$\frac{dy}{dx}\,|_{(0,1)} = \frac{1+1}{-2} = -1.$$

36. $2xy + e^{x+y} - 2 = 0,\ P\left(0, \ln 2\right)$
Solution:
$F\left(x,y\right) = 2xy + e^{x+y} - 2 \Longrightarrow F_x = 2y + e^{x+y}$ and $F_y = 2x + e^{x+y}$
$\Longrightarrow \dfrac{dy}{dx} = -\dfrac{F_x}{F_y} = -\dfrac{2y + e^{x+y}}{2x + e^{x+y}}$
$\Longrightarrow$ at $\left(x,y\right) = \left(0, \ln 2\right)$ we have
$\dfrac{dy}{dx}\mid_{(0,\ln 2)} = -\dfrac{2\ln 2 + 2}{0+2} = -\left(\ln 2 + 1\right)$

7. Directional Derivatives

In Exercises 37-40, find the directions in which f increases and decreases most rapidly at P_0 and find the derivative of f in each direction. Also, find the derivative of f at P_0 in the direction of the vector $\mathbf{v}$.

37. $f\left(x,y\right) = \cos x \cos y,\ P_0\left(\pi/4, \pi/4\right), \mathbf{v} = 3\mathbf{i} + 4\mathbf{j}$
Solution:
$\nabla f = \left(-\sin x \cos y\right)\mathbf{i} - \left(\cos x \sin y\right)\mathbf{j}$
$\Longrightarrow \nabla f \mid_{\left(\frac{\pi}{4},\frac{\pi}{4}\right)} = -\frac{1}{2}\mathbf{i} - \frac{1}{2}\mathbf{j}$
$\Longrightarrow \left|\nabla f\right| = \sqrt{\left(-\frac{1}{2}\right)^2 + \left(-\frac{1}{2}\right)^2} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2};$
$\mathbf{u} = \frac{\nabla f}{\left|\nabla f\right|} = -\frac{\sqrt{2}}{2}\mathbf{i} - \frac{\sqrt{2}}{2}\mathbf{j}$
$\Longrightarrow f$ increases most rapidly in the direction $\mathbf{u} = -\frac{\sqrt{2}}{2}\mathbf{i} - \frac{\sqrt{2}}{2}\mathbf{j}$
and decreases most rapidly in the direction $-\mathbf{u} = \dfrac{\sqrt{2}}{2}\mathbf{i} + \dfrac{\sqrt{2}}{2}\mathbf{j};$
$\left(\mathrm{D}_{\mathbf{u}}f\right)_{P_0} = \left|\nabla f\right| = \dfrac{\sqrt{2}}{2}$ and $\left(\mathrm{D}_{-\mathbf{u}}f\right)_{P_0} = -\left|\nabla f\right| = -\dfrac{\sqrt{2}}{2};$
$\mathbf{u}_1 = \frac{\mathbf{v}}{\left|\mathbf{v}\right|} = \frac{3\mathbf{i}+4\mathbf{j}}{\sqrt{3^2+4^2}} = \frac{3}{5}\mathbf{i} + 4\mathbf{j}$
$\Longrightarrow \left(\mathrm{D}_{\mathbf{u}_1}f\right)_{P_0} = \nabla f \cdot \mathbf{u}_1 = \left(-\dfrac{1}{2}\right)\left(\dfrac{3}{5}\right) + \left(-\dfrac{1}{2}\right)\left(\dfrac{4}{5}\right) = -\dfrac{7}{10}.$

38. $f\left(x,y\right) = x^2 e^{-2y},\ P_0\left(1,0\right), \mathbf{v} = \mathbf{i} + \mathbf{j}$
Solution:
$\nabla f = 2xe^{-2y}\mathbf{i} - 2x^2 e^{-2y}\mathbf{j}$
$\Longrightarrow \nabla f \mid_{(1,0)} = 2\mathbf{i} - 2\mathbf{j}$
$\Longrightarrow \left|\nabla f\right| = \sqrt{\left(2\right)^2 + \left(-2\right)^2} = 2\sqrt{2};$
$\mathbf{u} = \frac{\nabla f}{\left|\nabla f\right|} = \frac{1}{\sqrt{2}}\mathbf{i} - \frac{1}{\sqrt{2}}\mathbf{j}$
$\Longrightarrow f$ increases most rapidly in the direction $\mathbf{u} = \frac{\sqrt{2}}{2}\mathbf{i} - \frac{\sqrt{2}}{2}\mathbf{j}$
and decreases most rapidly in the direction $-\mathbf{u} = -\dfrac{\sqrt{2}}{2}\mathbf{i} + \dfrac{\sqrt{2}}{2}\mathbf{j};$
$\left(\mathrm{D}_{\mathbf{u}}f\right)_{P_0} = \left|\nabla f\right| = 2\sqrt{2}$ and $\left(\mathrm{D}_{-\mathbf{u}}f\right)_{P_0} = -\left|\nabla f\right| = -2\sqrt{2};$
$\mathbf{u}_1 = \frac{\mathbf{v}}{\left|\mathbf{v}\right|} = \frac{\mathbf{i}+\mathbf{j}}{\sqrt{1^2+1^2}} = \frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}$

$\Longrightarrow (\mathrm{D}_{\mathbf{u}_1} f)_{P_0} = \nabla f \cdot \mathbf{u}_1 = (2)\left(\frac{1}{\sqrt{2}}\right) + (-2)\left(\frac{1}{\sqrt{2}}\right) = 0.$

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39. $f(x,y,z) = \ln(2x+3y+6z)$, $P_0(-1,-1,1)$, $\mathbf{v} = 2\mathbf{i}+3\mathbf{j}+6\mathbf{k}$
Answer:
f increases most rapidly in the direction $\mathbf{u} = \frac{2}{7}\mathbf{i} - \frac{3}{7}\mathbf{j} + \frac{6}{7}\mathbf{k}$
and decreases most rapidly in the direction $-\mathbf{u} = -\dfrac{2}{7}\mathbf{i} + \dfrac{3}{7}\mathbf{j} - \dfrac{6}{7}\mathbf{k}$;
$(\mathrm{D}_{\mathbf{u}} f)_{P_0} = |\nabla f| = 7$ and $(\mathrm{D}_{-\mathbf{u}} f)_{P_0} = -|\nabla f| = -7$;
$\mathbf{u}_1 = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{2}{7}\mathbf{i} + \frac{3}{7}\mathbf{j} + \frac{6}{7}\mathbf{k}$
$\Longrightarrow (\mathrm{D}_{\mathbf{u}_1} f)_{P_0} = (\mathrm{D}_{\mathbf{u}} f)_{P_0} = 7.$

