ÇANKAYA UNIVERSITY
Department of Mathematics and Computer Science
MATH 237 Linear Algebra I

2^{nd} Midterm Practice Problems 4

December 18, 2007

17:40-19:10

---

1. Find a basis for the solution space of the homogeneous system

$$\begin{aligned} x+y+z-t &= 0\\ 2x-y-2z+3t &= 0\\ -x+2y-3z+4t &= 0\\ 2x+2y-4z+6t &= 0 \end{aligned}$$

---

2. Find a basis for the solution space of

$$\begin{aligned} 2x-y+3z+t &= 0\\ -5x+y+4z-t &= 0\\ -x-y+10z+t &= 0 \end{aligned}$$

---

3. Find a basis for the solution space of

$$\begin{bmatrix} 3 & -1 & 1 & 2 & 1\\ 4 & 4 & -2 & 1 & 2\\ 0 & 1 & 0 & 1 & -1\\ 1 & 5 & -3 & -1 & 1\\ 0 & 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1\\ x_2\\ x_3\\ x_4\\ x_5 \end{bmatrix} = \begin{bmatrix} 0\\ 0\\ 0\\ 0\\ 0 \end{bmatrix}$$

---

4. Find a basis for the subspace of $\mathbb{R}^4$ given by

$\{(a,b,c,d) : a-b+c+d=0, 2a+3b-d=0\}$..

---

5. Find a basis for the vector space of polynomials given by

$\{a+bx+cx^2+dx^3 : a+b+c=0, a+b+d=0\}$.

---

6. Find a basis for the column space of each of the following

a) $\begin{bmatrix} 2 & 5 & 3\\ -1 & 1 & 1\\ 1 & 1 & -1\\ 4 & -1 & 1 \end{bmatrix}$, b) $\begin{bmatrix} 1 & 0 & -1 & 0\\ 0 & 1 & 0 & -1\\ 2 & 1 & -1 & 1\\ 1 & -1 & 1 & 1 \end{bmatrix}$, c) $\begin{bmatrix} 1 & -1 & 3\\ 2 & 1 & 1\\ 1 & 1 & -1\\ 1 & -1 & 1 \end{bmatrix}$.

---

7. Find a basis for the space of 2×2 matrices

$$\begin{bmatrix} x & y \\ z & t \end{bmatrix}$$

satisfying $x + t = 0$.

---

8. Find a basis for the row space of the following matrices

$$\text{a)} \begin{bmatrix} 2 & -1 & 1 & 3 \\ 4 & 1 & 1 & -1 \\ 1 & 2 & -1 & 3 \\ 0 & 3 & -1 & -7 \end{bmatrix}, \text{b)} \begin{bmatrix} 1 & -1 & 1 & 3 & 4 \\ -1 & 1 & 1 & 4 & 1 \\ 3 & -1 & 2 & 0 & 1 \\ 3 & 1 & 4 & 7 & 6 \end{bmatrix}, \text{c)} \begin{bmatrix} -1 & 1 & 3 & 1 & 4 & 5 \\ 2 & 1 & -1 & 1 & 0 & 4 \\ -1 & 4 & 8 & 4 & 12 & 19 \end{bmatrix}.$$

---

9. Find a basis for the column space of each of the following

$$\text{a)} \begin{bmatrix} 2 & 5 & 3 \\ -1 & 1 & 1 \\ 1 & 1 & -1 \\ 4 & -1 & 1 \end{bmatrix}, \text{b)} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 2 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 \end{bmatrix}, \text{c)} \begin{bmatrix} 1 & -1 & 3 \\ 2 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}.$$

---

10. Find the rank of each matrix below:

$$\text{a)} \begin{bmatrix} 1 & -1 & 1 & 1 & 4 \\ 2 & 1 & 4 & -5 & 6 \\ -1 & 4 & 1 & -8 & -6 \\ 2 & 3 & 6 & -12 & 4 \\ 1 & 7 & 7 & -20 & -2 \end{bmatrix}, \text{b)} \begin{bmatrix} 2 & 1 & -1 & 4 \\ 1 & 1 & 3 & 4 \\ 3 & 2 & 2 & 8 \\ 1 & 0 & 1 & 0 \\ 4 & 3 & 5 & 12 \end{bmatrix}.$$

---

11. Find the order of the largest invertible submatrix of

$$\begin{bmatrix} 1 & 1 & -1 & 1 \\ 4 & 2 & 1 & 0 \\ 2 & 0 & 3 & -2 \\ 7 & 3 & 3 & -1 \end{bmatrix}.$$

---

12. Let $S = \{(1,1,1), (-1,1,-1), (3,2,3), (4,6,1), (1,0,4)\}$. Find all subsets of S which are bases for span(S) by considering
a) S as a subset of $\mathbb{R}^3$
b) S as a subset of $\mathbb{Z}_2^3$
c) S as a subset of $\mathbb{Z}_3^3$.

