ÇANKAYA UNIVERSITY
Department of Mathematics and Computer Science
MATH 351 Complex Analysis I
Practice Problems-1
First midterm

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1. Complex Numbers

1.1. **Section 4.** (p. 11)

3. Verify that $\sqrt{2}\,|z| \geq |\operatorname{Re} z| + |\operatorname{Im} z|$

Suggestion Reduce this inequality to $(|x| - |y|)^2 \geq 0$.

Solution:

Let $z = x + iy \Longrightarrow$ the inequality becomes $\sqrt{2}\sqrt{x^2+y^2} \geq |x| + |y|$

$$\begin{aligned}
&\iff \quad 2\left(x^2+y^2\right) \geq (|x|+|y|)^2 = x^2+y^2+2\,|x|\,|y| \\
&\iff \quad x^2+y^2-2\,|x|\,|y| \geq 0 \\
&\iff \quad (|x|-|y|)^2 \geq 0.
\end{aligned}$$

This last form of the inequality to be verified is obviously true since the left-hand side is a perfect square.

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4. In each case, sketch the set of points determined by the given condition:

(a) $|z-1+i| = 1$; (b) $|z+i| \leq 3$; (c) $|z-4i| \geq 4$.

Solution:

(a) $|z-1+i| = 1$; it's a circle with center $z_0 = (1,-1)$ and radius $R = 1$.

(b) $|z+i| \leq 3$; it's a disk with center $z_0 = (0,-1)$ and radius $R = 3$.

(c) $|z-4i| \geq 4$; it's the set of points outside the disk of radius $R = 4$ and center $z_0 = 4i$.

(p.13)

7. Use the established properties of moduli to show that when $|z_3| \neq |z_4|$,

$$\left|\frac{z_1+z_2}{z_3+z_4}\right| \leq \frac{|z_1|+|z_2|}{||z_3|-|z_4||}.$$

Solution:

$\left|\dfrac{z_1+z_2}{z_3+z_4}\right| = \dfrac{|z_1+z_2|}{|z_3+z_4|} \leq \dfrac{|z_1|+|z_2|}{||z_3|-|z_4||} \leq \dfrac{|z_1|+|z_2|}{||z_3|-|z_4||}$ by triangle inequality $|z_1+z_2| \leq |z_1|+|z_2|$ and the inequality $|z_3 \pm z_4| \geq ||z_3|-|z_4||$ (p.10).

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10. By factoring z^4-4z^2+3 into two quadratic factors, show that z lies on the circle $|z| = 2$, then

$$\left|\frac{1}{z^4-4z^2+3}\right| \leq \frac{1}{3}$$

Solution:
Factorizing $z^4 - 4z^2 + 3 = \left(z^2 - 1\right)\left(z^2 - 3\right)$

$$\left|\frac{1}{z^4 - 4z^2 + 3}\right| = \frac{1}{|z^4 - 4z^2 + 3|} = \frac{1}{|(z^2 - 1)(z^2 - 3)|} = \frac{1}{|z^2 - 1|\,|z^2 - 3|} \leq \frac{1}{\left||z|^2 - 1\right|\,\left||z|^2 - 3\right|}$$

$$= \frac{1}{(4 - 1)(4 - 3)} = \frac{1}{3}.$$

(p.21)
1. Find the principal argument Arg z when
(a) $z = \dfrac{i}{-2 - 2i}$; (b) $z = \left(\sqrt{3} - i\right)^6$.
Solution:
(a)

$$z = \frac{i}{-2 - 2i} = -\frac{1}{2}\frac{i}{1 + i}\frac{1 - i}{1 - i} = -\frac{1}{2}\frac{i - i^2}{1 - i^2} = -\frac{1}{4}(1 + i) = \frac{\sqrt{2}}{4}\left(-\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right) = \frac{\sqrt{2}}{4}e^{-3\pi i/4}$$

$\Longrightarrow \mathrm{Arg}(z) = -\frac{3\pi}{4}$
(b)

$$z = \left(\sqrt{3} - i\right)^6$$

Observe $\xi = \sqrt{3} - i \Longrightarrow |\xi| = \sqrt{3 + 1} = 2 \Longrightarrow \xi = 2\left(\dfrac{\sqrt{3}}{2} - \dfrac{i}{2}\right) = 2e^{-i\pi/6}$

$z = \xi^6 = \left(2e^{-i\pi/6}\right)^6 = 2^6 e^{-i\pi} = 2^6 e^{i\pi} = -64$ (since $-\pi = \pi + 2\pi$ and $e^{2\pi i} = 1$)
$\Longrightarrow \mathrm{Arg}(z) = \pi$.