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40. $f(x,y,z) = x^2 + 3xy - z^2 + 2y + z + 4$, $P_0(0,0,0)$, $\mathbf{v} = \mathbf{i}+\mathbf{j}+\mathbf{k}$
Solution:
$\nabla f = (2x+3y)\mathbf{i} + (3x+2)\mathbf{j} + (1-2z)$
$\Longrightarrow \nabla f\,|_{(0,0,0)} = 2\mathbf{j} + \mathbf{k}$;
$\mathbf{u} = \frac{\nabla f}{|\nabla f|} = \frac{2}{\sqrt{5}}\mathbf{j} + \frac{1}{\sqrt{5}}\mathbf{k}$
$\Longrightarrow f$ increases most rapidly in the direction $\mathbf{u} = \frac{2}{\sqrt{5}}\mathbf{j} + \frac{1}{\sqrt{5}}\mathbf{k}$
and decreases most rapidly in the direction $-\mathbf{u} = -\dfrac{2}{\sqrt{5}}\mathbf{j} - \dfrac{1}{\sqrt{5}}\mathbf{k}$;
$(\mathrm{D}_{\mathbf{u}} f)_{P_0} = |\nabla f| = \sqrt{5}$ and $(\mathrm{D}_{-\mathbf{u}} f)_{P_0} = -|\nabla f| = -\sqrt{5}$;
$\mathbf{u}_1 = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{\mathbf{i}+\mathbf{j}+\mathbf{k}}{\sqrt{1^2+1^2+1^2}} = \frac{1}{\sqrt{3}}\mathbf{i} + \frac{1}{\sqrt{3}}\mathbf{j} + \frac{1}{\sqrt{3}}\mathbf{k}$
$\Longrightarrow (\mathrm{D}_{\mathbf{u}_1} f)_{P_0} = \nabla f \cdot \mathbf{u}_1 = (0)\left(\frac{1}{\sqrt{3}}\right) + (2)\left(\frac{1}{\sqrt{3}}\right) + (1)\left(\frac{1}{\sqrt{3}}\right) = \sqrt{3}.$

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41. Find the derivative of $f(x,y,z) = xyz$ in the direction of the velocity vector of the helix

$$\mathbf{r}(t) = (\cos 3t)\,\mathbf{i} + (\sin 3t)\,\mathbf{j} + 3t\mathbf{k}$$

at $t = \dfrac{\pi}{3}$.
Solution:
$\mathbf{r}(t) = (\cos 3t)\,\mathbf{i} + (\sin 3t)\,\mathbf{j} + 3t\mathbf{k}$
$\Longrightarrow \mathbf{v}(t) = (-3\sin 3t)\,\mathbf{i} + (3\cos 3t)\,\mathbf{j} + 3\mathbf{k}$
$\Longrightarrow \mathbf{v}\left(\dfrac{\pi}{3}\right) = -3\mathbf{j} + 3\mathbf{k}$
$\Longrightarrow u = -\dfrac{1}{\sqrt{2}}\mathbf{j} + \dfrac{1}{\sqrt{2}}\mathbf{k}$;
$f(x,y,z) = xyz \Longrightarrow \nabla f = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$;
$t = \dfrac{\pi}{3}$ yields the point on the helix $(-1,0,\pi)$
$\Longrightarrow \nabla f\,|_{(-1,0,\pi)} = -\pi\mathbf{j} \Longrightarrow \nabla f \cdot \mathbf{u} = (-\pi\mathbf{j}) \cdot \left(-\dfrac{1}{\sqrt{2}}\mathbf{j} + \dfrac{1}{\sqrt{2}}\mathbf{k}\right) = \dfrac{\pi}{\sqrt{2}}.$

---

42. What is the largest value that the directional derivative of $f(x,y,z) = xyz$ can have at the point $(1,1,1)$?
Solution:
$f(x,y,z) = xyz \Longrightarrow \nabla f = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$; at $(1,1,1)$ we get $\nabla f = \mathbf{i}+\mathbf{j}+\mathbf{k} \Longrightarrow$ the maximum value of $D_{\mathbf{u}} f\,|_{(1,1,1)} = |\nabla f| = \sqrt{3}$

43. At the point $(1, 2)$, the function $f(x, y)$ has a derivative of 2 in the direction toward $(2, 2)$ and a derivative of -2 in the direction toward $(1, 1)$.
a. Find $f_x(1, 2)$ and $f_y(1, 2)$
b. Find the derivative of f at $(1, 2)$ in the direction toward the point $(4, 6)$.
Solution:
(a)
Let $\nabla f = a\mathbf{i} + b\mathbf{j}$ at $(1, 2)$. The direction toward $(2, 2)$ is determined by $\mathbf{v}_1 = (2-1)\,\mathbf{i} + (2-2)\,\mathbf{j} = \mathbf{i} = \mathbf{u}$ so that $\nabla f \cdot \mathbf{u} = 2 \Longrightarrow a = 2$. The direction toward $(1, 1)$ is determined by $\mathbf{v}_2 = (1-1)\,\mathbf{i} + (1-2)\,\mathbf{j} = -\mathbf{j} = \mathbf{u}$ so that $\nabla f \cdot \mathbf{u} = -2 \Longrightarrow -b = -2$.
Therefore
$\nabla f = 2\mathbf{i} + 2\mathbf{j}; f_x(1, 2) = f_y(1, 2) = 2.$
(b)
The direction toward $(4, 6)$ is determined by $\mathbf{v}_3 = (4-1)\,\mathbf{i} + (6-2)\,\mathbf{j} = 3\mathbf{i} + 4\mathbf{j} \Longrightarrow \mathbf{u} = \frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{j} \Longrightarrow \nabla f \cdot \mathbf{u} = \frac{14}{5}$

---

8. Gradients, Tangent Planes, and Normal Lines

In Exercises 45 and 46, sketch the surface $f(x, y, z) = c$ together with ∇f at the given points.

---

45. $x^2 + y^2 + z^2 = 0$; $(0, -1, \pm 1), (0, 0, 0)$
Solution: omitted

---

46. $y^2 + z^2 = 4$; $(2, \pm 2, 0)$, $(2, 0, \pm 2)$
Solution: omitted

In Exercises 47 and 48, find an equation for the plane tangent to the level surface $f(x, y, z) = c$ at the point P_0. Also, find parametric equations for the line that is normal to the surface at P_0.

---

47. $x^2 - y - 5z = 0$, $P_0(2, -1, 1)$
Solution:
$\nabla f = 2x\mathbf{i} - \mathbf{j} - 5\mathbf{k} \Longrightarrow \nabla f\,|_{(2,-1,1)} = 4\mathbf{i} - \mathbf{j} - 5\mathbf{k} \Longrightarrow$ Tangent Plane: $4(x-2) - (y+1) - 5(z-1) = 0 \Longrightarrow 4x - y - 5z = 4$; Normal Line: $x = 2 + 4t, y = -1 - t, z = 1 - 5t$.

