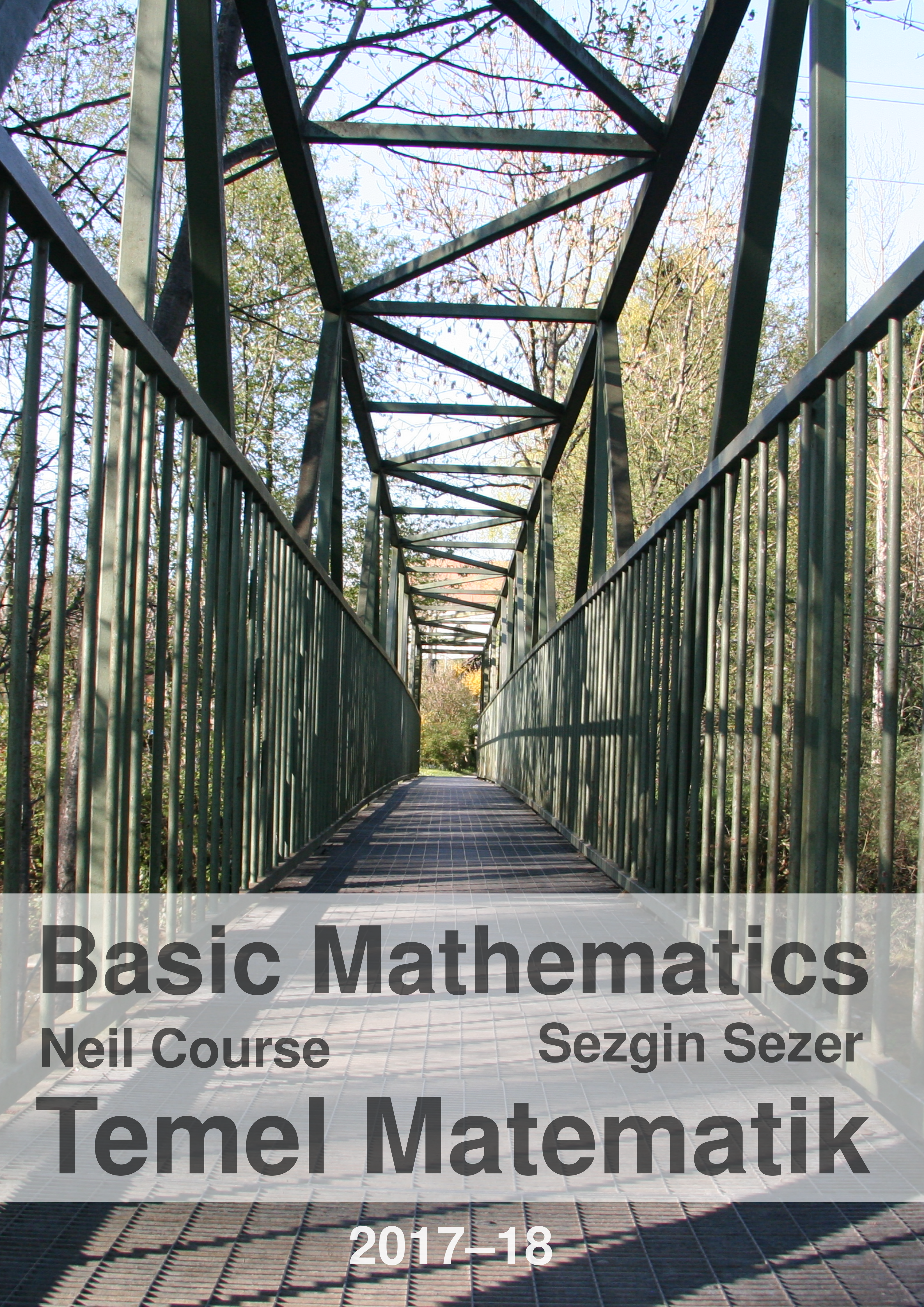

Basic Mathematics

Neil Course
Sezgin Sezer

Temel Matematik

2017–18

www.neilcourse.co.uk/math115.html

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Contents

Part I

Numbers and Functions

Numbers

The Natural Numbers

The set
$$\mathbb{N} = \{1, 2, 3, 4, 5, 6, \ldots\}$$
is called the set of ***natural numbers***. These are the first numbers that children learn. The symbol $\in$ means "***in***". For example

$$\begin{array}{rl} 2 \in \mathbb{N} & \text{means "2 is a natural number"} \\ 7 \in \mathbb{N} & \text{means "7 is a natural number"} \\ \frac{1}{2} \notin \mathbb{N} & \text{means "}\tfrac{1}{2}\text{ is \textbf{not} a natural number"} \\ 0 \notin \mathbb{N} & \text{means "0 is \textbf{not} a natural number"} \\ -5 \notin \mathbb{N} & \text{means "}-5\text{ is \textbf{not} a natural number"} \end{array}$$

In the natural numbers, we can do "+" and "×"

$$2 + 7 = 9 \in \mathbb{N}, \qquad 2 \times 7 = 14 \in \mathbb{N}.$$

However we can not do "−" because

$$2 - 7 \notin \mathbb{N}.$$

So we invent new numbers!

