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Department of Mathematics and Computer Science

MATH 237
Linear Algebra I
First Midterm Practice Problems (B)
November 13, 2007
17:40 – 19:30

1. Compute the following determinants

$$\text{a)} \begin{vmatrix} 2 & 2 \\ -3 & 6 \end{vmatrix}, \quad \text{b)} \begin{vmatrix} a & -3 \\ x & b \end{vmatrix}, \quad \text{c)} \begin{vmatrix} 1 & 0 & 0 & 0 & 0 \\ 7 & -5 & 0 & 0 & 0 \\ -9 & 6 & 4 & 0 & 0 \\ 3 & -2 & 0 & 1 & 0 \\ -2 & 3 & 2 & 5 & -6 \end{vmatrix}$$

2. Solve the following equations in $\mathbb{Z}_5$

$$\text{a)} \begin{vmatrix} x & 0 & 0 & 0 \\ x & x^2+1 & 0 & 0 \\ 3x-5 & x+1 & x & 0 \\ 1 & -x & -3 & x+2 \end{vmatrix} = 1, \quad \text{b)} \begin{vmatrix} 2x & x^2+1 & x \\ x & x^2+1 & x \\ 3x-5 & x+1 & 4x-5 \end{vmatrix} = 0$$

3. Given that

$$\text{(a)} \begin{vmatrix} x^2 & ax & 2x \\ -x & 1 & b \\ ax & 2 & 3b \end{vmatrix}, \quad \text{(b)} \begin{vmatrix} x & a+bx & 2 \\ -x & 1-bx & b \\ ax & 2+abx & 3b \end{vmatrix},$$

$$\text{(c)} \begin{vmatrix} -1 & -2a & -2 \\ -2 & 4 & 2b \\ a & 4 & 3b \end{vmatrix}, \quad \text{(d)} \begin{vmatrix} a+1 & a+2 & 2+3b \\ -1 & 1 & b \\ a & 2 & 3b \end{vmatrix}$$

4. Compute the determinant of each of the following matrices and interpret the results:

$$\text{(a)} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad \text{(b)} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad \text{(c)} \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

5. Evaluate the determinants of

$$\text{(a)} \begin{bmatrix} 2 & -5 & 4 & -1 & 1 \\ -7 & 3 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 41 & 10 & 0 & 0 \\ 12 & 10 & 0 & 0 & -20 \end{bmatrix}, \quad \text{(b)} \begin{bmatrix} 1 & -3 & 3 & -4 \\ 2 & 1 & -1 & -5 \\ -3 & 9 & -9 & 12 \\ 7 & 1 & -1 & 5 \end{bmatrix}, \quad \text{(c)} \begin{vmatrix} 1 & 4 & 1 & -4x \\ 3 & -1 & 3 & x \\ 2 & -1 & 7 & x \\ 5 & 1 & 1 & -x \end{vmatrix}.$$

6. Evaluate the determinant of

$$\begin{bmatrix} 1 & 4 & 2 & x+4x^2+x^3 \\ -1 & 2 & 3 & -x+2x^2+3x^3 \\ 3 & 1 & 2 & 3x+x^2+2x^3 \\ 0 & 1 & 1 & 3+x^2+x^3 \end{bmatrix}$$

7. Solve the following equation over $\mathbb{Z}_7$

$$\begin{vmatrix} x+1 & x+1 & 1 \\ -2x-3 & -x & -3 \\ 1 & 0 & 1 \end{vmatrix} = 3$$

8. Let E be an idempoten matrix (i.e., $E^2 = E$) different from I. Prove that $\det(A) = 0$.
9. Show that if c is a scalar and A is an $n \times n$ matrix, then $\det(cA) = c^n \det(A)$.
10. Show that the determinant of any real or complex skew-symmetric matrix of odd order is zero.
11. If Q is an orthogonal matrix (i.e., $Q^T = Q^{-1}$), then $\det(Q) = \pm 1$.
12. Verify that

$$\begin{vmatrix} b^2+ac & bc & c^2 \\ ab & 2ac & bc \\ a^2 & ab & b^2+ac \end{vmatrix} = \begin{vmatrix} b & c & 0 & d & e & f \\ a & 0 & c & x & y & z \\ 0 & a & b & u & v & w \\ 0 & 0 & 0 & b & c & 0 \\ 0 & 0 & 0 & a & 0 & c \\ 0 & 0 & 0 & 0 & a & b \end{vmatrix}$$

and compute the common value.

