



Your Name / Adınız - Soyadınız

Your Signature / İmza

Student ID # / Öğrenci No

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Professor's Name / Öğretim Üyesi

Your Department / Bölüm

- This exam is closed book.
- Give your answers in exact form (for example $\frac{\pi}{3}$ or $5\sqrt{3}$), except as noted in particular problems.
- Calculators, cell phones are not allowed.
- In order to receive credit, you must **show all of your work**. If you do not indicate the way in which you solved a problem, you may get little or no credit for it, even if your answer is correct. **Show your work in evaluating any limits, derivatives.**
- Place a box around your answer to each question.
- If you need more room, use the backs of the pages and indicate that you have done so.
- Do not ask the invigilator anything.
- This exam has 3 pages plus this cover sheet and 6 problems. Please make sure that your exam is complete.

| Problem | Points | Score |
|---------|--------|-------|
| 1 | 20 | |
| 2 | 15 | |
| 3 | 15 | |
| 4 | 15 | |
| 5 | 15 | |
| 6 | 20 | |
| Total: | 100 | |

. Time limit is 90 min.

Do not write in the table to the right.

1. Suppose $f(x) = \frac{x^2}{x+1}$.

- (a) 4 Points All critical points of f , and the intervals where f is increasing and decreasing;

Solution: p.212, pr.83 Differentiating, we have

$$f'(x) = \frac{x^2 + 2x}{(x+1)^2}$$

and

$$f''(x) = \frac{2}{(x+1)^3}.$$

Solving $f'(x) = 0$ gives us the critical numbers 0 and -2 . This splits the real line into 4 open subintervals, namely, $(-\infty, -2)$, $(-2, -1)$, $(-1, 0)$, and $(0, \infty)$. By considering test values on each of these intervals, we see that f is increasing on $(-\infty, -2) \cup (0, \infty)$ and decreasing on $(-2, -1) \cup (-1, 0)$.

- (b) 4 Points All inflection points of f , and the open intervals where f is concave up resp. concave down.

Solution: Since $f''(x) = \frac{2}{(x+1)^3} \neq 0$ and there is no domain point for f where the second derivative is undefined, there are no points of inflection for the graph of f . By looking at the sign for f'' , we see that the graph is concave down on $(-\infty, -1)$ and concave up on $(-1, +\infty)$.

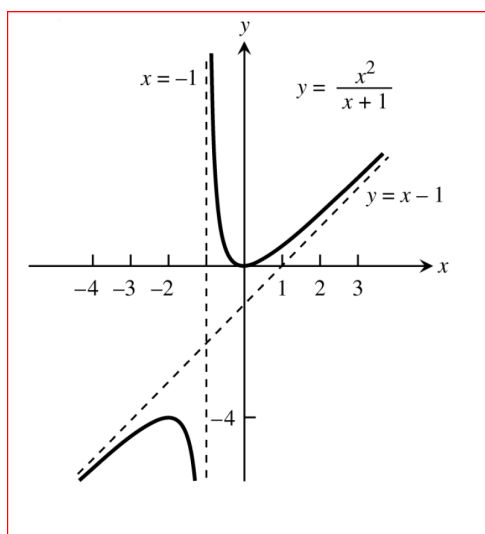
- (c) 4 Points Classify the critical points of f as either local maxima, local minima or neither.

Solution: At $x = 0$, the graph has a local minimum, namely, the point $(0, 0)$ is a point of local minimum and at $x = -2$, there is a local maximum and so the point $(-2, -4)$ is the point of local maximum.

- (d) 4 Points Find the asymptotes.

Solution: The vertical line $x = -1$ is a vertical asymptote as $\lim_{x \rightarrow -1^\pm} \frac{x^2}{x+1} = \pm\infty$. Also since by long division $\frac{x^2}{x+1} = x - 1 + \frac{1}{x+1}$ and $\lim_{x \rightarrow \pm\infty} \frac{1}{x+1} = 0$, the line $y = x - 1$ is the oblique asymptote.

- (e) 4 Points Sketch the graph of f using your results in (a), (b), (c) and (d).



Solution:

2. **15 Points** Find the volume of the largest right circular cone that can be inscribed in a sphere of radius 3.

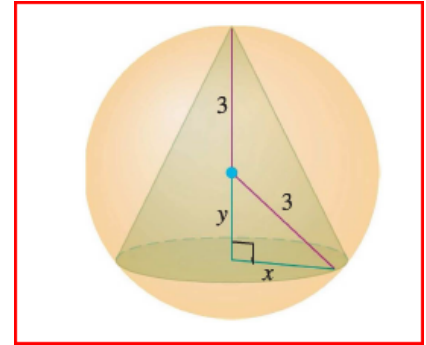
Solution:

p.220, pr.12

The volume of the cone is $V = \frac{1}{3}\pi r^2 h$ where $r = x = \sqrt{9 - y^2}$ and $h = y + 3$ (from the figure). Thus,
 $V(y) = \frac{\pi}{3}(9 - y^2)(y + 3) = \frac{\pi}{3}(27 + 9y - 3y^2 - y^3) \Rightarrow V'(y) = \frac{\pi}{3}(9 - 6y - 3y^2) = \pi(1 - y)(3 + y)$. The critical points are -3 and 1 , but -3 is not in the domain. Thus $V''(1) = \frac{\pi}{3}(-6 - (6)(1)) < 0 \Rightarrow$ at $y = 1$ we have a maximum volume of

$$V(1) = \frac{\pi}{3}(8)(4) = \frac{32\pi}{3}$$

cubic units.