The Integers

The set

$$\mathbb{Z} = \{\ldots, -4, -3, -2, -1, 0, 1, 2, 3, 4, \ldots\}$$

is called the set of ***integers***. We use a $\mathbb{Z}$ for the German word 'zahlen' (numbers). In $\mathbb{Z}$, we can do "+", "−" and "×" but we can not do "÷". For example $3 \in \mathbb{Z}$, $4 \in \mathbb{Z}$, $-5 \in \mathbb{Z}$ and

$$3 + 4 \in \mathbb{Z}, \quad 3 - 4 \in \mathbb{Z}, \quad 3 \times 4 \in \mathbb{Z}, \quad 3 \div 4 \notin \mathbb{Z},$$

$$3 + (-5) \in \mathbb{Z}, \quad 3 - (-5) \in \mathbb{Z}, \quad 3 \times (-5) \in \mathbb{Z}, \quad 3 \div (-5) \notin \mathbb{Z}.$$

So we invent new numbers!

Sayılar

Doğal sayılar

$$\mathbb{N} = \{1, 2, 3, 4, 5, 6, \ldots\}$$

kümesi ***doğal sayılar*** kümesi olarak adlandırırılır. Bunlar çocukluğumuzda ilk öğrenilen sayılardır. $\in$ sembolü "elemanıdır" anlamına gelir. Örneğin,

$$\begin{array}{rl} 2 \in \mathbb{N} & \text{demek "2 bir doğal sayıdır"} \\ 7 \in \mathbb{N} & \text{anlamı "7 bir doğal sayıdır"} \\ \frac{1}{2} \notin \mathbb{N} & \text{anlamı "}\tfrac{1}{2}\text{ bir doğal sayı \textbf{değildir}"} \\ 0 \notin \mathbb{N} & \text{anlamı "0 bir doğal sayı \textbf{değildir}"} \\ -5 \notin \mathbb{N} & \text{anlamı "}-5\text{ bir doğal sayı \textbf{değildir}"} \end{array}$$

Doğal sayılarla "+" ve "×" işlemlerini yaparız.

$$2 + 7 = 9 \in \mathbb{N}, \qquad 2 \times 7 = 14 \in \mathbb{N}.$$

Ne yazık ki "−" işlemini yapamayı, çünkü, örneğin

$$2 - 7 \notin \mathbb{N}.$$

dir.

Bu yüzden yeni sayılar keşfederiz.!

Tam sayılar

$$\mathbb{Z} = \{\ldots, -4, -3, -2, -1, 0, 1, 2, 3, 4, \ldots\}$$

kümesine ***tam sayılar*** denir. Bunu Almanca 'zahlen' (sayılar) kelimesinden $\mathbb{Z}$ ile gösteririz. $\mathbb{Z}$ içerisinde, "+", "−" ve "×" yapabiliriz ama "÷" yapamayız. Örneğin $3 \in \mathbb{Z}$, $4 \in \mathbb{Z}$, $-5 \in \mathbb{Z}$ ve

$$3 + 4 \in \mathbb{Z}, \quad 3 - 4 \in \mathbb{Z}, \quad 3 \times 4 \in \mathbb{Z}, \quad 3 \div 4 \notin \mathbb{Z},$$

$$3 + (-5) \in \mathbb{Z}, \quad 3 - (-5) \in \mathbb{Z}, \quad 3 \times (-5) \in \mathbb{Z}, \quad 3 \div (-5) \notin \mathbb{Z}.$$

Dolayısıyla yeni sayılar keşfederiz!

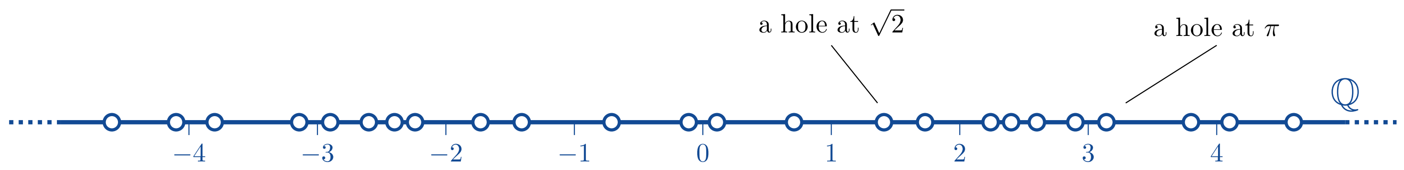


Figure 1.1: The Rational Numbers
Şekil 1.1: Rasyonel Sayılar

The Rational Numbers

The set

$$\mathbb{Q} = \{\text{all fractions}\} = \left\{ \frac{a}{b} \;\middle|\; a, b \in \mathbb{Z} \text{ and } b \neq 0 \right\}$$

is called the set of ***rational numbers***. We use a $\mathbb{Q}$ for the word 'quotient'. For example

$$0 = \frac{0}{1} \in \mathbb{Q} \qquad \frac{100}{13} \in \mathbb{Q}$$
$$1 = \frac{1}{1} \in \mathbb{Q} \qquad \sqrt{2} \notin \mathbb{Q}$$
$$\frac{3}{4} \in \mathbb{Q} \qquad -4 = \frac{8}{-2} \in \mathbb{Q}$$
$$\pi \notin \mathbb{Q} \qquad 0.12345 = \frac{12345}{100000} \in \mathbb{Q}.$$

In $\mathbb{Q}$ we can do "+", "−", "×" and "÷(by a number $\neq 0$)".

