ÇANKAYA UNIVERSITY

Department of Mathematics and Computer Science

MATH 237 Linear Algebra I

Final Exam Practice Problems B

January 7, 2008

13:00-14:50

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1. Let $L : \mathbb{R}^5 \longrightarrow \mathbb{R}^4$ be the linear transformation defined by

$$L(x, y, z, t, u) = (x - y, z - t, y - z - t + u, x - z, t - u).$$

(a) Is $(1, 1, 1, 1, 1)$ in $N(L)$?

(b) Is $(1, -1, 1, -1, 1)$ in $N(L)$?

(c) Is $(2, -1, 2)$ in $R(L)$?

(d) Is $(0, 0, 0, 0, 0)$ in $N(L)$?

(e) Is $(0, 0, 0)$ in $R(L)$?

(f) Is $(1, 0, 1)$ in $R(L)$?.

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2. Which conditions must be placed on $a, b,$ and c for (a, b, c, d) to be in the range of the operator L on $\mathbb{R}^4$ given by

$$L(x_1, x_2, x_3, x_4) = (2x_1 - x_2 + x_3 - x_4, x_2 + x_4, 2x_1 + x_3, x_1)$$

Use this to find a basis for $\operatorname{Im} L = R(T)$.

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3. Express the range of the linear transformation $L : \mathbb{R}^4 \longrightarrow \mathbb{R}^5$given by

$$L(x, y, z, t) = (2x + y + z, 2x - y - z, , x - z, z - t, y + z + t)$$

as the solution space of a homogeneous system.

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4. Is there a linear transformation $L : \mathbb{C}^3 \longrightarrow \mathbb{C}^4$ so that

$$L(1, 0, 0) = L(1, 0, i) = (1, 1, 1, 1), L(0, 1, 0) = (1, -1, 1, -1)?$$

If so find its nullspace and range as the solution spaces of homogeneous systems.

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5. What must be k if $(k, k + 1, k - 1)$ is in the image of the linear operator L on $\mathbb{R}^3$ defined by

$$L(x_1, x_2, x_3) = (x_1 + 2x_2 - x_3, -2x_1 + x_2 - x_3, -x_1 + 3x_2 - 2x_3)?$$

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6. Find a basis for the nullspace of the linear map $L : \mathbb{R}^4 \longrightarrow \mathbb{R}^2$ given by

$$L(x_1, x_2, x_3, x_4) = (x_1 + x_2, x_2 - x_4).$$

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7. Given the linear map $L : \mathrm{M}_{2\times 2}(\mathbb{R}) \longrightarrow \mathbb{R}^3$ defined by

$$L\begin{pmatrix} a & b \\ c & d \end{pmatrix} = (a+b+c, b+c+d, a+d)$$

(a) Find bases for the nullspace and range of L.
(b) Find its rank and nullity.

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8. Find the nullity and rank of the linear operator on $\mathrm{M}_{2\times 2}(\mathbb{R})$ given by $L(B) = AB - BA$ where

$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.

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9. Show that the differentiation operator is a linear operator on the subspace of functions spanned by $1, \sin x, \cos x, \sin 2x, \cos 2x$. Find its rank and nullity.

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10. Let L and T be the linear operators on $\mathbb{R}^4$ given by

$L(x_1, x_2, x_3, x_4) = (x_1 + x_2 - x_3, 2x_1 - x_2 + x_4, x_1 - 2x_2 + x_3 + x_4, 3x_2 - 2x_3 - x_4)$

$T(x_1, x_2, x_3, x_4) = (x_1 + x_2 + x_3 - x_4)(1, 1, 1, 1)$.

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11. Prove that the function

$$P(x) \longmapsto x^2 \frac{d}{dx}(P(x)) + x \int_0^1 P(t)\, dt$$

is a linear operator on real polynomial functions. Is this linear operator injective? Is it surjective?

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12. Prove that the mapping defined by
$L\left(a_0 + a_1 x + a_2 x^2\right) = (a_0 - a_1 + a_2, -2a_0 + a_1 + a_2, -3a_0 + 2a_1)$ is not onto.

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13. Prove that the mapping defined by
$L\left(a_0 + a_1 x + a_2 x^2\right) = (a_0 - a_1 + a_2, -2a_0 + a_1 + a_2, a_2)$ is an isomorphism between the space of polynomials of degree ≤ 2 and $\mathbb{R}^3$.

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14. Prove that the mapping $L : \mathbb{C} \longrightarrow \mathrm{M}_{2\times 2}(\mathbb{R})$ given by
$L(a + ib) = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$ is an injective linear transformation which preserves products.

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15. Which of the following form isomorphic pairs of vector spaces?
a) $\mathbb{R}^4$, b) $\mathrm{M}_{2\times 2}(\mathbb{R})$, c) $\mathrm{M}_{3\times 1}(\mathbb{R})$, $\mathrm{M}_{1\times 3}(\mathbb{R})$
e) the subspace of $\mathbb{R}^5$ spanned by $\{(1,1,1,1,1), (1,0,0,0,0), (0,1,0,0,0), (1,1,0,0,0), (1,1,1,0,0)\}$.

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16. Show that there is an isomorphism from W_1 to W_2 where
(a) W_1 is the solution space of

$x_1 + x_2 - x_3 + x_4 = 0$
$x_1 + x_2 + x_3 - x_4 = 0$
(b) W_2 is the vector space of real polynomials spanned by
$1 + x + x^2 + x^3, x - x^3 1 - x + x^2 + 3x^3$.
