ÇANKAYA UNIVERSITY

Department of Mathematics and Computer Science

MATH 237 Linear Algebra I

Final Exam Practice ProblemsC

January 7, 2008

13:00-14:50

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1. Let L and T be the linear transformations from $\mathbb{R}^3$ to $\mathbb{R}^2$ given by
$L(x_1, x_2, x_3) = (2x_1 - x_2 + x_3, x_1 - x_2 + x_3)$
$T(x_1, x_2, x_3) = (x_1 - x_3, x_2 - x_1)$.
(a) Compute $(-2L + T)(1, -1, 1)$.

(b) Find a basis for the nullspace and range of $L + 3T$.

(c) Find the rank of $L - T$.

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2. Show that the linear operators defined on $\mathbb{R}^3$ are linearly dependent:
$T_1(x_1, x_2, x_3) = (x_1 - x_2, -x_1, x_1 + 2x_2 + x_3)$
$T_2(x_1, x_2, x_3) = (-2x_2 + x_3, -2x_1 + x_2, 2x_1 + x_2)$
$T_3(x_1, x_2, x_3) = (x_1 + x_2 + x_3, x_1 + x_2 + x_3, -x_1 + x_2 + x_3)$.

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3. what must be k if the following linear maps from $\mathbb{R}^2$ to $\mathbb{R}^3$ are linearly dependent?
$T_1(x, y) = (x + y, x - 2y, x + ky)$
$T_2(x, y) = (kx + y, x - y, x + y)$
$T_3(x, y) = (x + y, x + ky, x + y)$.

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4. Find a basis for $\mathcal{L}\left(\mathbb{R}^3, \mathbb{C}\right)$ by giving the explicit formula $L(x_1, x_2, x_3)$ for each basis element.

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5. Use the standard bases to construct the canonical basis for $\mathcal{L}\left(\mathbb{R}^3, \mathbb{R}\right)$.

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6. Let D be the differentiation operator and let $\mathcal{J}$ be the integration operator
$\mathcal{J}(p) = \int_0^x p(t)\, dt$ on real polynomial functions.
(a) Prove that $-2D + 3\mathcal{J}$ is an injective operator.
(b) Find k for which $-D + k\mathcal{J}$ is not injective.

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7. Find a linear form on $\mathbb{R}^3$ so that

$$f(1, 0, 0) = 3, f(0, 1, 0) = -1, \text{ and } f(0, 0, 1) = 4$$

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8. Show that every linear form on $\mathbb{F}^n$ is of the form
$f(x_1, x_2, \cdots, x_n) = a_1x_1 + a_2x_2 + \cdots + a_nx_n$.

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9. Let be a vector space over a field $\mathbb{F}$ and let $L : V \longrightarrow \mathbb{F}^n$ be a linear transformation. Prove that the map $L_1 : V \longrightarrow \mathbb{F}$ defined by
$L_1(v) =$ the first component of $L(v)$
is a linear form on V.

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10. Find a linear form on $\mathbb{R}^3$ so that
$f(-1,1,-2) = 2, f(2,1,1) = f(0,0,1) = -1.$

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11. Is there a linear form on $\mathbb{C}^3$ so that
$f(i,1,-1) = i, f(1,1,1) = -i, f(3,2,0) = 1$?
If so find one of them.

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12. Is there a linear form on $\mathbb{C}^3$ so that
$f(i,1,-1) = i+1, f(i,-1,2) = -i-1, f(0,0,1) = 0$?
If so find one of them.

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13. Find a nonzero linear form f on $\mathbb{C}^3$ which annihilates $(1,1,-1)$ and $(-i,1,-1)$, that is to say determine f so that
$f(1,1,-1) = f(-i,1,-1) = 0.$

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14. Determine all linear forms on $\mathbb{R}^2$ which annihilate the subspace spanned by $(-1,1)$.

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15. Find the dual basis for the basis $\{(1,1,1),(1,1,0),(1,0,0)\}$ of $\mathbb{C}^3$.

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16. Find the dual basis for $\{1,x\}$ of the space of polynomials of degree ≤ 1.

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17. Given the set of vectors $v_1 = (0,1,1), v_2 = (1,0,1), v_3 = (1,1,0)$ and the linear forms

$$\begin{aligned} f_1 &= \frac{1}{2}(-x_1 + x_2 + x_3) \\ f_2 &= \frac{1}{2}(x_1 - x_2 + x_3) \\ f_3 &= \frac{1}{2}(x_1 + x_2 - x_3) \end{aligned}$$

Show that $\{v_1, v_2, v_3\}$ is a basis for V and $\{f_1, f_2, f_3\}$ is a basis for V^*.

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18. Let

$$\begin{aligned} f_1(x_1,x_2,x_3) &= 2x_1 - x_2 + x_3 \\ f_2(x_1,x_2,x_3) &= x_1 + x_2 \\ f_3(x_1,x_2,x_3) &= x_1 \end{aligned}$$

be linear forms on $\mathbb{R}^3$. Find a basis for $\mathbb{R}^3$ so that $\{f_1, f_2, f_3\}$ is the dual basis of this basis.

