ÇANKAYA UNIVERSITY
Department of Mathematics and Computer Science

MATH 351 Complex Analysis I
Final Exam
SOLUTIONS
August 8, 2008
9:00-11:00

Surname : ____________________
Name : ____________________
ID # : ____________________
Department : ____________________
Section : ____________________
Instructor : ____________________
Signature : ____________________

- The exam consists of 6 questions.
- Please read the questions carefully and write your answers under the corresponding questions. Be neat.
- Show all your work. Correct answers without sufficient explanation might not get full credit.
- Calculators are not allowed.

GOOD LUCK!

Please do not write below this line.

Q1	Q2	Q3	Q4	Q5	Q6	TOTAL
14	16	16	24	20	15	105

1.

a) Express $2e^{i\pi/4}$ in the standard form $a+ib$.

b) Express $\left(\dfrac{1-i}{\sqrt{3}+i}\right)^8$ in polar form $re^{i\theta}$.

Solution:

a)

$$2e^{i\pi/4} = 2\left(\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right) = 2\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right) = \sqrt{2} + i\sqrt{2}.$$

b)

$$\left(\frac{1-i}{\sqrt{3}+i}\right)^8 = \left(\frac{\sqrt{2}e^{-i\pi/4}}{2e^{i\pi/6}}\right)^8 = \left(\frac{1}{\sqrt{2}}\right)^8 \left(e^{-i5\pi/12}\right)^8 = \frac{1}{16}e^{-i10\pi/3} = \frac{1}{16}e^{i2\pi/3}.$$

2.
a) For what values of x, y is the function $f(x+iy) = xy + ix$ is differentiable? analytic?
b) Find a function analytic in the entire plane whose real part is $u(x,y) = x^3y - xy^3$.
Solution:
a)

$$\begin{aligned} u_x &= y, \quad u_y = x \\ v_x &= 1, \quad v_y = 0 \end{aligned}$$

Thus, by Cauchy-Riemann equations, if f is differentiable at $x+iy$, then $x=-1, y=0$. Since all partial derivatives are continuous, f is indeed *differentiable* at $x=-1, y=0$.
Since f is not differentiable in a neighborhood of this point, f is *nowhere* analytic.

b) Find harmonic conjugate v of u:
Since $v_y = u_x = 3x^2y - y^3$,

$$v = \int \left(3x^2y - y^3\right) \, dy = (3/2)\, x^2y^2 - y^4/4 + h(x),$$

where $h(x)$ can be determined from the equations:

$$v_x = 3xy^2 + h'(x), \quad v_x = -u_y = -x^3 + 3xy^2$$

thus, $h'(x) = -x^3$ and so $h(x) = -x^4/4 + C$, where C is a constant.
It follows that

$$v = (3/2)\, x^2y^2 - y^4/4 - x^4/4 + C,$$

is a harmonic conjugate for u and that $f(x,y) = u + iv = \left(x^3y - xy^3\right) + i\left((3/2)\, x^2y^2 - y^4/4 - x^4/4 + C\right)$ is an analytic function whose real part is $u(x,y) = x^3y - xy^3$.

3.

a) Let C be the unit circle traversed clockwise. Find the value of $\int_C z \sin z^2 \, dz$ without explicitly calculating the integral.

b) Let C be the circle of radius 1 centered at $2+i$ traversed counterclockwise. Find the value of $\int_C \frac{1}{z} \, dz$ without explicitly calculating the integral.

Solution:

a) We know that $f(z) = z \sin z^2$ is everywhere analytic so in particular, inside and on C, therefore by Cauchy-Goursat theorem, $\int_C z \sin z^2 = 0$.

b) The function $f(z) = \frac{1}{z}$ has one isolated singular point namely, $z = 0$, and it is analytic everywhere else, but $z = 0$ is outside the contour C, therefore by Cauchy-Goursat theorem, $\int_C \frac{1}{z} \, dz = 0$.