---

48. $x^2 + y^2 + z = 4$, $P_0(1, 1, 2)$
Answer:
Tangent Plane: $2x + 2y + z - 6 = 0$;
Normal Line: $x = 1 + 2t, y = 1 + 2t, z = 2 + t$.

In Exercises 49 and 50, find an equation for the plane tangent to the surface $z = f(x, y)$ at the given point.

49. $z = \ln\left(x^2+y^2\right)$, $(0,1,0)$
Solution:
$\dfrac{\partial z}{\partial x} = \dfrac{2x}{x^2+y^2} \Longrightarrow \dfrac{\partial z}{\partial x}\,|_{(0,1,0)}$
and
$\dfrac{\partial z}{\partial y} = \dfrac{2y}{x^2+y^2} \Longrightarrow \dfrac{\partial z}{\partial y}\,|_{(0,1,0)} = 2;$
thus the tangent plane is
$2\,(y-1)-(z-0)=0$ or $2y-z-2=0$

---

50. $z = 1/\left(x^2+y^2\right)$, $(1,1,1/2)$
Solution:
$\dfrac{\partial z}{\partial x} = -2x\left(x^2+y^2\right)^{-2} \Longrightarrow \dfrac{\partial z}{\partial x}\,|_{(1,1,1/2)} = -\dfrac{1}{2}$ and $\dfrac{\partial z}{\partial y} = -2y\left(x^2+y^2\right)^{-2} \Longrightarrow \dfrac{\partial z}{\partial y}\,|_{(1,1,1/2)} = -\dfrac{1}{2}$
; thus the tangent plane is $-\dfrac{1}{2}(x-1)-\dfrac{1}{2}(y-1)-\dfrac{1}{2}(z-1)=0$ or $x+y+2z-3=0$

In Exercises 51 and 52, find equations for the lines that are tangent and normal to the level surface $f(x,y)=c$ at the point P_0.

---

51. $y-\sin x = 1$, $P_0(\pi,1)$
Solution:
$\nabla f = (-\cos x)\,\mathbf{i}+\mathbf{j} \Longrightarrow \nabla f\,|_{(\pi,1)} = \mathbf{i}+\mathbf{j} \Longrightarrow$ the tangent line is $(x-\pi)+(y-1)=0 \Longrightarrow$ $x+y=\pi+1$; the normal line is $y-1=1\,(x-\pi) \Rightarrow y=x-\pi+1$

---

52. $\dfrac{y^2}{2}-\dfrac{x^2}{2}=\dfrac{3}{2}$, $P_0(1,2)$
Answer:
$y=-2x+4$

---

9. Tangent Lines to Curves

In Exercises 53 and 54, find parametric equations for the line that is tangent to the curve of intersection of the surfaces at the given point.

---

53. Surfaces: $x^2+2y+2z=4,\ y=1,$
Point: $(1,1,1/2)$
Solution:
Let $f(x,y,z)=x^2+2y+2z-4$ and $g(x,y,z)=y-1$. Then

$$\nabla f = 2x\mathbf{i}+2\mathbf{j}+2\mathbf{k}\,|_{(1,1,1/2)} = 2\mathbf{i}+2\mathbf{j}+2\mathbf{k} \text{ and } \nabla g = \mathbf{j} \Longrightarrow \nabla f\times\nabla g = \begin{vmatrix} i & j & k \\ 2 & 2 & 2 \\ 0 & 1 & 0 \end{vmatrix} = -2\mathbf{i}+2\mathbf{k}$$

$\Longrightarrow$ the line is $x = 1 - 2t, y = 1, z = \dfrac{1}{2} + 2t$

---

54. Surfaces: $x^2 + y^2 + z = 2,\ y = 1,$
Point: $(1/2, 1, 1/2)$
Answer:
$x = \dfrac{1}{2} - t, y = 1, z = \dfrac{1}{2} + t$

---

10. Linearizations

In Exercises 55 and 56, find the liearization $L(x, y)$ of the function $f(x, y)$ at the point P_0. Then find an upper bound for the error E in the approximation $f(x, y) \approx L(x, y)$ over the rectangle R.

---

55. $f(x, y) = \sin x \cos y,\ P_0(\pi/4, \pi/4)$
$R : \left|x - \dfrac{\pi}{4}\right| \le 0.1, \left|y - \dfrac{\pi}{4}\right| \le 0.1$
Solution:
$f(\pi/4, \pi/4) = \dfrac{1}{2},$
$f_x(\pi/4, \pi/4) = \cos x \cos y\ |_{(\pi/4,\pi/4)} = \dfrac{1}{2},$
$f_y(\pi/4, \pi/4) = -\sin x \sin y\ |_{(\pi/4,\pi/4)} = -\dfrac{1}{2},$
$\Longrightarrow L(x, y) = \dfrac{1}{2} + \dfrac{1}{2}(x - \pi/4) - \dfrac{1}{2}(y - \pi/4) = \dfrac{1}{2} + \dfrac{1}{2}x - \dfrac{1}{2}y;$
$f_{xx}(x, y) = -\sin x \cos y,$
$f_{yy}(x, y) = -\sin x \cos y,$
and
$f_{xy}(x, y) = -\cos x \sin y.$
Thus an upper bound for E depends on the bound M used for $|f_{xx}|, |f_{yy}|$, and $|f_{xy}|$.
With $M = \dfrac{\sqrt{2}}{2}$ we have $|E(x, y)| \le \dfrac{1}{2}\left(\dfrac{\sqrt{2}}{2}\right)\left(\left|x - \dfrac{\pi}{4}\right| + \left|y - \dfrac{\pi}{4}\right|\right)^2 \le \dfrac{\sqrt{2}}{4}(0.2)^2 \le 0.0142,$
$M = 1,$
$|E(x, y)| \le \dfrac{1}{2}(1)\left(\left|x - \dfrac{\pi}{4}\right| + \left|y - \dfrac{\pi}{4}\right|\right)^2 = \dfrac{1}{2}(6)(0.2)^2 = 0.02$

---

56. $f(x, y) = xy - 3y^2 + 2,\ P_0(1, 1)$
$R : |x - 1| \le 0.1, |y - 1| \le 0.2$
Solution:
$f(1, 1) = 0,$
$f_x(1, 1) = y\ |_{(1,1)} = 1,$
$f_y(1, 1) = x - 6y\ |_{(1,1)} = -5,$
$\Longrightarrow L(x, y) = (x - 1) - 5(y - 1) = x - 5y + 4;$
$f_{xx}(x, y) = 0,$
$f_{yy}(x, y) = -6,$
and
$f_{xy}(x, y) = 1$

$\Longrightarrow$ maximum of $|f_{xx}|, |f_{yy}|$, and $|f_{xy}|$ is 6 $\Longrightarrow M = 6$
$\Longrightarrow |E(x,y)| \leq \frac{1}{2}(6)\left(|x-1| + |y-1|^2\right) = \frac{1}{2}(6)(0.1+0.2)^2 = 0.27$

Find the linearizations of the functions in Exercises 57 and 58 at the given points.