13. Compute

$$\begin{vmatrix} 1+x_1 & x_1 & x_1 & \cdots & x_1 \\ x_2 & 1+x_2 & x_2 & \cdots & x_2 \\ x_3 & x_3 & 1+x_3 & \cdots & \\ & & \cdots & & \\ x_n & x_n & x_n & \cdots & 1+x_n \end{vmatrix}$$

14. Compute

$$\begin{vmatrix} a & b & \cdots & b & b \\ b & a & \cdots & b & b \\ & & \cdots & & \\ b & b & \cdots & a & b \\ b & b & \cdots & b & a \end{vmatrix}$$

15. Evaluate

$$\begin{vmatrix} 0 & 1 & \cdots & n-2 & n-1 \\ 1 & 0 & \cdots & n-3 & n-2 \\ & & \cdots & & \\ n-2 & n-3 & \cdots & 0 & 1 \\ n-1 & n-2 & \cdots & 1 & 0 \end{vmatrix}$$

16. Evaluate the following and factorize the result:

$$\begin{vmatrix} a & b & c & d \\ b & c & d & a \\ c & d & a & b \\ d & a & b & c \end{vmatrix}$$

17. Compute the adjoint of

$$(a)\begin{bmatrix} 2 & -3 & 1 \\ 0 & 1 & -1 \\ 3 & 1 & 7 \end{bmatrix},\quad (b)\begin{bmatrix} 2 & 0 & 0 & 1 \\ -2 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \\ 1 & 0 & 2 & 0 \end{bmatrix}$$

18. The inverse of a certain matrix A is given as

$$A^{-1} = \begin{bmatrix} 2 & 2 & 3 & 3 \\ -1 & -2 & 1 & 6 \\ 1 & -3 & -2 & 5 \\ 4 & 4 & 5 & 4 \end{bmatrix}$$

use this information to find $\det(A)$ and $adj\,(A)$.

19. For an $n \times n$ matrix prove that
(a) $adj\,(A^t) = (adj\,(A))^T$
(b) $adj\,(cA) = c^{n-1} adj\,(A)$ for any scalar c.

20. Compute $(adj\,(A))^{-1}$ and $\det(adj\,(A))$ for

$$\begin{bmatrix} 1 & -3 & 0 & 0 & 0 \\ 3 & -6 & 0 & 0 & 0 \\ 2 & -4 & 2 & 4 & 0 \\ -2 & 3 & 5 & 1 & -5 \\ 0 & 9 & 3 & 4 & 7 \end{bmatrix}$$

21. Show that the matrix

$$\begin{bmatrix} 2 & 1 & 1 \\ 0 & 3 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

cannot be the adjoint of any reaal matrix.

22. Compute $adj\,(adj\,(A))$ for

$$\begin{bmatrix} 4 & -3 & 9 & 2 \\ 1 & -3 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 2 & -1 \end{bmatrix}.$$

23. Establish a formula for the Vandermonde determinant

$$\begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ x_1 & x_2 & x_3 & \cdots & x_n \\ x_1^2 & x_2^2 & x_3^2 & \cdots & x_n^2 \\ & & \cdots & & \\ x_1^{n-1} & x_2^{n-1} & x_3^{n-1} & \cdots & x_n^{n-1} \end{vmatrix}$$

Use this to evaluate the complex determinant

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & i & -i \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -i & i \end{vmatrix}.$$

24. Prove that if A and B are square matrices, then

$$\begin{vmatrix} A & 0 \\ \star & B \end{vmatrix} = \begin{vmatrix} A & \star \\ 0 & B \end{vmatrix} = |A|\,|B|$$

and generalize this result.