---

13. Is $(0,-1,2,-7)$ in span $\{(1,2,-3,4), (1,1,-1,-3), (2,3,-4,1)\}$?

---

14. Find the number of subsets of

$S = \{(1,-1,1), (-1,1,-1), (3,-4,2), (4,-3,5)\}$
which are bases for span(S) over $\mathbb{R}$.

---

15. Find the number of all bases for $\mathbb{Z}_2^3$.

---

16. Express

span $\{(1,-3,2,1),(1,1,1,1),(3,-5,5,3),(2,-2,3,2)\}$

as the solution space of a homogeneous system of linear system.

---

17. Express the space spanned by

$$\left\{\begin{bmatrix}1\\-1\\2\\4\end{bmatrix},\begin{bmatrix}3\\1\\-1\\1\end{bmatrix},\begin{bmatrix}-2\\-2\\3\\3\end{bmatrix}\right\}$$

as the solution space of a homogeneous system.

---

18. Find a basis for the column space of

$$\text{a)}\begin{bmatrix}2&4&0&4&2\\1&1&-1&0&0\\-1&-1&1&0&0\\3&1&-5&-4&-2\\4&0&-8&-8&-4\end{bmatrix},\quad\text{b)}\begin{bmatrix}0&1&3&-2\\1&4&1&4\\-1&1&-1&1\\1&-1&4&-4\\-1&1&5&-1\\3&0&1&2\end{bmatrix}.$$

---

19. Determine whether $(2,-1,1,4)$, $(-1,1,5,-2)$, $(-3,2,4,-6)$,are linearly independent. If not, write one vector as a linear combination of the others.

---

20. What must be α if the vectors

$(1,-\alpha,1,1),\ (-1,\alpha+2,-2,\alpha-1),\ (0,2,-1,\alpha^2+\alpha-4)$.

---

21. Which of the following sets of vectors span $\mathbb{R}^3$?
a) $\{(3,-1,4),\ (5,-1,7),\ (4,-2,5),\ (1,-1,1)\}$
b) $\{(2,-1,1),\ (1,4,-1),\ (-1,5,-2),\ (0,9,-3)\}$
c) $\{(0,0,0),\ (1,-1,1),\ (4,2,-1)\}$.

---

22. Determine the following subsets of $\mathbb{R}^4$ are linearly dependent or independent. In case it is linearly dependent express one vector as a linear combination of the preceding vectors, in case it is linearly independent find a basis for $\mathbb{R}^4$
a) $\{(1,-1,2,3),\ (-1,4,1,1),\ (1,2,5,7)\}$
b) $\{(1,-1,1,1),\ (2,1,-1,1),\ (1,0,0,1)\}$
c) $\{(2,-1,1,1),\ (1,-1,1,-1)\}$.

---

23. Find a basis for the subspace of $\mathbb{R}^3$ generated by

$$(1,-1,4)\ (3,-1,4),\ (1,1,-4),\ (4,-2,8).$$

24.
a) Find a basis for the subspace of $\mathbb{R}^4$ generated by

$$(-2,1,-2,1),(1,0,1,0),(-1,1,-1,1).$$

b) Show that $(1,1,1,1)$ is contained in this subspace and express it in terms of basis vectors in a).
c) What condition must be satisfied by a,b,c,d if (a,b,c,d) is in this subspace?