---

57. $f(x,y,z) = xy + 2yz - 3xz$ at $(1,0,0)$ and $(1,1,0)$.
Solution:
$f(1,0,0) = 0$, $f_x(1,0,0) = y - 3z\,|_{(1,0,0)} = 0$,
$f_y(1,0,0) = x + 2z\,|_{(1,0,0)} = 1$
$f_z(1,0,0) = 2y - 3x\,|_{(1,0,0)} = -3$
$\Longrightarrow L(x,y,z) = 0(x-1) + (y-0) - 3(z-0) = y - 3z$;
$f(1,1,0) = 1$,
$f_x(1,1,0) = 1$
$f_y(1,1,0) = 1$
$f_z(1,1,0) = -1$
$\Longrightarrow L(x,y,z) = 1 + (x-1) + (y-1) - 1(z-0) = x + y - z - 1.$.

---

58. $f(x,y,z) = \sqrt{2}\cos x \sin(y+z)$ at $(0,0,\pi/4)$ and $(\pi/4,\pi/4,0)$.
Solution:
$f(0,0,\pi/4) = 1$, $f_x(0,0,\pi/4) = -\sqrt{2}\sin x \sin(y+z)\,|_{(0,0,\pi/4)} = 0$,
$f_y(0,0,\pi/4) = \sqrt{2}\cos x \cos(y+z)\,|_{(0,0,\pi/4)} = 1$
$f_z(0,0,\pi/4) = \sqrt{2}\cos x \cos(y+z)\,|_{(0,0,\pi/4)} = 1$
$\Longrightarrow L(x,y,z) = 1 + 1(y-0) + 1\left(z - \frac{\pi}{4}\right) = 1 + y + z - \frac{\pi}{4}$;
$f(\pi/4,\pi/4,0) = \frac{\sqrt{2}}{2}$,
$f_x(\pi/4,\pi/4,0) = -\frac{\sqrt{2}}{2}$
$f_y(\pi/4,\pi/4,0) = \frac{\sqrt{2}}{2}$
$f_z(\pi/4,\pi/4,0) = \frac{\sqrt{2}}{2}$
$\Longrightarrow L(x,y,z) = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\left(x - \frac{\pi}{4}\right) + \frac{\sqrt{2}}{2}\left(y - \frac{\pi}{4}\right) + \frac{\sqrt{2}}{2}(z-0) = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}x + \frac{\sqrt{2}}{2}y + \frac{\sqrt{2}}{2}z.$

---

11. Local Extrema

Test the functions in Exercises 65-70 for local maxima and minima and saddle points. Find each function's value at these points.

---

65. $f(x,y) = x^2 - xy + y^2 + 2x + 2y - 4$.
Solution:
$f_x(x,y) = 2x - y + 2 = 0$
and
$f_x(x,y) = -x + 2y + 2 = 0$

$\Longrightarrow x = -2$ and $y = -2 \Longrightarrow (-2, -2)$ is the critical point;
$f_{xx}(-2, -2) = 2$
$f_{yy}(-2, -2) = 2$
$f_{xy}(-2, -2) = -1$
$\Longrightarrow f_{xx}f_{yy} - f_{xy}^2 = 3 > 0$ and $f_{xx} > 0 \Longrightarrow$ local minimum value of $f(-2, -2) = -8$.

---

66. $f(x, y) = 5x^2 + 4xy - 2y^2 + 4x - 4y$
Answer:
The critical point is $(0, -1)$
saddle point with $f(0, -1) = 2$

---

67. $f(x, y) = 2x^3 + 3xy + 2y^3$
Answer:
The critical points are $(0, 0)$ and $\left(-\frac{1}{2}, -\frac{1}{2}\right)$
saddle point with $f\left(-\frac{1}{2}, -\frac{1}{2}\right) = \frac{1}{4}$

---

68. $f(x, y) = x^3 + y^3 - 3xy + 15$
Answer:
The critical points are $(0, 0)$ and $(1, 1)$
saddle point with $f(0, 0) = 15$
local minimum value of $f(1, 1) = 14$

---

69. $f(x, y) = x^3 + y^3 + 3x^2 - 3y^2$
Answer:
The critical points are $(0, 0), (0, 2), (-2, 0)$ and $(-2, 2)$
saddle point with $f(0, 0) = 0$
local minimum value of $f(0, 2) = -4$
local maximum value of $f(-2, 0) = 4$
saddle point with $f(-2, 2) = 0$

---

70. $f(x, y) = x^4 - 8x^2 + 3y^2 - 6y$
Answer:
The critical points are $(0, 1), (2, 1)$, and $(-2, 1)$
saddle point with $f(0, 1) = -3$
local minimum value of $f(2, 1) = -19$
local minimum value of $f(-2, 1) = -19$

---

12. Absolute Extrema

In Exercises 71-78, find the absolute maximum and minimum values of f on the region R.

---

71. $f(x, y) = x^2 + xy + y^2 - 3x + 3y$

R is the triangular region cut from the first quadrant by the line $x+y=4$
Solution:
Let O$(0,0)$,A$(0,4)$,B$(4,0)$.
(i) On OA, $f(x,y)=f(0,y)=y^2+3y$ for $0\le y\le 4$
$\Longrightarrow f'(0,y)=2y+3=0\Longrightarrow y=-\dfrac{3}{2}$.
But $\left(0,-\dfrac{3}{2}\right)$ is not in the region.
Endpoints: $f(0,0)=0$ and $f(0,4)=28$.
(ii) On AB, $f(x,y)=f(x,-x+4)=x^2-10x+28$
for $0\le x\le 4\Longrightarrow f'(x,-x+4)=2x-10=0$
$\Longrightarrow x=5, y=-1$.
But $(5,-1)$ is not in the region.
Endpoints: $f(4,0)=4$ and $f(0,4)=28$.
(iii) On OB, $f(x,y)=f(x,0)=x^2-3x$ for $0\le x\le 4\Longrightarrow f'(x,0)=2x-3\Longrightarrow x=\dfrac{3}{2}$ and for $y=0\Longrightarrow\left(\dfrac{3}{2},0\right)$ is a critical point with for $f\left(\dfrac{3}{2},0\right)=-\dfrac{9}{4}$.
Endpoints: $f(0,0)=0$ and $f(4,0)=4$.
(iv) For the interior of the triangular region, $f_x(x,y)=2x+y-3=0$ and $f_y(x,y)=x+2y+3=0\Longrightarrow x=3$ and $y=-3$.
But $(3,-3)$ is not in the region. Therefore the absolute maximum is 28 at $(0,4)$ and the absolute minimum is $-\dfrac{9}{4}$ at $\left(\dfrac{3}{2},0\right)$.