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4. Evaluate the following integrals:

(a) $\displaystyle\int_{|z-1|=1} \frac{z}{z^2-1}\,dz$, (b) $\displaystyle\int_{|z|=2} \frac{ze^z}{(z-1)^3}\,dz$, (c) $\displaystyle\int_{|z|=1} \frac{z\sin z}{(z-2)^3}\,dz$

Solution:

a) Let $f(z) = \dfrac{z}{z+1}$. Then $f(z)$ is analytic inside and on C. Therefore, by the Cauchy Integral Formula, we have $\displaystyle\int_{|z-1|=1} \frac{z}{z^2-1}\,dz = \int_{|z-1|=1} \frac{f(z)}{z-1}\,dz = 2\pi i f(1) = 2\pi i\left[\frac{z}{z+1}\right]_{z=1} = 2\pi i\frac{1}{1+1} = \pi i.$

b) Let $g(z) = ze^z$. Then $g(z)$ is analytic inside and on C. Hence, by the Cauchy Integral Formula, we have $\displaystyle\int_{|z|=2} \frac{ze^z}{(z-1)^3}\,dz = \frac{2\pi i}{2!}g''(1) = \pi i\left[2e^z + ze^z\right]_{z=1} = \pi i\left[2e^1 + e^1\right] = 3\pi i e.$

c) $\displaystyle\int_{|z|=1} \frac{z\sin z}{(z-2)^3}\,dz = 0$, by the Cauchy-Goursat theorem since the integrand $\dfrac{z\sin z}{(z-2)^3}$ is analytic at all points in the interior and on C.

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5. Evaluate the following contour integrals

a) $\displaystyle\int_C \left(z+z^2\right)\ dz$ where C is the straight line segment from $z=1$ to $z=i$.

b) $\displaystyle\int_C \sqrt{z}\ dz$ where C is the segment of the ellipse $\dfrac{x^2}{9}+\dfrac{y^2}{4}=1$ from $z=3$ to $z=2i$. (use the principal branch of $\sqrt{z}$).

Solution:

a)

$f(z)=z+z^2$ has antiderivative $F(z)=\dfrac{1}{2}z^2+\dfrac{1}{3}z^3$ in $\mathbb{C}$. Therefore,

$$\int_C f(z)\ dz=\left[\frac{1}{2}z^2+\frac{1}{3}z^3\right]_1^i=-\frac{1}{2}-\frac{i}{3}-\left(\frac{1}{2}+\frac{1}{3}\right)=-\frac{4}{3}-\frac{i}{3}.$$

b)

$f(z)=\sqrt{z}$ (principal branch) has antiderivative $F(z)=\dfrac{2}{3}z^{3/2}$ (principal branch).
Therefore,

$$\begin{aligned}\int_C f(z)\ dz &= \left[\frac{2}{3}z^{3/2}\right]_3^{2i}=\frac{2}{3}\left((2i)^{3/2}-3^{3/2}\right)\\ &= \frac{2}{3}\left(2^{3/2}e^{i3\pi/4}-3^{3/2}\right)\\ &= -\frac{4}{3}-2\sqrt{3}+\frac{4}{3}i\end{aligned}$$

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6. Find the Taylor series representation for $f(z) = \dfrac{z^2}{(2+z)^2}$, indicate its domain of convergence.

Solution:

$f(z) = \dfrac{z^2}{(2+z)^2}$;

We start with

$$\frac{1}{2+z} = \frac{1}{2}\frac{1}{1-\left(-\frac{z}{2}\right)} = \frac{1}{2}\sum_{n=0}^{\infty}\left(-\frac{z}{2}\right)^n = \sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+1}}z^n \quad \text{for } \left|\frac{z}{2}\right| < 1 \text{ i.e., for } |z| < 2.$$

Next we differentiate:

$$\begin{aligned}
\frac{d}{dz}\left(\frac{1}{2+z}\right) &= \frac{d}{dz}\left(\sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+1}}z^n\right), \text{ for } |z| < 2 \\
-\frac{1}{(2+z)^2} &= \sum_{n=1}^{\infty}\frac{(-1)^n}{2^{n+1}}nz^{n-1}, \text{ for } |z| < 2 \\
-\frac{1}{(2+z)^2} &= \sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{2^{n+2}}(n+1)z^n, \text{ for } |z| < 2 \\
\frac{1}{(2+z)^2} &= \sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+2}}(n+1)z^n, \text{ for } |z| < 2 \\
\frac{z^2}{(2+z)^2} &= \sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+2}}(n+1)z^{n+2}, \text{ for } |z| < 2
\end{aligned}$$
