ÇANKAYA UNIVERSITY
Department of Mathematics and Computer Science

MATH 237
Fall 2007
Linear Algebra I

Final Exam
January 7, 2008
13:00-14:50

Surname : ______________________
Name : ______________________
ID # : ______________________
Department : ______________________
Section : ______________________
Instructor : ______________________
Signature : ______________________

- The exam consists of 6 questions.
- Please read the questions carefully and write your answers under the corresponding questions. Be neat.
- Show all your work. Correct answers without sufficient explanation might not get full credit.
- Calculators are not allowed.

GOOD LUCK!

Please do not write below this line.

Q1	Q2	Q3	Q4	Q5	Q6	TOTAL
20	15	21	20	20	14	110

1. (20 pts.) Mark each of the following assertions True (T) or False (F). Justify your answer: give a proof or a counterexample.

a) The function $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ defined by $T(a_1, a_2) = (1, a_2)$ is linear.

Solution:
FALSE. Since $T(0,0) = (1,0) \neq (0,0)$, T does not take the zero vector to the zero vector, and so T cannot be linear.

b) If $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is given by $T(a_1, a_2, a_3) = (a_1 - a_2, 2a_3)$, then $\operatorname{rank}(T) = 1$.

Solution:
FALSE

$$\mathit{span}\{T(1,0,0), T(0,1,0), T(0,0,1)\} = \mathit{span}\{(1,0), (-1,0), (0,2)\} = \mathit{span}\{(1,0), (0,2)\}.$$

Hence $\operatorname{rank}(T) = 2$.

c) The function $h : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $h(x) = x^2$ is a linear transformation.

Solution:
FALSE. Since $h(1+1) = 2^2 \neq h(1) + h(1)$.

d) There exists a linear transformation $T : \mathbf{M}_{3\times 1}(\mathbb{R}) \longrightarrow \mathbf{M}_{2\times 1}(\mathbb{R})$ for which

$$T\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, T\left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, T\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 3 \end{bmatrix}.$$

Solution:
FALSE The three conditions together violate linearity. Note that the vectors in the two last equations add up to the vector in the first equation.

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2. (15 pts.) Let $\mathbb{F}$ be a field.

Define $T : \mathbf{M}_{2\times 2}(\mathbb{F}) \longrightarrow \mathbf{M}_{2\times 1}(\mathbb{F})$ by $T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a \\ d \end{bmatrix}$.

Determine if T is a linear transformation. Give a proof.

Solution:

Let $\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}, \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} \in \mathbf{M}_{2\times 2}(\mathbb{F})$, $\lambda \in \mathbb{F}$.

$$T\left(\lambda \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} + \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}\right) = T\left(\begin{bmatrix} \lambda a_1 + a_2 & \lambda b_1 + b_2 \\ \lambda c_1 + c_2 & \lambda d_1 + d_2 \end{bmatrix}\right) = \begin{bmatrix} \lambda a_1 + a_2 \\ \lambda d_1 + d_2 \end{bmatrix}$$

$$= \lambda \begin{bmatrix} a_1 \\ d_1 \end{bmatrix} + \begin{bmatrix} a_2 \\ d_2 \end{bmatrix} = \lambda T\left(\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}\right) + T\left(\begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}\right).$$

Thus T is linear.

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3. (21 pts.) Let $T : \mathbb{R}^2 \longrightarrow \mathbf{P}_2(\mathbb{R})$ be a linear transformation such that $T(-1,4) = x^2 - 3$ and $T(-2,9) = x + 1$

(a) Find $T(7,-2)$

(b) Find a vector $v = (a_1, a_2) \in \mathbb{R}^2$ for which $T(v) = 3x^2 - 2x - 11$.

(c) Find a polynomial $p(x) \in \mathbf{P}_2(\mathbb{R})$ but $p(x) \notin R(T)$.

Solution:

a)

First note that $\{(-1,4),(-2,9)\}$ is a basis for $\mathbb{R}^2$. Then we find the scalars c_1, c_2 satisfying $(7,-2) = c_1(-1,4) + c_2(-2,9)$.