---

72. $f(x,y)=x^2-y^2-2x+4y+1$
R is the rectangular region in the first quadrant bounded by the coordinate axes and the lines $x=4, y=2$
Solution:
Let O$(0,0)$,A$(0,2)$,B$(4,2)$,C$(4,0)$.
(i) On OA, $f(x,y)=f(0,y)=-y^2+4y+1$ for $0\le y\le 2$
$\Longrightarrow f'(0,y)=-2y+4=0\Longrightarrow y=2$.
But $(0,2)$ is not in the interior of OA.
Endpoints: $f(0,0)=1$ and $f(0,2)=5$.
(ii) On AB, $f(x,y)=f(x,2)=x^2-2x+5$
for $0\le x\le 4\Longrightarrow f'(x,2)=2x-2=0$
$\Longrightarrow x=1, y=2$.
$(1,2)$ is an interior critical point of AB with $f(1,2)=4$.
Endpoints: $f(1,2)=4$ and $f(0,2)=5$.
(iii) On BC, $f(x,y)=f(4,y)=-y^2+4y+9$ for $0\le y\le 2\Longrightarrow f'(4,y)=-2y+4=0\Longrightarrow$ $y=2$ and $x=4$
But $(4,2)$ is not in the interior of BC.
Endpoints: $f(4,0)=9$ and $f(4,2)=13$.
(iv) On OC, $f(x,y)=f(x,0)=x^2-2x+1$ for $0\le x\le 4\Longrightarrow f'(x,0)=2x-2\Longrightarrow x=1$ and $y=0\Longrightarrow(1,0)$ is an interior critical point of OC with $f(1,0)=0$.
Endpoints: $f(0,0)=1$ and $f(4,0)=9$.
(v) For the interior of the rectangular region, $f_x(x,y)=2x-2=0$ and $f_y(x,y)=-2y+4=0\Longrightarrow x=1$ and $y=2$.
But $(1,2)$ is not in the interior of the region. Therefore the absolute maximum is 13 at $(4,2)$ and the absolute minimum is 0 at $(1,0)$.

73. $f(x, y) = y^2 - xy - 3y + 2x$
R is the square enclosed by the lines $x = \pm 2, y = \pm 2$
Answer:
Absolute maximum: 18 at $(2, -2)$; absolute minimum is $-\dfrac{17}{4}$ at $\left(-2, \dfrac{1}{2}\right)$

---

74. $f(x, y) = 2x + 2y - x^2 - y^2$
R is the square region bounded by the coordinate axes and the lines $x = 2, y = 2$ in the first quadrant.
Answer:
Absolute maximum: 2 at $(1, 1)$; absolute minimum is 0 at the four corners $(0, 0), (0, 2), (2, 2)$ and $(2, 0)$

---

75. $f(x, y) = x^2 - y^2 - 2x + 4y$
R is the triangular region bounded below by the coordinate axes and the lines $x = 2, y = 2$ in the first quadrant.
Answer:
Absolute maximum: 8 at $(-2, 0)$; absolute minimum is -1 at $(1, 0)$.

---

76. $f(x, y) = 2x + 2y - x^2 - y^2$
R is the square region bounded by the coordinate axes and the lines $x = 2, y = 2$ in the first quadrant.
Answer:
Absolute maximum: 18 at $(1, 1)$; absolute minimum is -32 at $(2, -2)$.

---

77. $f(x, y) = 2x + 2y - x^2 - y^2$
R is the square region bounded by the coordinate axes and the lines $x = 2, y = 2$ in the first quadrant.
Answer:
Absolute maximum: 4 at $(1, 0)$; absolute minimum is -4 at $(0, -1)$.

---

78. $f(x, y) = 2x + 2y - x^2 - y^2$
R is the square region bounded by the coordinate axes and the lines $x = 2, y = 2$ in the first quadrant.

---

13. Lagrange Multipliers

79. Find the extreme values of $f(x, y) = x^2 + y^2$ on the circle $x^2 + y^2 = 1$.

---

80. Find the extreme values of $f(x, y) = xy$ on the circle $x^2 + y^2 = 1$.

---

81. Find the extreme values of $f(x, y) = x^2 + 3y^2 + 2y$ on the unit disk $x^2 + y^2 \leq 1$.

---

82. Find the extreme values of $f(x, y) = x^2 + y^2 - 3x - xy$ on the disk $x^2 + y^2 \leq 9$.

---

83. Find the extreme values of $f(x, y, z) = x - y + z$ on the unit sphere $x^2 + y^2 + z^2 = 1$.

---

84. Find the points on the surface $z^2 - xy = 4$ closest to the origin..

---

85. A closed rectangular box is to have volume V cm^3. The cost of the material used in the box is a cents/cm^2 for top and bottom, b cents/cm^2 for front and back, and c cents/cm^2 for the remaining sides. What dimensions minimize the total cost of materials?

---

86. Find the plane $x/a + y/b + z/c = 1$ that passes through the point $(2, 1, 2)$ and cuts off the least volume from the first octant.

---

87. Find the extreme values of $f(x, y, z) = x(y + z)$ on the curve of intersection of the right circular cylinder $x^2 + y^2 = 1$ and the hyperbolic cylinder $xz = 1$.

---

88. Find the point closest to the origin on the curve of intersection of the plane $x + y + z = 1$ and the cone $z^2 = 2x^2 + 2y^2$.

---

14. Partial Derivatives with Constrained Variables

In Exercises 89 and 90, begin by drawing a diagram that shows the relations among the variables.

---

89. If $w = x^2 e^{yz}$ and $z = x^2 - y^2$ find
a. $\left(\dfrac{\partial w}{\partial y}\right)_z$ **b.** $\left(\dfrac{\partial w}{\partial z}\right)_x$ **c.** $\left(\dfrac{\partial w}{\partial y}\right)_y$

Solution:
(a) y, z are independent with $w = x^2 e^{yz}$
and

$$z = x^2 - y^2 \Longrightarrow \frac{\partial w}{\partial y} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial y} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial y} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial y} = \left(2xe^{yz}\right)\frac{\partial x}{\partial y} + \left(zx^2e^{yz}\right)(1) + \left(yx^2e^{yz}\right)(0);$$

$$z = x^2 - y^2 \Longrightarrow 0 = 2x\frac{\partial x}{\partial y} - 2y \Longrightarrow \frac{\partial x}{\partial y} = \frac{y}{x};$$

therefore

$$\left(\frac{\partial w}{\partial y}\right)_z = \left(2xe^{yz}\right)\left(\frac{y}{x}\right) + zx^2e^{yz} = \left(2y + zx^2\right)e^{yz}$$

