ÇANKAYA UNIVERSITY
Department of Mathematics and Computer Science

MATH 237
Linear Algebra I
2^{nd} Midterm

SOLUTIONS

December 18, 2007
17:40-19:15

Surname : ______________________
Name : ______________________
ID # : ______________________
Department : ______________________
Section : ______________________
Instructor : ______________________
Signature : ______________________

- The exam consists of 6 questions.
- Please read the questions carefully and write your answers under the corresponding questions. Be neat.
- Show all your work. Correct answers without sufficient explanation might not get full credit.
- Calculators are not allowed.

GOOD LUCK!

Please do not write below this line.

Q1	Q2	Q3	Q4	Q5	Q6	TOTAL
20	20	20	20	20	10	110

1. Are the following sets in $\mathbb{R}^3$ vector subspaces? Give reasons.

(a) $\left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 : 2x - 2y + z = 0 \right\}$ **<u>YES</u>** **NO**

It is given by a linear equation equal to 0.You can also think about it as the nullspace of the matrix $\begin{bmatrix} 2 & -2 & 1 \end{bmatrix}$.

(b) $\left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 : x^2 - y^2 + z = 0 \right\}$ **YES** **<u>NO</u>**

The vector $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ is in the set, but if you multiply by -1 $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ is not.

(c) $\left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 : 2x - 2y + z = 1 \right\}$ **YES** **<u>NO</u>**

It is given by a linear equation not set equal to 0. In particular, it doesn't contain the 0 vector.

(d) $\left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 : x = y \text{ AND } x = 2z \right\}$ **<u>YES</u>** **NO**

We can think about it as the nullspace of the matrix $\begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -2 \end{bmatrix}$

(e) $\left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 : x = y \text{ OR } x = 2z \right\}$ **YES** **<u>NO</u>**

Take for example $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$ which is not in the set.

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2. Find k if $\alpha = \left(k,\ 2k+1,\ 3k-1,\ k^2+3k-9\right) \in \mathbb{R}^4$ is in the following subspace of $\mathbb{R}^4$

$$\operatorname{span}\left\{(1,1,1,1), (-1,-1,-2,-1), (1,2,3,2), (1,0,4,2)\right\}.$$

Solution: $\alpha = \left(k,\ 2k+1,\ 3k-1,\ k^2+3k-9\right) \in \operatorname{span}\left\{(1,1,1,1), (-1,-1,-2,-1), (1,2,3,2), (1,0,$
if and only if

$\alpha = \left(k,\ 2k+1,\ 3k-1,\ k^2+3k-9\right) = d_1(1,1,1,1) + d_2(-1,-1,-2,-1) + d_3(1,2,3,2) + d_4(1,0,4,2)$ for some scalars d_1, d_2, d_3, d_4.

This is possible iff

$$\begin{aligned} d_1 - d_2 + d_3 + d_4 &= k \\ d_1 - d_2 + 2d_3 + 0d_4 &= 2k+1 \\ d_1 - 2d_2 + 3d_3 + 4d_4 &= 3k-1 \\ d_1 - d_2 + 2d_3 + 2d_4 &= k^2+3k-9. \end{aligned}$$

We form the following augmented matrix and reduce it to an echelon matrix

$$\widetilde{A} = \begin{bmatrix} 1 & -1 & 1 & 1 & k \\ 1 & -1 & 2 & 0 & 2k+1 \\ 1 & -2 & 3 & 4 & 3k-1 \\ 1 & -1 & 2 & 2 & k^2+3k-9 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -1 & 1 & 1 & k \\ 0 & 0 & 1 & -1 & k+1 \\ 0 & -1 & 2 & 3 & 2k-1 \\ 0 & 0 & 1 & 1 & k^2+2k-9 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} 1 & -1 & 1 & 1 & k \\ 0 & -1 & 2 & 3 & 2k-1 \\ 0 & 0 & 1 & -1 & k+1 \\ 0 & 0 & 1 & 1 & k^2+2k-9 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -1 & 1 & 1 & k \\ 0 & -1 & 2 & 3 & 2k-1 \\ 0 & 0 & 1 & -1 & k+1 \\ 0 & 0 & 0 & 2 & k^2+k-10 \end{bmatrix}$$

$$d_4 = \frac{1}{2}\left(k^2+k-10\right), d_3 = k+1+\frac{1}{2}\left(k^2+k-10\right) = \frac{1}{2}\left(k^2+3k-8\right),$$
$$d_2 = 2k-1-2\left(\frac{1}{2}\left(k^2+3k-8\right)\right) - 3\left(\frac{1}{2}\left(k^2+k-10\right)\right) = \frac{1}{2}\left(-5k^2-5k+44\right)$$

This last matrix has no bad row and so the above linear system is consistent for all values of $k \in \mathbb{R}$. So the vector α lies in $\operatorname{span}\left\{(1,1,1,1), (-1,-1,-2,-1), (1,2,3,2), (1,0,4,2)\right\}$ for all values of k.

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3. Find the dimension of the vector space of polynomials generated by

$$\left\{x+x^2,\ x-x^2+x^3,\ 2-x-x^3,\ x+1\right\}.$$

Solution: First $\left\{x+x^2,\ x-x^2+x^3\right\}$ is linearly independent.

Next

$2-x-x^3 \notin \operatorname{span}\left\{x+x^2,\ x-x^2+x^3\right\}$

and this shows that

$\left\{x+x^2,\ x-x^2+x^3, 2-x-x^3\right\}$ is linearly independent.

