ÇANKAYA UNIVERSITY
Department of Mathematics and Computer Science

MATH 351 Complex Analysis I
Second Midterm
SOLUTIONS
August 4, 2008
9:00-10:30

Surname : ______________________
Name : ______________________
ID # : ______________________
Department : ______________________
Section : ______________________
Instructor : ______________________
Signature : ______________________

- The exam consists of 6 questions.
- Please read the questions carefully and write your answers under the corresponding questions. Be neat.
- Show all your work. Correct answers without sufficient explanation might not get full credit.
- Calculators are not allowed.

GOOD LUCK!

Please do not write below this line.

Q1	Q2	Q3	Q4	Q5	Q6	TOTAL
11	18	20	20	16	20	105

1. Find all the zeros of the function $f(z) = 2 + \cos z$. (Hint: if they exist, they must be nonreal.)

Solution:

Following the hint, write $z = x + iy$ with real and imaginary parts $x, y \in \mathbb{R}$. But then

$$\cos z = \cos(x + iy) = \cos x \cos iy - \sin x \sin iy = \cos x \cosh y - i \sin x \sinh y,$$

since $\cos iy = \cosh y$ and $\sin iy = i \sinh y$. To solve $2 + \cos z = 0$ is thus equivalent to finding $z = x + iy$ such that $\cos y \cosh y = -2$ and $\sin x \sinh y = 0$.

Now $\sin x \sinh y = 0$ if and only if either $\sinh y = 0$ or $\sin x = 0$. The first case is excluded because it requires $y = 0$, so $\cosh y = 1$, so $\cos x = -2$ which cannot happen.

The second case is equivalent to $x = k\pi$ for $k \in \mathbb{Z}$. Now $\cosh y = \frac{1}{2}\left(e^y + e^{-y}\right) \geq 1$ for all real y with equality if and only if $y = 0$; otherwise, $\cosh y = C$ has two distinct real roots for every $C > 1$. We conclude that

$$-2 = \cos x \cosh y = \cos k\pi \cosh y = (-1)^k \cosh y$$

has a solution if and only if $x = k\pi$ for some odd integer k and y is one of the two real roots of $\cosh y = 2$.

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2. Find all of values of $\tan^{-1}(1+i)$.
Solution:

$$\begin{aligned}
\tan^{-1}(1+i) &= \frac{i}{2}\log\left(\frac{i+1+i}{i-1-i}\right) \\
&= \frac{i}{2}\log(-1-2i) \\
&= \frac{i}{2}\left(\ln\sqrt{5}+i\arg(-1-2i)\right) \\
&= -\frac{1}{2}\arg(-1-2i)+i\frac{\ln\sqrt{5}}{2} \\
&= -\frac{1}{2}\arg(-1-2i)+i\ln 5
\end{aligned}$$

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3. Evaluate the line integral $\int_C |z|^2 \ dz$ where C is the line segment from the point 0 to the point $1+i$.

Solution:

Since $f(z) := |z|^2 = x^2 + y^2$, for $z(t) = t + it, (0 \le t \le 1)$ is the parametrization of C then we have $z'(t) = (1+i) \ dt$, $f(z(t)) = t^2 + t^2 = 2t^2$. Therefore

$$\begin{aligned}
\int_C |z|^2 \ dz &= \int_0^1 2t^2 (1+i) \ dt \\
&= 2(1+i) \int_0^1 t^2 \ dt \\
&= 2(1+i) \left[\frac{1}{3} t^3\right]_0^1 \\
&= \frac{2}{3}(1+i).
\end{aligned}$$

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4. By finding an antiderivative, evaluate the integral

$$\int_0^{\pi+2i} \cos\left(\frac{z}{2}\right)\, dz.$$

Solution:

$$\int_0^{\pi+2i} \cos\left(\frac{z}{2}\right)\, dz = \left[2\sin\left(\frac{z}{2}\right)\right]_0^{\pi+2i} = 2\sin\left(\frac{\pi+2i}{2}\right) - 2\sin\left(\frac{0}{2}\right)$$

$$= 2\frac{e^{i\left(\frac{\pi}{2}+i\right)} - e^{-i\left(\frac{\pi}{2}+i\right)}}{2i} = -i\left(e^{i\pi/2}e^{-1} - e^{-i\pi/2}e\right)$$

$$= -i\left(\frac{i}{e} + ie\right) = \frac{1}{e} + e = e + \frac{1}{e}.$$

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5. Use Cauchy's Integral Formula to evaluate $\displaystyle\int_{|z-1|=1} \frac{\cos(2\pi z)}{z^2-1}\,dz$ where the integration path is oriented in the standard counterclockwise direction.

Solution:

Let $f(z) = \dfrac{\cos(2\pi z)}{z+1}$. Then $f(z)$ is analytic at all points both interior to and on the contour C. Therefore, by the Cauchy Integral Formula, we have

$$\begin{aligned}
\int_{|z-1|=1} \frac{\cos(2\pi z)}{z^2-1}\,dz &= 2\pi i f(1) \\
&= 2\pi i \left[\frac{\cos(2\pi z)}{z+1}\right]_{z=1} \\
&= 2\pi i \frac{\cos(2\pi(1))}{1+1} \\
&= 2\pi i \frac{1}{1+1} = \pi i.
\end{aligned}$$

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6. Find the value of the integral $\int_C \frac{z-b}{z-a}\,dz$ where C is the unit circle traversed once counterclockwise. Be sure to consider the cases $|a|<1$ and $|a|>1$.

Solution 1:

If $|a|>1$, then the integrand is analytic on $|z|<|a|$ and Cauchy-Goursat Theorem says that

$$\int_C \frac{z-b}{z-a}\,dz = 0.$$

If $|a|<1$, then define $f(z)=(z-b)$ which is analytic on $\mathbb{C}$. Then Cauchy Integral Formula says that

$$\begin{aligned}\int_C \frac{z-b}{z-a}\,dz &= \int_C \frac{f(z)}{z-a}\,dz \\ &= 2\pi i f(a) \\ &= 2\pi i (a-b).\end{aligned}$$

Solution 2:

If $|a|<1$, then we could notice that $z-b=(z-a)+(a-b)$ and therefore

$$\begin{aligned}\int_C \frac{z-b}{z-a}\,dz &= \int_C dz + (a-b)\int_C \frac{1}{z-a}\,dz \\ &= 2\pi i (a-b).\end{aligned}$$

There are other ways to do this as well, but these two methods are the simplest.