(b) z, x are independent with $w = x^2 e^{yz}$
and

$z=x^2-y^2 \Longrightarrow \dfrac{\partial w}{\partial z}=\dfrac{\partial w}{\partial x}\dfrac{\partial x}{\partial z}+\dfrac{\partial w}{\partial y}\dfrac{\partial y}{\partial z}+\dfrac{\partial w}{\partial z}\dfrac{\partial z}{\partial z}=(2xe^{yz})(0)+\left(zx^2e^{yz}\right)\dfrac{\partial y}{\partial z}+\left(yx^2e^{yz}\right)(1);$

$z=x^2-y^2 \Longrightarrow 1=0-2y\dfrac{\partial y}{\partial z}\Longrightarrow \dfrac{\partial y}{\partial z}=-\dfrac{1}{2y};$

therefore

$\left(\dfrac{\partial w}{\partial z}\right)_x=\left(zx^2e^{yz}\right)\left(-\dfrac{1}{2y}\right)+yx^2e^{yz}=x^2e^{yz}\left(y-\dfrac{z}{2y}\right)$

(c) z,y are independent with $w=x^2e^{yz}$

and

$z=x^2-y^2 \Longrightarrow \dfrac{\partial w}{\partial z}=\dfrac{\partial w}{\partial x}\dfrac{\partial x}{\partial z}+\dfrac{\partial w}{\partial y}\dfrac{\partial y}{\partial z}+\dfrac{\partial w}{\partial z}\dfrac{\partial z}{\partial z}=(2xe^{yz})\left(\dfrac{\partial x}{\partial z}\right)+\left(zx^2e^{yz}\right)(0)+\left(yx^2e^{yz}\right)(1);$

$z=x^2-y^2 \Longrightarrow 1=2x\dfrac{\partial x}{\partial z}-0\Longrightarrow \dfrac{\partial x}{\partial z}=\dfrac{1}{2x};$

therefore

$\left(\dfrac{\partial w}{\partial z}\right)_x=(2xe^{yz})\left(\dfrac{1}{2x}\right)+yx^2e^{yz}=\left(1+x^2\right)e^{yz}$

---

90. Let $U=f(P,V,T)$ be the internal energy of a gas that obeys the ideal gas law $PV=nRT$ ($n$and R are constants). Find

a. $\left(\dfrac{\partial U}{\partial T}\right)_P$ **b.** $\left(\dfrac{\partial U}{\partial V}\right)_T$

Solution:

(a) T,P are independent with $U=f(P,V,T)$ and $PV=nRT\Longrightarrow \dfrac{\partial U}{\partial T}=\dfrac{\partial U}{\partial P}\dfrac{\partial P}{\partial T}+\dfrac{\partial U}{\partial V}\dfrac{\partial V}{\partial T}+\dfrac{\partial U}{\partial T}\dfrac{\partial T}{\partial T}$

$=\dfrac{\partial U}{\partial P}(0)+\dfrac{\partial U}{\partial V}\dfrac{\partial V}{\partial T}+\dfrac{\partial U}{\partial T}(1)$

$PV=nRT\Longrightarrow P\dfrac{\partial V}{\partial T}=nR\Longrightarrow \dfrac{\partial V}{\partial T}=\dfrac{nR}{P};$

therefore,

$\left(\dfrac{\partial U}{\partial T}\right)_P=\left(\dfrac{\partial U}{\partial V}\right)\left(\dfrac{nR}{P}\right)+\dfrac{\partial U}{\partial T}$

(b) V,T are independent with $U=f(P,V,T)$ and $PV=nRT\Longrightarrow \dfrac{\partial U}{\partial V}=\dfrac{\partial U}{\partial P}\dfrac{\partial P}{\partial V}+\dfrac{\partial U}{\partial V}\dfrac{\partial V}{\partial V}+\dfrac{\partial U}{\partial T}\dfrac{\partial T}{\partial V}$

$=\left(\dfrac{\partial U}{\partial P}\right)\left(\dfrac{\partial P}{\partial V}\right)+\dfrac{\partial U}{\partial V}(1)+\dfrac{\partial U}{\partial T}(0)$

$PV=nRT\Longrightarrow V\dfrac{\partial P}{\partial V}+P=(nR)\dfrac{\partial T}{\partial V}=0\Longrightarrow;\frac{\partial P}{\partial V}=-\frac{P}{V}$

therefore,

$\left(\dfrac{\partial U}{\partial V}\right)_P=\left(\dfrac{\partial U}{\partial P}\right)\left(-\dfrac{P}{V}\right)+\dfrac{\partial U}{\partial V}$

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15. Theory and Examples

91. Let $w=f(r,\theta), r=\sqrt{x^2+y^2}$, and $\theta=\tan^{-1}(y/x)$. Find $\partial w/\partial x$ and $\partial w/\partial y$ and express your answers in terms of r and θ.

Solution:

Note that

$x = r\cos\theta$ and $y = r\sin\theta \Longrightarrow r = \sqrt{x^2+y^2}$ and $\theta = \tan^{-1}(y/x)$.

Thus

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial w}{\partial \theta}\frac{\partial \theta}{\partial x} = \frac{\partial w}{\partial r}\left(\frac{x}{\sqrt{x^2+y^2}}\right) + \frac{\partial w}{\partial \theta}\left(\frac{-y}{x^2+y^2}\right) = (\cos\theta)\frac{\partial w}{\partial r} - \left(\frac{\sin\theta}{r}\right)\frac{\partial w}{\partial \theta};$$

$$\frac{\partial w}{\partial y} = \frac{\partial w}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial w}{\partial \theta}\frac{\partial \theta}{\partial y} = \frac{\partial w}{\partial r}\left(\frac{y}{\sqrt{x^2+y^2}}\right) + \frac{\partial w}{\partial \theta}\left(\frac{x}{x^2+y^2}\right) = (\sin\theta)\frac{\partial w}{\partial r} - \left(\frac{\cos\theta}{r}\right)\frac{\partial w}{\partial \theta}$$

---

92. Let $z = f(u,v)\,, u = ax+by$, and $v = ax-by$. Express z_x and z_y in terms of f_u, f_v and the constants a and b.

Solution:

$z_x = f_u\dfrac{\partial u}{\partial x} + f_v\dfrac{\partial v}{\partial x} = af_u + af_v,$

and

$z_y = f_u\dfrac{\partial u}{\partial y} + f_v\dfrac{\partial v}{\partial y} = bf_u - bf_v,$

---

93. If a and b are constants, $w = u^3\tanh u + \cos u$, and $u = ax+by$, show that

$$a\frac{\partial w}{\partial y} = b\frac{\partial w}{\partial x}$$

Solution:

$\dfrac{\partial u}{\partial y} = b$ and $\dfrac{\partial u}{\partial x} = a \Longrightarrow \dfrac{\partial w}{\partial x} = \dfrac{dw}{du}\dfrac{\partial u}{\partial x} = a\dfrac{dw}{du}$

and

$\dfrac{\partial w}{\partial y} = \dfrac{dw}{du}\dfrac{\partial u}{\partial y} = b\dfrac{dw}{du} \Longrightarrow \dfrac{1}{a}\dfrac{\partial w}{\partial x} = \dfrac{dw}{du}$

and

$\dfrac{1}{b}\dfrac{\partial w}{\partial y} = \dfrac{dw}{du} \Longrightarrow \dfrac{1}{a}\dfrac{\partial w}{\partial x} = \dfrac{1}{b}\dfrac{\partial w}{\partial y} \Longrightarrow b\dfrac{\partial w}{\partial x} = a\dfrac{\partial w}{\partial y}$

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94. If $w = \ln\left(x^2+y^2+2z\right), x = r+s, y = r-s, z = 2rs$ find w_r and w_s by the Chain Rule. Then check your answer another way.