This is true if and only if $\begin{array}{c} -c_1 - 2c_2 = 7 \\ 4c_1 + 9c_2 = -2 \end{array}$, i.e., $\begin{array}{c} -4c_1 - 8c_2 = 28 \\ 4c_1 + 9c_2 = -2 \end{array} \Longrightarrow \begin{array}{c} c_1 = -59 \\ c_2 = 26 \end{array}$.

Hence

$T(7,-2) = (-59)\,T(-1,4) + (26)\,T(-2,9) = (-59)\left(x^2 - 3\right) + (26)(x+1)$

$= -59x^2 + 26x + 203.$

b)

The vector $3x^2 - 2x - 11 \in span\left(\left\{x^2 - 3, x + 1\right\}\right)$ iff $3x^2 - 2x - 11 = c_1\left(x^2 - 3\right) + c_2(x+1)$.

This is possible iff $3x^2 - 2x - 11 = c_1x^2 + c_2x + c_2 - 3c_1$.

Hence $c_1 = 3, c_2 = -2$.

Thus $v = 3(-1,4) + (-2)(-2,9) = (1,-6)$.

c)

We want $p(x) \notin span\left\{x^2 - 3, x + 1\right\}$.

For this, $ax^2 + bx + c \in span\left(\left\{x^2 - 3, x + 1\right\}\right)$ iff $ax^2 + bx + c = \lambda_1\left(x^2 - 3\right) + \lambda_2(x+1) \Longleftrightarrow a = \lambda_1, b = \lambda_2, c = -3\lambda_1 + \lambda_2$.

Now let $\lambda_1 = \lambda_2 = 1$. Then $a = b = 1, c = -2$.

Now it is easy to check that $p(x) = x^2 + x + 1 \notin span\left(\left\{x^2 - 3, x + 1\right\}\right)$, i.e., $p(x) = x^2 + x + 1 \notin R(T)$.

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4. (20 pts.) Let $T : \mathbf{M}_{2\times 2}(\mathbb{R}) \longrightarrow \mathbf{P}_1(\mathbb{R})$ be the linear transformation defined by

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = (a+d) + (b+c)\,x.$$

Consider the following two bases for $\mathbf{M}_{2\times 2}(\mathbb{R})$ and $\mathbf{P}_1(\mathbb{R})$ respectively:

$$\beta = \left\{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}\right\} \text{ and } \gamma = \{1, 1+x\}.$$

a) Find $[T]_\beta^\gamma$

b) Find $\left[T\left(\begin{bmatrix} 3 & 1 \\ 7 & 5 \end{bmatrix}\right)\right]_\gamma$ only by using part a).

Solution:

a)

$$T\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 2 = c_1 + c_2(1+x) = 2 + 0(1+x) \Longleftrightarrow c_1 = 2, c_2 = 0.$$

$$T\left(\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\right) = 1 + x = d_1 + d_2(1+x) = 0 + 1(1+x) \Longleftrightarrow d_1 = 0, d_2 = 1.$$

$$T\left(\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}\right) = 1 + x = g_1 + g_2(1+x) = 0 + 1(1+x) \Longleftrightarrow g_1 = 0, g_2 = 1.$$

$$T\left(\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}\right) = 1 + 2x = f_1 + f_2(1+x) = -1 + 2(1+x) \Longleftrightarrow f_1 = -1, f_2 = 2.$$

$$[T]_\beta^\gamma = \begin{bmatrix} 2 & 0 & 0 & -1 \\ 0 & 1 & 1 & 2 \end{bmatrix}.$$

b)

$$T\left(\begin{bmatrix} 3 & 1 \\ 7 & 5 \end{bmatrix}\right) = (3+5) + (1+7)\,x = 8 + 8x = 0 + 8(1+x) \Longrightarrow \left[T\left(\begin{bmatrix} 3 & 1 \\ 7 & 5 \end{bmatrix}\right)\right]_\gamma = \begin{bmatrix} 0 \\ 8 \end{bmatrix}$$

or by using part a), we get

$$\left[T\left(\begin{bmatrix} 3 & 1 \\ 7 & 5 \end{bmatrix}\right)\right]_\gamma = [T]_\beta^\gamma \left[\begin{bmatrix} 3 & 1 \\ 7 & 5 \end{bmatrix}\right]_\beta = \begin{bmatrix} 2 & 0 & 0 & -1 \\ 0 & 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} -2 \\ 5 \\ 11 \\ -4 \end{bmatrix} = \begin{bmatrix} 0 \\ 8 \end{bmatrix}$$

