ÇANKAYA UNIVERSITY
Department of Mathematics and Computer Science

MATH 237
Linear Algebra I

1^{st} Midterm
November 13, 2007
17:40-19:00

Surname : ______________________
Name : ______________________
ID # : ______________________
Department : ______________________
Section : ______________________
Instructor : ______________________
Signature : ______________________

- The exam consists of 6 questions.
- Please read the questions carefully and write your answers under the corresponding questions. Be neat.
- Show all your work. Correct answers without sufficient explanation might not get full credit.
- Calculators are not allowed.

GOOD LUCK!

Please do not write below this line.

Q1	Q2	Q3	Q4	Q5	Q6	TOTAL
20	20	20	20	20	10	110

1. Given the system $\mathcal{S}$ of linear equations

$$\begin{aligned} x_1 + 2x_2 + ax_3 + 2x_4 &= 1 \\ x_1 + 3x_3 + 4x_4 &= b \\ 2x_1 + x_2 + (a+b)\,x_3 + 7x_4 &= 2 \end{aligned}$$

i) Find the value(s) of a and b so that the system has
- a) no solution;
- b) a unique solution;
- c) infinitely many solutions.

ii) Find all solutions in the case $a = 7, b = 1$.

Solution:

(i)

$$\left[\begin{array}{cccc|c} 1 & 2 & a & 2 & 1 \\ 1 & 0 & 3 & 4 & b \\ 2 & 1 & a+b & 7 & 2 \end{array}\right] \longrightarrow \left[\begin{array}{cccc|c} 1 & 2 & a & 2 & 1 \\ 0 & -2 & 3-a & 2 & b-1 \\ 0 & -3 & -a+b & 3 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cccc|c} 1 & 2 & a & 2 & 1 \\ 0 & 1 & 3-b & -1 & b-1 \\ 0 & -3 & -a+b & 3 & 0 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{cccc|c} 1 & 2 & a & 2 & 1 \\ 0 & 1 & 3-b & -1 & b-1 \\ 0 & 0 & 9-a-2b & 0 & 3b-3 \end{array}\right]$$

This last matrix has no bad row iff $9 - a - 2b = 0$ and $3b - 3 = 0$, i.e., if $a = 7, b = 1$.

a) The system has no solution if $9 - a - 2b = 0$ and $b \neq 1$.

b) The system has infinitely many solutions if $a = 7$ and $b \neq 1$.

c) There are no values of a and b that yields unique solution.

(ii)

$$\left[\begin{array}{cccc|c} 1 & 2 & 7 & 2 & 1 \\ 0 & 1 & 3-1 & -1 & 1-1 \\ 0 & 0 & 9-7-2 & 0 & 3\,(1)-3 \end{array}\right] \longrightarrow \left[\begin{array}{cccc|c} 1 & 2 & 7 & 2 & 1 \\ 0 & 1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cccc|c} 1 & 0 & 3 & 4 & 1 \\ 0 & 1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{array}{ccccl} x_1 & & +3u_1 & +4u_2 & = 1 \\ & x_2 & +2u_1 & -u_2 & = 0 \end{array} \qquad \begin{array}{ll} x_1 & = 1 - 3u_1 - 4u_2 \\ x_2 & = -2u_1 + u_2 \end{array}$$

$$A = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 - 3u_1 - 4u_2 \\ -2u_1 + u_2 x_3 \\ u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + u_1 \begin{bmatrix} -3 \\ -2 \\ 1 \\ 0 \end{bmatrix} + u_2 \begin{bmatrix} -4 \\ 1 \\ 0 \\ 1 \end{bmatrix} \text{ where } u_1, u_2 \in \mathbb{R}$$

2.

a) Compute the following product

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ -5 & -4 & -3 & -2 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{bmatrix}$$

No explanation is necessary.

b) Let U be the matrix below. Find all solutions to the homogeneous system $Ux = 0$.

$$U = \begin{bmatrix} 1 & 1 & 1 & -2 & 0 \\ 0 & 0 & 1 & 7 & 5 \\ 0 & 0 & 0 & 0 & 7 \end{bmatrix}.$$

Solution:

a)