Solution:

$\dfrac{\partial w}{\partial x} = \dfrac{2x}{x^2+y^2+2z} = \dfrac{2(r+s)}{(r+s)^2+(r-s)^2+2(2rs)} = \dfrac{2(r+s)}{2(r^2+2rs+s^2)} = \dfrac{1}{r+s},$

$\dfrac{\partial w}{\partial y} = \dfrac{2y}{x^2+y^2+2z} = \dfrac{2(r-s)}{2(r+s)^2},$

and

$\dfrac{\partial w}{\partial z} = \dfrac{2}{x^2+y^2+2z} = \dfrac{1}{(r+s)^2} \Longrightarrow \dfrac{\partial w}{\partial r} = \dfrac{\partial w}{\partial x}\dfrac{\partial x}{\partial r} + \dfrac{\partial w}{\partial y}\dfrac{\partial y}{\partial r} + \dfrac{\partial w}{\partial z}\dfrac{\partial z}{\partial r} = \dfrac{1}{r+s} + \dfrac{r-s}{(r+s)^2} + \left[\dfrac{1}{(r+s)^2}\right](2s) = \dfrac{2r+2s}{(r+s)^2} = \dfrac{2}{r+s}$

and

$\dfrac{\partial w}{\partial s} = \dfrac{\partial w}{\partial x}\dfrac{\partial x}{\partial s} + \dfrac{\partial w}{\partial y}\dfrac{\partial y}{\partial s} + \dfrac{\partial w}{\partial z}\dfrac{\partial z}{\partial s} = \dfrac{1}{r+s} + \dfrac{r-s}{(r+s)^2} + \left[\dfrac{1}{(r+s)^2}\right](2r) = \dfrac{2}{r+s}$

---

95. The quations $e^u \cos v - x = 0$ and $e^u \sin v - y = 0$ define u and v as differentiable functions of x and u. Show that the angle between the vectors

$$\frac{\partial u}{\partial x}\mathbf{i} + \frac{\partial u}{\partial y}\mathbf{j} \qquad \text{and} \qquad \frac{\partial v}{\partial x}\mathbf{i} + \frac{\partial v}{\partial y}\mathbf{j}$$

is constant.

Solution:

$e^u \cos v - x = 0 \Longrightarrow (e^u \cos v)\dfrac{\partial u}{\partial x} - (e^u \sin v)\dfrac{\partial v}{\partial x} = 1$; $e^u \sin v - y = 0 \Longrightarrow (e^u \sin v)\dfrac{\partial u}{\partial x} - (e^u \cos v)\dfrac{\partial v}{\partial x} = 0$.

Solving this system yields $\dfrac{\partial u}{\partial x} = e^{-u}\cos v$ and $\dfrac{\partial v}{\partial x} = e^{-u}\sin v$.

Similarly, $e^u \cos v - x = 0 \Longrightarrow (e^u \cos v)\dfrac{\partial u}{\partial y} - (e^u \sin v)\dfrac{\partial v}{\partial y} = 0$

and

$e^u \sin v - y = 0 \Longrightarrow (e^u \sin v)\dfrac{\partial u}{\partial y} + (e^u \cos v)\dfrac{\partial v}{\partial y} = 1.$

Solving this system yields $\dfrac{\partial u}{\partial y} = e^{-u}\sin v$ and $\dfrac{\partial v}{\partial x} = e^{-u}\cos v$.

Therefore $\left(\dfrac{\partial u}{\partial x}\mathbf{i} + \dfrac{\partial u}{\partial y}\mathbf{j}\right)\cdot\left(\dfrac{\partial v}{\partial x}\mathbf{i} + \dfrac{\partial v}{\partial y}\mathbf{j}\right) = \left[\left(e^{-u}\cos v\right)\mathbf{i} + \left(e^{-u}\sin v\right)\mathbf{j}\right]\cdot\left[\left(-e^{-u}\sin v\right)\mathbf{i} + \left(e^{-u}\cos v\right)\mathbf{j}\right] = 0 \Longrightarrow$ the vectors are orthogonal $\Longrightarrow$ the angle between the vectors is the constant $\dfrac{\pi}{2}$.

---

96. Introducing polar coordinates $x = r\cos\theta$ and $y = r\sin\theta$ changes $f(x, y)$ to $g(r, \theta)$. Find the value of $\dfrac{\partial^2 g}{\partial x^2}$ at the point $(r, \theta) = (2, \pi/2)$, given that

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 f}{\partial y^2} = 1$$

at that point.

Solution:

$\dfrac{\partial g}{\partial \theta} = \dfrac{\partial f}{\partial x}\dfrac{\partial x}{\partial \theta} + \dfrac{\partial f}{\partial y}\dfrac{\partial y}{\partial \theta} = (-r\sin\theta)\dfrac{\partial f}{\partial x} + (r\cos\theta)\dfrac{\partial f}{\partial y}$

$\Longrightarrow \dfrac{\partial^2 g}{\partial \theta^2} = (-r\sin\theta)\left(\dfrac{\partial^2 f}{\partial x^2}\dfrac{\partial x}{\partial \theta} + \dfrac{\partial^2 f}{\partial y \partial x}\dfrac{\partial y}{\partial \theta}\right) - (r\cos\theta)\dfrac{\partial f}{\partial x} + (r\cos\theta)\left(\dfrac{\partial^2 f}{\partial y \partial x}\dfrac{\partial x}{\partial \theta} + \dfrac{\partial^2 f}{\partial y^2}\dfrac{\partial y}{\partial \theta}\right) - (r\sin\theta)\dfrac{\partial f}{\partial y}$

$= (-r\sin\theta)\left(\dfrac{\partial x}{\partial \theta} + \dfrac{\partial y}{\partial \theta}\right) - (r\cos\theta) + (r\cos\theta)\left(\dfrac{\partial x}{\partial \theta} + \dfrac{\partial y}{\partial \theta}\right) - (r\sin\theta)$

$= (-r\sin\theta + r\cos\theta)(-r\sin\theta + r\cos\theta) - (r\sin\theta + r\cos\theta) = (-2)(-2) - (0+2) = 4 - 2 = 2$ at $(r, \theta) = \left(2, \dfrac{\pi}{2}\right)$.