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5. (20 pts.) Consider the linear tansformation $T : \mathbf{P}_3(\mathbb{R}) \longrightarrow \mathbb{R}^2$ given by

$$T(p) = (p''(0), p'(0))$$

a) Find a basis for $N(T)$, compute the nullity of T.
b) Find a basis for $R(T)$, compute the rank of T
c) Determine whether T is one-to-one or onto.

Solution

a)
Let $p(x) = a + bx + cx^2 + dx^3 \in \mathbf{P}_3(\mathbb{R})$.

Then $p'(x) = b + 2cx + 3dx^2, p''(x) = 2c + 6dx$, and so $p'(0) = b, p''(0) = 2c$.

Thus we see that

$p(x) \in N(T) \iff (p''(0), p'(0)) = (0,0) \iff (b, 2c) = (0,0) \iff b = c = 0$, and a, d can be anything.

Therefore $p(x) \in N(T) \iff p(x) = a + dx^3 \in span(\{1, x^3\}) = N(T)$. So $\{1, x^3\}$ is a basis for $N(T)$.

b)

We have

$$R(T) = span(\{T(x), T(x^2)\}) = span(\{(0,1), (2,0)\}) = \mathbb{R}^2.$$

We conclude that $\{(0,1), (2,0)\}$ is a basis for $R(T)$. Thus $\text{rank}(T) = 2$.

c)
Since $\text{nullity}(T) = 2 \neq 0$, T is not one-to-one.

Nevertheless, $\text{rank}(T) = 2 = \dim(\mathbb{R}^2) \implies R(T) = \mathbb{R}^2$ shows that T is onto.

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6. (14 pts.) Suppose $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is a linear transformation with $N(T) = R(T)$.
(a) Show that $T(T(v)) = 0$ for all $v \in \mathbb{R}^2$.
(b) Show that there is a vector $v \in \mathbb{R}^2$ such that $\{v, T(v)\}$ is a basis for $\mathbb{R}^2$. (Hint: show first that $T(v) \neq 0$ for some $v \in \mathbb{R}^2$.)
(c) Determine $[T]_\beta$ for the ordered basis $\beta = \{v, T(v)\}$ from part (b)

Solution:

(a)
For any $v \in \mathbb{R}^2$, we have $T(v) \in R(T)$. Since $N(T) = R(T)$, we get $T(v) \in N(T)$. Hence $T(T(v)) = 0$ and this holds for each $v \in \mathbb{R}^2$.

(b)
If $T(v) = 0$ for each $v \in \mathbb{R}^2$, then $N(T) = \mathbb{R}^2$ and $R(T) = \{0\}$, but this is contrary to the hypothesis that $N(T) = R(T)$. So $T(v) \neq 0$ for some $v \in \mathbb{R}^2$. We now consider the subset $\{v, T(v)\}$. This set is not linearly dependent, for if $T(v) = cv$ for some scalar $c \neq 0$. We apply T to both sides $T(T(v)) = cT(v)$ and so we get $0 = cT(v) \Longrightarrow T(v) = 0$, contradiction. So $T(v) \neq cv$ for any scalar c. Hence the subset $\{v, T(v)\}$ must be linearly independent. Since $\dim\left(\mathbb{R}^2\right) = 2$, we see that $\{v, T(v)\}$ is a basis for $\mathbb{R}^2$.

(c)
We compute images of each vector in the basis $\beta = \{v, T(v)\}$. Thus $T(v) = (0)v + (1)T(v)$ and also $T(T(v)) = 0 = (0)v + (0)T(v)$. Thus the matrix that represents T relative to the basis is $\beta = \{v, T(v)\}$

$$[T]_\beta = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$