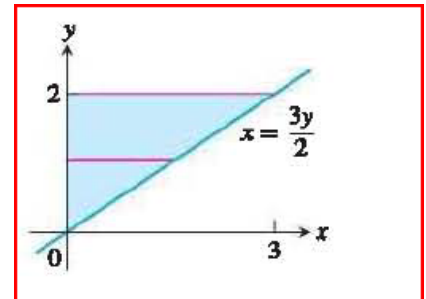


3. **15 Points** Find the volume of the solid generated by revolving the shaded region about y-axis.

Solution:

p.414, pr.108

$$\begin{aligned} R(y) = x = \frac{3y}{2} \Rightarrow V &= \int_0^2 \pi [R(y)]^2 dy \\ &= \pi \int_0^2 \left(\frac{3y}{2}\right)^2 dy \\ &= \pi \int_0^2 \left(\frac{9}{4}\right) y^2 dy \\ &= \pi \left[\frac{3}{4}y^3\right]_0^2 \\ &= \pi \cdot \frac{3}{4} \cdot 8 = 6\pi. \end{aligned}$$



4. **15 Points** Using definite integral for $y = \sqrt{r^2 - x^2}$, $0 \leq x \leq \frac{r}{\sqrt{2}}$ verify that the circumference of the circle of radius r is $2\pi r$.

Solution:

$$\begin{aligned} y &= \sqrt{r^2 - x^2} \Rightarrow \frac{dy}{dx} = \frac{-x}{\sqrt{r^2 - x^2}} \Rightarrow \left(\frac{dx}{dy}\right)^2 = \frac{1}{2y - 1} \Rightarrow L = \int_0^{r/\sqrt{2}} \sqrt{1 + \left(\frac{-x}{\sqrt{r^2 - x^2}}\right)^2} dx \\ &= \int_0^{r/\sqrt{2}} \sqrt{1 + \frac{x^2}{r^2 - x^2}} dx = \int_0^{r/\sqrt{2}} \frac{r}{\sqrt{r^2 - x^2}} dx = \left[r \sin^{-1}\left(\frac{x}{r}\right)\right]_0^{r/\sqrt{2}} \\ \rightarrow L &= r \sin^{-1}\left(\frac{r\sqrt{2}}{r}\right) - r \sin^{-1}(0) = r\left(\frac{\pi}{4}\right) = \frac{\pi r}{4} \end{aligned}$$

The total circumference of the circle is $C = 8L = 8\left(\frac{\pi r}{4}\right) = 2\pi r$.

5. (a) **6 Points** $e^{2x} = \sin(x + 3y) \Rightarrow \frac{dy}{dx} = ?$

Solution: We differentiate the equality implicitly.

$$\begin{aligned}\frac{d}{dx}e^{2x} &= \frac{d}{dx}\sin(x+3y) \Rightarrow 2e^{2x} = \cos(x+3y)\left(1+3\frac{dy}{dx}\right) \\ \Rightarrow 2e^{2x} &= \cos(x+3y) + 3\cos(x+3y)\frac{dy}{dx} \\ \frac{dy}{dx} &= \frac{2e^{2x} - \cos(x+3y)}{3\cos(x+3y)}\end{aligned}$$

(b) 9 Points $\int_0^{x^2} f(t) dt = x \cos(\pi x) \Rightarrow f(4) = ?$

Solution: By the Fundamental Theorem of Calculus Part 1, we have

$$\frac{d}{dx} \int_0^{x^2} f(t) dt = \frac{d}{dx} (x \cos(\pi x)).$$

Hence we have

$$f(x^2)2x = \cos(\pi x) - x\pi \sin(\pi x).$$

Now plug $x = 2$. Hence

$$f(2^2)(2)(2) = \cos(2\pi) - 2\pi \sin(2\pi) = 1 - 2\pi(0) \Rightarrow f(4) = \frac{1}{4}$$

6. (a) 10 Points $\int_1^e \frac{\sqrt{\ln x}}{x} dx = ?$

Solution: Let $u = \ln x$ and so $du = \frac{1}{x} dx$. When $x = 1$ we have $u = \ln 1 = 0$ and also when $x = e$, we have $u = \ln e = 1$. Hence

$$\int_1^e \frac{\sqrt{\ln x}}{x} dx = \int_0^1 \sqrt{u} du = \left[\frac{u^{3/2}}{3/2} \right]_0^1 = \frac{2}{3}$$

(b) 10 Points $\int x 3^{x^2} dx = ?$

Solution: Let $u = x^2$ and so $du = 2x dx$. Then we have

$$\int x 3^{x^2} dx = \frac{1}{2} \int 3^u du = \frac{1}{2} \int 3^u du = \frac{1}{2} \frac{3^u}{\ln 3} + C = \frac{1}{2} \frac{3^{x^2}}{\ln 3} + C$$