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97. Find the points on the surface

$$(y+z)^2 + (z-x)^2 = 16$$

where the normal line is parallel to the yz-plane.

Solution:

$(y+z)^2+(z-x)^2=16 \Longrightarrow \nabla f=-2(z-x)\mathbf{i}+2(y+z)\mathbf{j}+2(y+2z-x)\mathbf{k}$; if the normal line is parallel to the yz-plane, then x is constant $\Longrightarrow \dfrac{\partial f}{\partial x}=0 \Longrightarrow -2(z-x)=0 \Longrightarrow z=x \Longrightarrow (y+z)^2+(z-z)^2=16 \Longrightarrow y+z=\pm 4$.
Let $x=t \Longrightarrow z=t \Longrightarrow y=-t\pm 4$.
Therefore the points are $(t,-t\pm 4,t)$, t a real number.

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98. Find the points on the surface

$$xy+yz+zx-x-z^2=0$$

where the tangent plane is parallel to the xy-plane.
Solution:
Let $f(x,y,z)=xy+yz+zx-x-z^2=0$. If the tangent plane is parallel to the xy-plane, then ∇f is perpendicular to the xy-plane $\Longrightarrow \nabla f\cdot\mathbf{i}=0$ and $\nabla f\cdot\mathbf{j}=0$.
Now $\nabla f=(y+z-1)\mathbf{i}+(x+z)\mathbf{j}+(y+x-2z)\mathbf{k}$ so that $\nabla f\cdot\mathbf{i}=y+z-1=0 \Longrightarrow y+z=1 \Longrightarrow y=1-z$, and $\nabla f\cdot\mathbf{j}=x+z=0 \Longrightarrow x=-z$. Then
$-z(1-z)+(1-z)z+z(-z)-(-z)-z^2=0 \Longrightarrow z-2z^2=0 \Longrightarrow z=\dfrac{1}{2}$ or $z=0$.
Now $z=\dfrac{1}{2} \Longrightarrow x=-\dfrac{1}{2}$ and $y=\dfrac{1}{2} \Longrightarrow \left(-\dfrac{1}{2},\dfrac{1}{2},\dfrac{1}{2}\right)$ is one desired point; $z=0 \Longrightarrow x=0$ and $y=1 \Longrightarrow (0,1,0)$ is a second desired point.

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99. Suppose that $\nabla f(x,y,z)$ is always parallel to the position vector $x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$. Show that $f(0,0,a)=f(0,0,-a)$ for any a.
Solution:
$\nabla f=\lambda(x\mathbf{i}+y\mathbf{j}+z\mathbf{k}) \Longrightarrow \dfrac{\partial f}{\partial x}=\lambda x \Longrightarrow f(x,y,z)=\dfrac{1}{2}\lambda x^2+g(y,z)$ for some function g $\Longrightarrow \lambda y=\dfrac{\partial f}{\partial y}=\dfrac{\partial g}{\partial y} \Longrightarrow g(y,z)=\dfrac{1}{2}\lambda y^2+h(z)$ for some function $h \Longrightarrow \lambda z=\dfrac{\partial f}{\partial z}=\dfrac{\partial g}{\partial z}=h'(z) \Longrightarrow h(z)=\dfrac{1}{2}\lambda z^2+C$ for some arbitrary constant $C \Longrightarrow g(y,z)=\dfrac{1}{2}\lambda y^2+\left(\dfrac{1}{2}\lambda z^2+C\right) \Longrightarrow f(x,y,z)=\dfrac{1}{2}\lambda x^2+\dfrac{1}{2}\lambda y^2+\dfrac{1}{2}\lambda z^2+C \Longrightarrow f(0,0,a)=\dfrac{1}{2}\lambda a^2+C$ and $f(0,0,-a)=\dfrac{1}{2}\lambda(-a)^2+C \Longrightarrow f(0,0,a)=f(0,0,-a)$ for any constant a, as claimed.

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100. The one-sided directional derivative of f at $P(x_0,y_0,z_0)$ in the direction $u=u_1\mathbf{i}+u_2\mathbf{j}+u_3\mathbf{k}$ is the number

$$\lim_{s\to 0^+}\frac{f(x_0+su_1,y_0+su_2,z_0+su_3)-f(x_0,y_0,z_0)}{s}.$$

Show that the one-sided directional derivative of

$$f(x,y,z)=\sqrt{x^2+y^2+z^2}$$

at the origin equals 1 in any direction but that f has no gradient vector at the origin.
Solution:
$$\left(\frac{df}{ds}\right)_{\mathbf{u},\,(0,0,0)}=\lim_{s\to 0}\frac{f(0+su_1,0+su_2,0+su_3)-f(0,0,0)}{s},\ s>0$$
$$=\lim_{s\to 0}\frac{\sqrt{(su_1)^2+(su_2)^2+(su_3)^2}-0}{s},\ s>0$$

$= \lim_{s\to 0} \frac{s\sqrt{u_1^2+u_2^2+u_3^2}}{s} = \lim_{s\to 0} |\mathbf{u}| = 1;$

however, $\nabla f = \frac{x}{\sqrt{x^2+y^2+z^2}}\mathbf{i} + \frac{y}{\sqrt{x^2+y^2+z^2}}\mathbf{j} + \frac{z}{\sqrt{x^2+y^2+z^2}}\mathbf{k}$ fails to exist at the origin $(0,0,0)$.

---

101. Show that the line normal to the surface $xy + z = 2$ at the point $(1,1,1)$ passes through the origin.
Solution:

Let $f(x,y,z) = xy + z - 2 \Longrightarrow \nabla f = y\mathbf{i} + x\mathbf{j} + \mathbf{k}$.

At $(1,1,1)$, we have $\nabla f = \mathbf{i} + \mathbf{j} + \mathbf{k} \Longrightarrow$ the normal line is

$x = 1 + t, y = 1 + t, z = 1 + t$, so at $t = -1 \Longrightarrow x = 0, y = 0, z = 0$ and the normal line passes through the origin.

102.
a. Find a vector normal to the surface $x^2 - y^2 + z^2 = 4$ at $(2,-3,3)$.
b. Find equations for the tangent plane and normal line at $(2,-3,3)$.
Solution:
a. $f(x,y,z) = x^2 - y^2 + z^2 = 4$
$\Longrightarrow \nabla f = 2x\mathbf{i} - 2y\mathbf{j} + 2z\mathbf{k} \Longrightarrow$ at $(2,-3,3)$
the gradient is $\Longrightarrow \nabla f = 4\mathbf{i}+6\mathbf{j} + 6\mathbf{k}$ which is normal to the surface.
b. Tangent plane: $4x + 6y + 6z = 8$ or
$2x + 3y + 3z = 4$
Normal line: $x = 2 + 4t, y = -3 + 6t, z = 3 + 6t$