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19. Consider the following linear forms on real polynomial functions of degree ≤ 1:

$$f_1(p) = \int_{-1}^{0} p(t)\,dt, \quad f_2(p) = \int_{0}^{1} p(t)\,dt.$$

Prove that $\{f_1, f_2\}$ is a basis for the dual space by exhibiting the basis for polynomials of which it is the dual.

20. Prove that there are no real $n \times n$ matrices A and B such that $AB - BA = I$.

21. Prove that if $T : \mathrm{M}_{n\times n}(\mathbb{R}) \longrightarrow \mathbb{R}$ is a linear form for which $T(AB) = T(BA)$ for all $A, B \in \mathrm{M}_{2\times 2}(\mathbb{R})$ and $T(I) = 2$, then T is the trace function. Generalize this result to $\mathrm{M}_{n\times n}(\mathbb{R})$.

22. Let V be an n-dimensional vector space. Prove that for any basis $\{f_1, f_2, \cdots, f_n\}$ of the dual space $V^\star$. There exists a basis $\{v_1, v_2, \cdots, v_n\}$ for V such that

$$f_i(v_j) = \delta_{ij}.$$

23. Let V be a finite dimensional vector space and let v be a nonzero vector in V.
a) Prove that there exists $f \in V^\star$ such that $f(v) \neq 0$.
b) Prove that the map $L_v : V^\star \longrightarrow \mathbb{F}$ defined by $L_v(f) = f(v)$ is a linear form on $V^\star$
c) Prove that the map $L : V \longrightarrow (V^\star)^\star$ given by $L(v) = L_v$ is a vector space isomorphism whish allows us to view vectors in V as linear forms on $V^\star$.

24. Let V be a finite dimensional vector space and let $\{v_1, v_2, \cdots v_n\}$ and $\{f_1, f_2, \cdots f_n\}$ be bases for V and $V^\star$ respectively. Prove that the matrix A defined by

$$A_{ij} = f_i(v_j)$$

is invertible. Deduce from this fact that each basis $\mathcal{B}$ has a dual $\mathcal{B}^\star$ and for each basis $\widetilde{\mathcal{B}}$ of $V^\star$ there is a basis $\mathcal{B}$ for V such that $\widetilde{\mathcal{B}}$ is the dual of $\mathcal{B}$.

25. Let V be a vector space of dimension n, and let $\{f_1, f_2, \cdots f_n\}$be a basis for $V^\star$. Show that the map $T : V \longrightarrow \mathbb{F}^n$ defined by

$$T(v) = (f_1(v), f_2(v), \cdots, f_n(v))$$

is an isomorphism.

26. Let $L : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ and $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ be linear transformations defined by

$$\begin{aligned} L(x_1, x_2) &= (x_1 + x_2, x_1 - x_2, x_1) \\ T(x_1, x_2, x_3) &= (-x_1 + x_2, x_1 + x_3). \end{aligned}$$

Compute the products LT and TL.

27. Let $L : \mathbb{C}^3 \longrightarrow \mathbb{C}^4$ and $T : \mathbb{C}^3 \longrightarrow \mathbb{C}^3$ be linear transformations defined by

$$\begin{aligned} L(x_1, x_2, x_3) &= (x_1, x_1 + x_2, x_1 + x_2 + x_3, x_2 - x_3) \\ T(x_1, x_2, x_3) &= (x_1 + x_2, x_2 + x_3, x_3 + x_1). \end{aligned}$$

Compute each of the following when it is defined:
a) LT, b) TL, c) $LT + L$, d) $LT + T$, e) $L(T + I)$.

28. Show that the linear transformation $L : \mathbb{R}^4 \longrightarrow \mathrm{M}_{2\times 2}(\mathbb{R})$ given by

$$L(x_1, x_2, x_3, x_4) = \begin{bmatrix} x_1 & x_1 + x_2 \\ x_1 + x_2 + x_3 & x_1 + x_2 + x_4 \end{bmatrix}$$

is invertible. Find its inverse by describing $L^{-1}\left(\begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}\right)$.

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29. Show that

$$L\left(x_1, x_2, x_3, x_4\right) = \left(x_1 + x_2 - x_3, 2x_1 + x_2 + x_3, x_1 + x_2 + x_3, x_4\right)$$

is an invertible linear operator and compute its inverse.

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30. Let D be the differentiation operator on differentiable functions. Find $N\left(D^2 + D\right)$ and determine a basis for this nullspace

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31. Find an operator of the form $L = D^2 + aD + bI$ for which
a) $\cos x \in N\left(L\right)$
b) $x^2 \in N\left(L\right)$
c) $xe^x \in N\left(L\right)$.
