ÇANKAYA UNIVERSITY
Department of Mathematics and Computer Science

MATH 351 Complex Analysis I
First Midterm
SOLUTIONS
July 14, 2008
9:00-10:30

Surname : ____________________
Name : ____________________
ID # : ____________________
Department : ____________________
Section : ____________________
Instructor : ____________________
Signature : ____________________

- The exam consists of 6 questions.
- Please read the questions carefully and write your answers under the corresponding questions. Be neat.
- Show all your work. Correct answers without sufficient explanation might <u>not</u> get full credit.
- Calculators are <u>not</u> allowed.

GOOD LUCK!

Please do <u>not</u> write below this line.

Q1	Q2	Q3	Q4	Q5	Q6	TOTAL
11	18	20	20	16	20	105

1. Find all complex numbers z that are complex conjugates of their own squares i.e., $\overline{z} = z^2$.
Solution:
We need to solve the equation $\overline{z} = z^2$; that is, we need to find all pairs (x, y) of real numbers such that $x - iy = x^2 - y^2 + i \cdot 2xy$. Since 1 and i are linearly independent, this means we need to solve the two equations

$$\begin{aligned} x &= x^2 - y^2 \\ -y &= 2xy \end{aligned}$$

for x and y. For $y = 0$, we get $x = 0$ or $x = 1$. For $y \neq 0$, we get $x = -\frac{1}{2}$ and $y = \pm\frac{1}{2}$. So there are four solutions, namely, $0, 1, -\frac{1}{2} + i\frac{1}{2}, \frac{1}{2} - i\frac{1}{2}$.

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2. Find all of the roots of $(-8i)^{1/3}$ in the form $a+ib$ and point out which is the principal root.
Solution:
Since $-8i = 8\exp\left[i\left(-\frac{\pi}{2}+2k\pi\right)\right]$ $(k=0,1,2)$, the three cube roots of the number $z_0 = -8i$ are

$$(-8i)^{1/3} = 2\exp\left[i\left(-\frac{\pi}{6}+\frac{2k\pi}{3}\right)\right] \qquad (k=0,1,2)\,.$$

the principal one being

$$c_0 = 2\exp\left(i\frac{-\pi}{6}\right) = 2\left(\frac{\sqrt{3}}{2} - i\frac{1}{2} = \sqrt{3} - i\right).$$

The others are

$$c_1 = 2\exp\left(i\frac{\pi}{2}\right) = 2i$$

and

$$c_2 = 2\exp\left(i\frac{7\pi}{6}\right) = 2\left(-\frac{\sqrt{3}}{2} - i\frac{1}{2}\right) = -\left(\sqrt{3}+1\right).$$

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3. Determine where $f'(z)$ exists and find its value when

(a) $f(z) = \dfrac{1}{z}$;

(b) $f(z) = x^2 + iy^2$.

Solution:

(a)

$f(z) = \dfrac{1}{z} = \dfrac{1}{z} \cdot \dfrac{\overline{z}}{\overline{z}} = \dfrac{\overline{z}}{|z|^2} = \dfrac{x}{x^2+y^2} + i\dfrac{-y}{x^2+y^2}$. So

$$u = \frac{x}{x^2+y^2} \quad \text{and} \quad v = \frac{-y}{x^2+y^2}.$$

Since

$$u_x = \frac{y^2-x^2}{(x^2+y^2)^2} = v_y \qquad \text{and} \qquad u_y = \frac{-2xy}{(x^2+y^2)^2} = -v_x \qquad x^2+y^2 \neq 0,$$

$f'(z)$ exists when $z \neq 0$. Moreover, when $z \neq 0$,

$$\begin{aligned} f'(z) &= u_x + iv_x = \frac{y^2-x^2}{(x^2+y^2)^2} + i\frac{2xy}{(x^2+y^2)^2} = -\frac{x^2 - i2xy - y^2}{(x^2+y^2)^2} \\ &= -\frac{(x-iy)^2}{(x^2+y^2)^2} = -\frac{(\overline{z})^2}{(z\overline{z})^2} = -\frac{(\overline{z})^2}{z^2(\overline{z})^2} = -\frac{1}{z^2}. \end{aligned}$$

(b)

$f(z) = x^2 + iy^2$. Hence $u = x^2$ and $v = y^2$. Now

$$u_x = v_y \Longrightarrow 2x = 2y \Longrightarrow y = x \quad \text{and} \quad u_y = -v_x \Longrightarrow 0 = 0.$$

So $f'(z)$ exists only when $y = x$, and we find that

$$f'(x+ix) = u_x(x,x) + iv_x(x,x) = 2x + i0 = 2x.$$

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4. Determine if the following functions are analytic
(a) $f(z) = 3x + y + i(3y - x)$
(b) $f(z) = 2xy + i(x^2 - y^2)$.
Solution:
(a)
$f(z) = \underbrace{3x + y} + i\underbrace{(3y - x)} = u + iv$ where $u = 3x + y$ and $v = 3y - x$ is entire since

$$u_x = 3 = v_y \quad \text{and} \quad u_y = 1 = -v_x.$$

(b) $f(z) = \underbrace{2xy} + i\underbrace{(x^2 - y^2)} = u + iv$ where $u = 2xy$ and $v = x^2 - y^2$ is nowhere analytic since

$$u_x = v_y \Longrightarrow 2y = -2y \Longrightarrow y = 0 \quad \text{and} \quad u_y = -v_x \Longrightarrow 2x = -2x \Longrightarrow x = 0,$$

which means that the Cauchy-Riemann equations hold only at the point $z = (0, 0) = 0$.

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5. Show that $u(x,y) = 2x - x^3 + 3xy^2$ is harmonic in some domain and find a harmonic conjugate $v(x,y)$.

Solution:

It is straightforward to show that $u_{xx} + u_{yy} = 0$. To find a harmonic conjugate $v(x,y)$, we start with $u_x(x,y) = 2 - 3x^2 + 3y^2$. Now

$$u_x = v_y \Longrightarrow v_y = 2 - 3x^2 + 3y^2 \Longrightarrow v(x,y) = 2y - 3x^2y + y^3 + \phi(x).$$

Then

$$u_y = -v_x \Longrightarrow 6xy = 6xy - \phi'(x) \Longrightarrow \phi'(x) = 0 \Longrightarrow \phi(x) = c.$$

Consquently,

$$v(x,y) = 2y - 3x^2y + y^3 + c.$$

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6. Find all values of z such that $e^z = 1 + \sqrt{3}i$.
Solution:
Write $e^z = 1 + \sqrt{3}i$ as $e^x e^{iy} = 2e^{i(\pi/3)}$, from which we see that

$$e^x = 2 \quad \text{and} \quad y = \frac{\pi}{3} + 2n\pi \qquad (n = 0, \pm 1, \pm 2, \cdots).$$

That is,

$$x = \ln 2 \quad \text{and} \quad y = \left(2n + \frac{1}{3}\right)\pi \qquad (n = 0, \pm 1, \pm 2, \cdots).$$

Consequently

$$z = \ln 2 + \left(2n + \frac{1}{3}\right)\pi i \qquad (n = 0, \pm 1, \pm 2, \cdots).$$