---

25. Find the dimension of the vector space of polynomials generated by $x+x^2, x-x^2+x^3, 2-x-x^3, x+1$.

---

26. Find a basis for $\mathbb{R}^4$ containing the vector $(1,2,-3,0)$.

---

27. Find a basis for $\mathbb{R}^5$ containing the vectors

$$(1,-1,1,1,-1) \text{ and } (0,1,1,1,0).$$

---

28. Find a basis for the vector space of 2×3 matrices containing

$$\begin{bmatrix} -1 & 2 & 1 \\ 4 & 1 & -1 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

---

29. Find a basis for the vector space of polynomials of degree ≤ 3 containing $1+x+x^2,\ x+x^2$.

---

30. Find a basis for the vector space of 2×2 symmetric matrices.

---

31. Find a basis for $\mathbb{R}^3$ containing $\{(1,-3,2),(4,-5,0)\}$.

---

32. Complete the set $\{(1,-1,5,3),(-1,2,-1,0)\}$ to a basis for $\mathbb{R}^4$.

---

33. Supplement the set of matrices

$$\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}$$

to a basis of the space of 2×2 matrices.

---

34. Find a basis of $\mathbb{R}^3$ which is a subset of

$$\{(1,-2,1),(-2,4,-2),(1,1,1),(2,-1,2),(1,1,0)\}$$

---

35. Find a basis for polynomials of degree ≤ 3 consisting of polynomials from

$$x-x^2,\ 1-x+x^2,\ 1-x^2+x^3,\ x,\ 1-x.$$

---

36. Find the coordinate matrix of the function $\cos^2 x$ relative to the ordered basis $\{1, \cos x, \sin x, \cos 2x,$ s

---

37. Find a basis for the solution space of $y''+y'=0$.

---

38. Find a basis for the solution space of $y''-y'=0$.

---

39. What must be k if the functions $\cos x + \sin x - 3\cos 2x + k \sin 2x$, $k\cos x - \sin x + k\cos 2x + 2\sin 2x$, $-\cos x + \sin x - 5\cos 2x$ are linearly dependent. For this value express the last function as a linear combination of the first two functions.

---

40. What is the dimension of the solution space of $y''' + 2y'' = 0$?

---

41. Describe the intersection of the solution spaces of

$$\begin{aligned} x_1 + 2x_2 + x_3 - x_4 &= 0 \\ 2x_1 - 3x_2 - x_3 - x_4 &= 0 \\ x_1 - 5x_2 - 2x_3 + 2x_4 &= 0 \end{aligned}$$

and

$$\begin{aligned} 3x_1 - 8x_2 - 3x_3 + 3x_4 &= 0 \\ -x_1 - 2x_2 - x_3 - x_4 &= 0 \\ 7x_2 + 3x_3 - 3x_4 &= 0 \end{aligned}$$

---

42. Let W_1 be the subspace of $\mathbb{R}^4$ spanned by $(1, -1, 2, 1)$ and let

$$W_2 = \{(x_1, x_2, x_3, x_4) \mid x_1 + x_2 - x_3 - x_4 = 0\}$$

(a) Find a spanning set for $W_1 + W_2$.
(b) Find bases for $W_1 \cap W_2$ and for $W_1 + W_2$
(c) Is the sum $W_1 + W_2$ a direct sum?

---

43. Let $W_1 =< x + x^2, x - x^2, x >$ and $W_2 =< x, x^2 + 1 >$. Find $\dim(W_1 + W_2)$ and $\dim(W_1 \cap W_2)$.

---

44. Let W_1 and W_2 be the spaces of odd and even fuctions defined on $\mathbb{R}$ respectively. Prove the $W_1 + W_2$ is a direct sum.

---

45. Let $W = \{(a, a, a) \mid a \in \mathbb{R}\}$. Find 5 distinct elements of $\mathbb{R}^3/W$.

---

46. Describe $\overline{V}$ if V is $\mathbb{F}[x]$ and W is the subspace $\{x^2 P(x) \mid P(x) \in \mathbb{F}[x]\}$ and find its dimension.

---

47. Find a basis for $\mathbb{R}^3/W$ where W is the subspace $\{(x, y, z) \in \mathbb{R}^3 \mid y - z = 0\}$.

---

48. Find $\dim \mathbb{R}^4/W$ for the subspace W spanned by $(2, 1, -1, 3), (-1, 1, 1, 4), (1, 2, 0, 7), (0, 1, 0, -1)$.

---

49. What must be W if $\dim V/W = \dim V$ for a finite dimensional vector space V?

---

50. For that for the finite dimensional subspaces W_1 and W_2 of a vector space prove the equaity

$$\dim W_1/W_1 \cap W_2 + \dim W_2/W_1 \cap W_2 = \dim(W_1 + W_2) - \dim(W_1 \cap W_2)$$