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1. Derive the following trigonometric identities:
(a) $\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$, (b) $\sin 3\theta = 3\cos^2\theta - \sin^3\theta$.
Solution:
(a)

$$z = \frac{i}{-2 - 2i} = -\frac{1}{2}\frac{i}{1 + i}\frac{1 - i}{1 - i} = -\frac{1}{2}\frac{i - i^2}{1 - i^2} = -\frac{1}{4}(1 + i) = \frac{\sqrt{2}}{4}\left(-\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right) = \frac{\sqrt{2}}{4}e^{-3\pi i/4}$$

$\Longrightarrow \mathrm{Arg}(z) = -\frac{3\pi}{4}$
(b)

$$z = \left(\sqrt{3} - i\right)^6$$

Observe $\xi = \sqrt{3} - i \Longrightarrow |\xi| = \sqrt{3 + 1} = 2 \Longrightarrow \xi = 2\left(\dfrac{\sqrt{3}}{2} - \dfrac{i}{2}\right) = 2e^{-i\pi/6}$

$z = \xi^6 = \left(2e^{-i\pi/6}\right)^6 = 2^6 e^{-i\pi} = 2^6 e^{i\pi} = -64$ (since $-\pi = \pi + 2\pi$ and $e^{2\pi i} = 1$)
$\Longrightarrow \mathrm{Arg}(z) = \pi$.

(p.73)
1. Verify that each of these functions is entire:
(a)$f(z) = 3x + y + i(3y - x)$; (b)$f(z) = \sin x\cosh y + i\cos x\sinh y$;
(c)$f(z) = e^{-y}\sin x - ie^{-y}\cos x$; (d)$f(z) = \left(z^2 - 2\right)e^{-x}e^{-iy}$.
Solution:

(a) $f(z) = \underbrace{3x+y}+i\underbrace{(3y-x)}$ is entire since

$$u_x = 3 = v_y \text{ and } u_y = 1 = -v_x$$

(b) $f(z) = \underbrace{\sin x \cosh y}+i\underbrace{\cos x \sinh y}$ is entire since

$$u_x = \cos x \cosh y = v_y \text{ and } u_y = \sin x \sinh y = -v_x.$$

(c) $f(z) = \underbrace{e^{-y}\sin x}+i\underbrace{\left(-e^{-y}\cos x\right)}$ is entire since

$$u_x = e^{-y}\cos x = v_y \text{ and } u_y = -e^{-y}\sin x = -v_x.$$

(d) $f(z) = \left(z^2-2\right)e^{-x}e^{-iy}$ is entire since it is the product of entire functions

$$g(z) = z^2 - 2 \text{ and } h(z) = e^{-x}e^{-iy} = e^{-x}(\cos y - i\sin y) = \underbrace{e^{-x}\cos y}+i\underbrace{\left(-e^{-x}\sin y\right)}.$$

The function g is entire since it is a polynomial, and h is entire since

$$u_x = -e^{-x}\cos y = v_y \text{ and } u_y = -e^{-x}\sin y = -v_x.$$

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2. Show that each of these functions is nowhere analytic:
(a) $f(z) = xy + iy$
(b) $f(z) = 2xy + i\left(x^2 - y^2\right)$.
(c) $f(z) = e^y e^{ix}$
Solution:
(a)
$f(z) = xy + iy$ is nowhere analytic since

$$u_x = v_y \Longrightarrow y = 1 \text{ and } u_y = -v_x \Longrightarrow x = 0,$$

which means that the Cauchy-Riemann equations hold only at the point $z = (0,1) = i$.
(b)
$f(z) = e^y e^{ix} = e^y(\cos x + i\sin x)$ is nowhere analytic since

$$u_x = v_y \Longrightarrow -e^y \sin x = e^y \sin x \Longrightarrow 2e^y \sin x = 0 \Longrightarrow \sin x = 0$$

and

$$u_y = -v_x \Longrightarrow e^y \cos x = -e^y \cos x \Longrightarrow 2e^y \cos x = 0 \Longrightarrow \cos x = 0.$$

More precisely, the roots of the equation $\sin x = 0$ are $n\pi$ $(n = 0, \pm 1, \pm 2, \cdots)$, and $\cos(n\pi) = (-1)^n \neq 0$. Consequently, the Cauchy-Riemann equations are not satisfied anywhere.