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ -5 & -4 & -3 & -2 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 1+2+3+4+5 & 0+2+6+12+20 \\ -5-4-3-2-1 & 0-4-6-6-4 \end{bmatrix} = \begin{bmatrix} 15 & 40 \\ -15 & -20 \end{bmatrix}.$$

b)

$$U = \begin{bmatrix} 1 & 1 & 1 & -2 & 0 \\ 0 & 0 & 1 & 7 & 5 \\ 0 & 0 & 0 & 0 & 7 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 0 & -9 & -5 \\ 0 & 0 & 1 & 7 & 5 \\ 0 & 0 & 0 & 0 & 7 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 0 & -9 & 0 \\ 0 & 0 & 1 & 7 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{aligned} x_1 + u_1 - 9u_2 &= 0 \\ x_3 + 7u_2 &= 0 \\ x_5 &= 0 \end{aligned} \implies \begin{aligned} x_1 &= -u_1 + 9u_2 \\ x_3 &= -7u_2 \\ x_5 &= 0 \end{aligned}$$

Thus all solutions are of the form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -u_1 + 9u_2 \\ u_1 \\ -7u_2 \\ u_2 \\ 0 \end{bmatrix} = u_1 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + u_2 \begin{bmatrix} 9 \\ 0 \\ -7 \\ 1 \\ 0 \end{bmatrix} \text{ where } u_1, u_2 \in \mathbb{R}$$

3. Compute the determinant

$$\det \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}.$$

Note. You must show your work to receive credit for this problem.

Solution:

$$\det \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} = - \begin{vmatrix} -1 & 2 & -1 & 0 \\ 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{vmatrix} = - \begin{vmatrix} -1 & 2 & -1 & 0 \\ 0 & 3 & -3 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{vmatrix}$$

$$= (-1)\,(-1) \begin{vmatrix} -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 3 & -3 & 0 \\ 0 & 0 & -1 & 2 \end{vmatrix} = (-1)\,(-1) \begin{vmatrix} -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & 3 & -3 \\ 0 & 0 & -1 & 2 \end{vmatrix}$$

$$= (-1)\,(-1)\,(-1) \begin{vmatrix} -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 3 & -3 \end{vmatrix}$$

$$= (-1)\,(-1)\,(-1) \begin{vmatrix} -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & 3 \end{vmatrix} = (-1)\,(-1)\,(-1)\,(-1)\,(-1)\,(-1)\,3 = 3.$$

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4. Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 5 & 6 \\ 2 & 6 & c \end{bmatrix}$.

a) Find the number c that makes this matrix not invertible.
b) If $c = 20$ factor the matrix into $A = LU$ (lower triangular L and upper triangular U).
c) If $B^2 = 0$, the zero matrix, explain why B is not invertible.

Solution:

a)

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 5 & 6 \\ 2 & 6 & c \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & 3 & 3 \\ 0 & 2 & c-6 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 2 & c-6 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & c-8 \end{bmatrix}$$

Hence if $c = 10$, then the matrix A is not invertible.

b) Using our multipliers from our elimination

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 5 & 6 \\ 2 & 6 & 20 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 2/3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 3 & 3 \\ 0 & 0 & 12 \end{bmatrix}$$

c) If B were invertible with inverse B^{-1}, then $B^{-1}B^2 = B^{-1}0$, this would imply that $B = 0$, itself the zero matrix. But the zero matrix cannot be invertible.

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5. Let

$$A = \left[\begin{array}{rrrr} 2 & 1 & -4 & 11 \\ 1 & -2 & -7 & 3 \\ -3 & 1 & 11 & -14 \end{array}\right] \text{ and } B = \left[\begin{array}{rrrr} 3 & -2 & -13 & 13 \\ 1 & 4 & 5 & 9 \\ -7 & 9 & 39 & -26 \end{array}\right].$$

a) Show that A and B have the same row reduced echelon form R.
b) Find invertible matrices P_1 and P_2 such that $R = P_1 A$ and $R = P_2 B$.
c) Find an invertible matrix P such that $A = PB$.