We now check whether or not $x+1 \in \operatorname{span}\left\{x+x^2,\ x-x^2+x^3, 2-x-x^3\right\}$.

This is true iff there are scalars c_1, c_2, c_3 such that

$x+1 = c_1\left(x+x^2\right) + c_2\left(\ x-x^2+x^3\right) + c_3\left(2-x-x^3\right)$ which in turn implies that

$x+1 = c_1x + c_1x^2 + c_2\ x - c_2x^2 + c_2x^3 + c_32 - c_3x - c_3x^3 = \left(c_2-c_3\right)x^3 + \left(c_1-c_2\right)x^2 + \left(c_1+c_2-c_3\right)x + 2c_3$

$c_2 - c_3 = 0, c_1 - c_2 = 0, c_1 + c_2 - c_3 = 1, 2c_3 = 1$

but this system is inconsistent, and so

$x+1 \notin \operatorname{span}\left\{x+x^2,\ x-x^2+x^3, 2-x-x^3\right\}$.

This shows that the generating set

$$\left\{x+x^2,\ x-x^2+x^3,\ 2-x-x^3,\ x+1\right\}.$$

is itself linearly independent and hence is a basis. Therefore the dimension is 4.

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4. Consider the matrix

$$A = \begin{bmatrix} 1 & 2 & -1 & 0 & 0 \\ 1 & 2 & 0 & 2 & 2 \\ 1 & 2 & -1 & 0 & 0 \\ 2 & 4 & 0 & 4 & 4 \end{bmatrix}.$$

(a) Find a basis for the row space of A.
(b) Find a basis for the nullspace $N(A)$.
(b) Find a basis for the column space $C(A)$.

Solution:

$$A = \begin{bmatrix} 1 & 2 & -1 & 0 & 0 \\ 1 & 2 & 0 & 2 & 2 \\ 1 & 2 & -1 & 0 & 0 \\ 2 & 4 & 0 & 4 & 4 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 4 & 4 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 0 & 2 & 2 \\ 0 & 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) A basis for the row space is $\left\{\begin{bmatrix} 1 & 2 & 0 & 2 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & 2 & 2 \end{bmatrix}\right\}$, i.e., $\operatorname{rank}(A) = 2$.

(b) A basis for the column space is $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -1 \\ 0 \end{bmatrix} \right\}$

(c) A vector $\begin{bmatrix} x \\ y \\ z \\ t \\ u \end{bmatrix}$ is in the nullspace $N(A)$ if and only if

$$\begin{bmatrix} x \\ y \\ z \\ t \\ u \end{bmatrix} = \begin{bmatrix} x = -2y - 2t - 2u \\ y \\ z = -2t - 2u \\ t \\ u \end{bmatrix} = y \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -2 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} + u \begin{bmatrix} -2 \\ 0 \\ -2 \\ 0 \\ 1 \end{bmatrix}$$

Hence a basis for $N(A)$ is $\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ -2 \\ 0 \\ 1 \end{bmatrix} \right\}$.

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5. Complete the set

$$S = \left\{ \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \right\}$$

to a basis of $M_{2\times 2}(\mathbb{R})$.

Solution: First we ask if $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \in \text{span} \left\{ \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \right\}$?, i.e., are there scalars c_1, c_2 such that

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = c_1 \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

This is equivalent to the system $1 = -c_1 + c_2, 0 = c_1, 0 = c_1, 0 = c_1 + c_2$, but this system is inconsistent, and this implies that

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \notin \text{span} \left\{ \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \right\}.$$

Hence $S \cup \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right\} = \left\{ \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right\}$ is linearly independent.

We need one more vector.

We now ask if $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \in \text{span} \left\{ \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right\}$? , i.e., are there scalars d_1, d_2, d_3 such that

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = d_1 \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} + d_2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + d_3 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

This is equivalent to the system $0 = -d_1 + d_2 + d_3, 1 = d_1, 0 = d_1, 0 = d_1 + d_2$, but this system is inconsistent too.

Hence we conclude that $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \notin \text{span} \left\{ \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right\}.$

Thus the set $S \cup \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right\} = \left\{ \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right\}$ is a linearly

independent set having exactly 4 vectors. Since $\dim(M_{2\times 2}(\mathbb{R})) = 4$, we see that this last subset

$\left\{ \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right\} \supset S$ is a basis containing the given set S.

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6. Find a basis and the dimension for the solution space of the system

$$\begin{aligned} 2x - y + 3z + t &= 0 \\ -5x + y + 4z - t &= 0 \\ -x - y + 10z + t &= 0. \end{aligned}$$

Solution:

$$A = \begin{bmatrix} 2 & -1 & 3 & 1 \\ -5 & 1 & 4 & -1 \\ -1 & -1 & 10 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 2 & -1 & 3 & 1 \\ -1 & -1 & 10 & 1 \\ -1 & -1 & 10 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 0 & -3 & 23 & 3 \\ -1 & -1 & 10 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} 1 & 1 & -10 & -1 \\ 0 & 1 & -23/3 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 7/3 & 0 \\ 0 & 1 & -23/3 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

so $\begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} \frac{-7}{3}s \\ \frac{23}{3}s + t \\ s \\ t \end{bmatrix} = s \begin{bmatrix} \frac{-7}{3} \\ \frac{23}{3} \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$

thus a basis for the space of solutions is $\left\{ \begin{bmatrix} -7 \\ 23 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}$

and its dimension is 2.
