ÇANKAYA UNIVERSITY
Department of Mathematics and Computer Science

MATH 237
Fall 2007
Linear Algebra I
Make-up Exam
January 24, 2008
10:00-11:50

Surname : ______________________
Name : ______________________
ID # : ______________________
Department : ______________________
Section : ______________________
Instructor : ______________________
Signature : ______________________

- The exam consists of 6 questions.
- Please read the questions carefully and write your answers under the corresponding questions. Be neat.
- Show all your work. Correct answers without sufficient explanation might not get full credit.
- Calculators are not allowed.

GOOD LUCK!

Please do not write below this line.

Q1	Q2	Q3	Q4	Q5	Q6	TOTAL
20	15	21	20	20	14	110

1. (20 pts.) Mark each of the following assertions True (T) or False (F). Justify your answer: give a proof or a counterexample.

a) The function $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ defined by $T(a_1, a_2) = (2a_1 + 3a_2, 7a_1 - 5a_2)$ is one-to-one.

b) If $T_b : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ is given by $T_b(x, y) = (x, 0, 0)$, then $\operatorname{rank}(T_b) = 1$.

c) The function $T_c : \mathbb{R}^2 \longrightarrow \mathbb{R}$ defined by $T_c(x, y) = x + 2y$ is an onto linear transformation.

d) There exists a linear transformation $T : \mathbf{P}_2(\mathbb{R}) \longrightarrow \mathbf{M}_{2\times 1}(\mathbb{R})$ for which

$T\left(1 + x + x^2\right) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, T\left(1 + x^2\right) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, T(x) = \begin{bmatrix} 2 \\ 3 \end{bmatrix}.$

2.(15 pts.) Let $\mathbb{F}$ be a field.

Define $T : \mathbf{M}_{2\times2}(\mathbb{F}) \longrightarrow \mathbf{P}_1(\mathbb{R})$ by $T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ax - d.$

Determine if T is a linear transformation. Give a proof.

3.(21 pts.) Let $T : \mathbf{P}_2(\mathbb{R}) \longrightarrow \mathbf{P}_3(\mathbb{R})$ be the linear transformation defined by

$$T(p(x)) = xp(x-3).$$

Consider the following two bases for $\mathbf{P}_2(\mathbb{R})$ and $\mathbf{P}_3(\mathbb{R})$ respectively.
$\mathcal{B} = \left\{1, x, x^2\right\}, \mathcal{D} = \left\{1, x-3, (x-3)^2, (x-3)^3\right\}$.
(a) Find $[T]_{\mathcal{B}}^{\mathcal{D}}$.
(b) Use part (a) to calculate $T\left(a + bx + cx^2\right)$ expressed in the basis $\mathcal{D}$.

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4. (20 pts.)

a) Let $T : \mathbf{P}_2(\mathbb{R}) \longrightarrow \mathbf{P}_2(\mathbb{R})$ be defined by

$$T(x+1) = x, T(x-1) = 1, T\left(x^2\right) = 0$$

Find $T\left(2 + 3x - x^2\right)$.

b) Let $T : V \longrightarrow W, S : W \longrightarrow U$ be linear transformations. Suppose that T and S are both onto. Show that $S \circ T$ is also onto.

5.(20 pts.) Consider the linear tansformation $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ such that

$$T\left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}, T\left(\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ 0 \\ -2 \end{bmatrix}, T\left(\begin{bmatrix} -2 \\ 7 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}.$$

a) Find $T\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right)$.

b) Find a basis for $N(T)$, compute the nullity of T.

c) Find a basis for $R(T)$, compute the rank of T

d) Determine whether T is one-to-one or onto.

6.(14 pts.) Let V be a vector space and let $T : V \longrightarrow V$ be linear. Prove that $T \circ T = T_0$ if and only if $R(T) \subseteq N(T)$.