Solution:

a) $$\left[\, A \mid I \,\right] = \left[\begin{array}{rrrr|rrr} 2 & 1 & -4 & 11 & 1 & 0 & 0 \\ 1 & -2 & -7 & 3 & 0 & 1 & 0 \\ -3 & 1 & 11 & -14 & 0 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{rrrr|rrr} 1 & -2 & -7 & 3 & 0 & 1 & 0 \\ 2 & 1 & -4 & 11 & 1 & 0 & 0 \\ -3 & 1 & 11 & -14 & 0 & 0 & 1 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{rrrr|rrr} 1 & -2 & -7 & 3 & 0 & 1 & 0 \\ 0 & 5 & 10 & 5 & 1 & -2 & 0 \\ 0 & -5 & -10 & -5 & 0 & 3 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{rrrr|rrr} 1 & -2 & -7 & 3 & 0 & 1 & 0 \\ 0 & 5 & 10 & 5 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{rrrr|rrr} 1 & -2 & -7 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & \frac{1}{5} & \frac{-2}{5} & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{rrrr|rrr} 1 & 0 & -3 & 5 & \frac{2}{5} & \frac{1}{5} & 0 \\ 0 & 1 & 2 & 1 & \frac{1}{5} & \frac{-2}{5} & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{array}\right] = R \text{ so } P_1 = \left[\begin{array}{rrr} \frac{2}{5} & \frac{1}{5} & 0 \\ \frac{1}{5} & \frac{-2}{5} & 0 \\ 1 & 1 & 1 \end{array}\right]$$

$$\left[\, B \mid I \,\right] = \left[\begin{array}{rrrr|rrr} 3 & -2 & -13 & 13 & 1 & 0 & 0 \\ 1 & 4 & 5 & 9 & 0 & 1 & 0 \\ -7 & 9 & 39 & -26 & 0 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{rrrr|rrr} 1 & 4 & 5 & 9 & 0 & 1 & 0 \\ 3 & -2 & -13 & 13 & 1 & 0 & 0 \\ -7 & 9 & 39 & -26 & 0 & 0 & 1 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{rrrr|rrr} 1 & 4 & 5 & 9 & 0 & 1 & 0 \\ 0 & -14 & -28 & -14 & 1 & -3 & 0 \\ 0 & 37 & 74 & 37 & 0 & 7 & 1 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{rrrr|rrr} 1 & 4 & 5 & 9 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & \frac{-1}{14} & \frac{3}{14} & 0 \\ 0 & 1 & 2 & 1 & 0 & \frac{7}{37} & \frac{1}{37} \end{array}\right] \longrightarrow \left[\begin{array}{rrrr|rrr} 1 & 0 & -3 & 5 & \frac{4}{14} & 1 - \frac{12}{14} & 0 \\ 0 & 1 & 2 & 1 & \frac{-1}{14} & \frac{3}{14} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{14} & \frac{7}{37} - \frac{3}{14} & \frac{1}{37} \end{array}\right]$$

$$\text{so } P_2 = \left[\begin{array}{rrr} \frac{2}{7} & \frac{1}{7} & 0 \\ -\frac{1}{14} & \frac{3}{14} & 0 \\ \frac{1}{14} & -\frac{13}{518} & \frac{1}{37} \end{array}\right]$$

Hence A and B have the same row reduced echelon form R. Also $A = P_1 R$ and $B = P_2 R$. Thus, $R = P_1^{-1} A$ and $R = P_2^{-1} B$ which gives $P_1^{-1} A = P_2^{-1} B$. This implies that $A = P_1 P_2^{-1} B$. Therefore we let $P = P_1 P_2^{-1}$.

6. (Bonus) Suppose the matrix A has row reduced echelon form R:

$$A = \begin{bmatrix} 1 & 2 & 1 & b \\ 2 & a & 1 & 8 \\ & \text{row } 3 & & \end{bmatrix}, \; R = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

a) What can you say about row 3 of A?
b) What are the numbers a and b?

Solution:

a) Because row 3 of R is all zeros, row 3 of A must be a linear combination of rows 1 and 2 of A. The three rows of A are linearly dependent.

b) After one step of elimination we have

$$\begin{bmatrix} 1 & 2 & 1 & b \\ 0 & a-4 & -1 & 8-2b \\ & & \text{row } 3 & \end{bmatrix}$$

Looking at R we see that the second column of A is not a basic column, so $a = 4$. Continuing with elimination, we get to

$$\begin{bmatrix} 1 & 2 & 0 & 8-b \\ 0 & 0 & 1 & 2b-8 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Comparing this to R we see that $b = 5$.