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4. In each case, determine the singular points of the function and state why the function is analytic everywhere except at those points:
(a) $f(z) = \dfrac{2z+1}{z\left(z^2+1\right)}$; (b) $f(z) = \dfrac{z^3+i}{z^2-3z+2}$; (c) $f(z) = \dfrac{z^2+1}{(z+2)\left(z^2+2z+2\right)}$.
Solution: (a)
$f(z) = \dfrac{2z+1}{z\left(z^2+1\right)}$; this function is the quotient of two polynomials (a) $f(z) = \dfrac{P(z)}{Q(z)}$, hence it's analytic in any domain throughout which (a) $Q(z) \neq 0$.
$\longrightarrow z\left(z^2+1\right) = 0$ iff (a) $z = 0$ or (a) $z = \pm i$ (ant the numerator does not vanish at these points)
$\Longrightarrow$ singular points: $z = 0, \pm i$ (They are poles, i.e., $\lim_{z \to 0, \pm i} |f(z)| = +\infty$)
(b)
$f(z) = \dfrac{z^3+i}{z^2-3z+2}$ similarly as above, check check where the denominator vanishes:

$z^2 - 3z + 2 = 0 \iff z = \dfrac{3 \pm \sqrt{9-8}}{2} = \dfrac{3 \pm 1}{2} \implies z_1 = 2, z_2 = 1$
and the numerator does not vanish at these points.
$\implies$ singular points $z = 1, 2$ (poles)

(c) $f(z) = \dfrac{z^2+1}{(z+2)(z^2+2z+2)}$
$(z+2)(z^2+2z+2) = 0$ iff $z = -2$ or $z^2 + 2z + 2 = 0 \iff z = -1 \pm \sqrt{1-2} = -1 \pm i$
$\implies$ singular points (poles) $z = -2, -1 \pm i$

(p.78)
7. Let a function $f(z)$ be analytic in a domain D. Prove that $f(z)$ must be constant throughout D if
(a) $f(z)$ is real-valued for all $z \in D$;
(b) $|f(z)|$ is constant throughout D.
Solution:
(a)
Suppose $f(z) \in \mathbb{R}$ for all $z \in D \implies v(x,y) = 0$ on D
$\implies u_x(x,y) = v_y(x,y) = 0$
and
$u_y(x,y) = -v_x(x,y) = 0$ on D.
$\implies \nabla u(x,y) = 0$ on D.
$u(x,y) =$ constant on $D \implies f(z) =$ constant on D.
(b)
Suppose
$|f(z)| = c$ for all $z \in D$
If $c = 0 \implies f(z) = 0$ on D, hence it's constant.
If $c \neq 0 \implies |f(z)|^2 = c^2 \iff f(z)\overline{f(z)} = c^2 \iff \overline{f(z)} = \dfrac{c^2}{f(z)}$
$\implies$ both f and $\overline{f}$ are analytic in D (since $\overline{f} = \dfrac{c^2}{f}$ and $f \neq 0$)
$\implies f(z) =$ constant on D.
(otherway to solve it)
$f(z) = c$ and suppose $c \neq 0 \implies u^2 + v^2 = c^2$
$\implies 2uu_x + 2vv_x = 0$
$2uu_y + 2vv_y = 0$
$\longrightarrow 0 = (uu_x + vv_x)^2 + (uu_y + vv_y)^2 = u^2u_x^2 + v^2v_x^2 + 2uvu_xv_x + u^2u_y^2 + v^2v_y^2 + 2uvu_yv_y$
$\longrightarrow (u^2+v^2)(u_x^2+u_y^2) = 0 \implies u_x^2 + u_y^2 = 0$
$\implies u_x = u_y = 0$ & (by C-R equations) $\implies v_x = v_y = 0$
$\implies u =$ constant $v =$ constant $\implies f(z) =$ constant

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