ÇANKAYA UNIVERSITY

Department of Mathematics and Computer Science

MATH 237
Fall 2007
Linear Algebra I

Final Exam

January 7, 2008
13:00-14:50

Surname : ____________________
Name : ____________________
ID # : ____________________
Department : ____________________
Section : ____________________
Instructor : ____________________
Signature : ____________________

- The exam consists of 6 questions.
- Please read the questions carefully and write your answers under the corresponding questions. Be neat.
- Show all your work. Correct answers without sufficient explanation might <u>not</u> get full credit.
- Calculators are <u>not</u> allowed.

GOOD LUCK!

Please do <u>not</u> write below this line.

Q1	Q2	Q3	Q4	Q5	Q6	TOTAL
20	15	21	20	20	14	110

1. (20 pts.) Mark each of the following assertions True (T) or False (F). Justify your answer: give a proof or a counterexample.

a) The function $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ defined by $T(a_1, a_2) = (1, a_2)$ is linear.

b) If $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is given by $T(a_1, a_2, a_3) = (a_1 - a_2, 2a_3)$, then $\operatorname{rank}(T) = 1$.

c) The function $h : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $h(x) = x^2$ is a linear transformation.

d) There exists a linear transformation $T : \mathbf{M}_{3\times 1}(\mathbb{R}) \longrightarrow \mathbf{M}_{2\times 1}(\mathbb{R})$ for which

$$T\left(\begin{bmatrix}1\\1\\1\end{bmatrix}\right) = \begin{bmatrix}1\\1\end{bmatrix}, T\left(\begin{bmatrix}1\\0\\1\end{bmatrix}\right) = \begin{bmatrix}1\\2\end{bmatrix}, T\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right) = \begin{bmatrix}2\\3\end{bmatrix}.$$

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2. (15 pts.) Let $\mathbb{F}$ be a field.

Define $T : \mathbf{M}_{2\times 2}(\mathbb{F}) \longrightarrow \mathbf{M}_{2\times 1}(\mathbb{F})$ by $T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a \\ d \end{bmatrix}$.

Determine if T is a linear transformation. Give a proof..

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3. (21 pts.) Let $T : \mathbb{R}^2 \longrightarrow \mathbf{P}_2(\mathbb{R})$ be a linear transformation such that $T(-1,4) = x^2 - 3$ and $T(-2,9) = x + 1$

(a) Find $T(7,-2)$

(b) Find a vector $v = (a_1, a_2) \in \mathbb{R}^2$ for which $T(v) = 3x^2 - 2x - 11$.

(c) Find a polynomial $p(x) \in \mathbf{P}_2(\mathbb{R})$ but $p(x) \notin R(T)$..

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4. (20 pts.) Let $T : \mathbf{M}_{2\times 2}(\mathbb{R}) \longrightarrow \mathbf{P}_1(\mathbb{R})$ be the linear transformation defined by

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = (a+d) + (b+c)\,x.$$

Consider the following two bases for $\mathbf{M}_{2\times 2}(\mathbb{R})$ and $\mathbf{P}_1(\mathbb{R})$ respectively:

$$\beta = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \right\} \text{ and } \gamma = \{1, 1+x\}.$$

a) Find $[T]_\beta^\gamma$

b) Find $\left[T\left(\begin{bmatrix} 3 & 1 \\ 7 & 5 \end{bmatrix}\right)\right]_\gamma$ only by using part a)..

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5. (20 pts.) Consider the linear tansformation $T : \mathbf{P}_3(\mathbb{R}) \longrightarrow \mathbb{R}^2$ given by

$$T(p) = (p''(0), p'(0))$$

a) Find a basis for $N(T)$, compute the nullity of T.
b) Find a basis for $R(T)$, compute the rank of T
c) Determine whether T is one-to-one or onto.

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6. (14 pts.) Suppose $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is a linear transformation with $N(T) = R(T)$.
(a) Show that $T(T(v)) = 0$ for all $v \in \mathbb{R}^2$.
(b) Show that there is a vector $v \in \mathbb{R}^2$ such that $\{v, T(v)\}$ is a basis for $\mathbb{R}^2$. (Hint: show first that $T(v) \neq 0$ for some $v \in \mathbb{R}^2$.)
(c) Determine $[T]_\beta$ for the ordered basis $\beta = \{v, T(v)\}$ from part (b).