Are we happy now?

No!

Why?

Because if we draw all the rational numbers in a line, then the line has lots of holes in it – see figure 1.1. In fact, $\mathbb{Q}$ has ∞ many holes in it.

So we invent new numbers!

The Real Numbers

The set

$$\mathbb{R} = \{\text{all numbers which can be written as a decimal}\}$$

is called the set of ***real numbers***. For example

$$0 = 0.0 \in \mathbb{R} \qquad \frac{100}{13} = 7.692307\ldots \in \mathbb{R}$$
$$\frac{23}{99} = 0.232323\ldots \in \mathbb{R} \qquad \sqrt{2} = 1.414213\ldots \in \mathbb{R}$$
$$\frac{3}{4} = 0.75 \in \mathbb{R} \qquad \frac{123}{999} = 0.123123\ldots \in \mathbb{R}$$
$$\pi = 3.141592\ldots \in \mathbb{R} \qquad \frac{12345}{100000} = 0.12345 \in \mathbb{R}.$$

The real numbers are complete – this means that if we draw all the real numbers in a line, then there are no holes in the line. See figure 1.2 on page 5.

Are we happy now?

Yes!

Rasyonel Sayılar

$$\mathbb{Q} = \{\text{tüm kesirler}\} = \left\{ \frac{a}{b} \;\middle|\; a, b \in \mathbb{Z} \text{ ve } b \neq 0 \right\}$$

kümesine ***rasyonel sayılar*** denir. Bunu $\mathbb{Q}$ ile gösteririz. Örneğin

$$0 = \frac{0}{1} \in \mathbb{Q} \qquad \frac{100}{13} \in \mathbb{Q}$$
$$1 = \frac{1}{1} \in \mathbb{Q} \qquad \sqrt{2} \notin \mathbb{Q}$$
$$\frac{3}{4} \in \mathbb{Q} \qquad -4 = \frac{8}{-2} \in \mathbb{Q}$$
$$\pi \notin \mathbb{Q} \qquad 0.12345 = \frac{12345}{100000} \in \mathbb{Q}.$$

$\mathbb{Q}$ daki sayılarla "+", "−", "×" ve ($\neq 0$ sayılarla) "÷ yapabiliriz".

Şimdi oldu mu?

Hayır!

Neden?

Çünkü rasyonel sayıları bir sayı doğrusu üzerinde gösterirsek, o zaman – şekil 1.1 deki gibi bir sürü rasyonel olmayan sayının karşılık geldiği nokta buluruz . Aslında, $\mathbb{Q}$ da ∞ sayıda delik bulmak mümkündür.

Böylece hala yeni sayılara ihtiyacımız var!

Reel Sayılar

$$\mathbb{R} = \{\text{ondalık olarak yazılabilen sayılar}\}$$

kümesine ***reel sayılar*** kümesi denir. Örneğin

$$0 = 0.0 \in \mathbb{R} \qquad \frac{100}{13} = 7.692307\ldots \in \mathbb{R}$$
$$\frac{23}{99} = 0.232323\ldots \in \mathbb{R} \qquad \sqrt{2} = 1.414213\ldots \in \mathbb{R}$$
$$\frac{3}{4} = 0.75 \in \mathbb{R} \qquad \frac{123}{999} = 0.123123\ldots \in \mathbb{R}$$
$$\pi = 3.141592\ldots \in \mathbb{R} \qquad \frac{12345}{100000} = 0.12345 \in \mathbb{R}.$$

Reel sayılar tamdır – yani bütün reel sayıları sayı eksnınde gösterecek olursak, eksen üzerinde reel sayı karşılık gelmeyen nokta kalmadığını görürüz. Sayfa 5 şekil 1.2i inceleyiniz.

Şimdi tamam mı?

Evet!

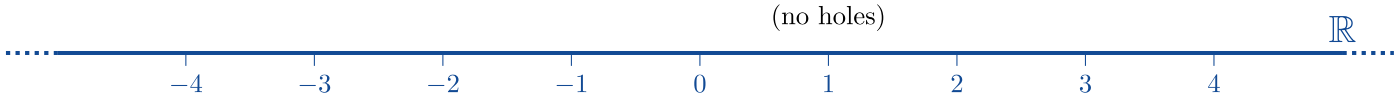


Figure 1.2: The Real Numbers
Şekil 1.2: Reel Sayılar

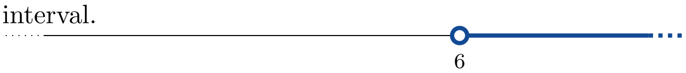
Intervals

A subset of $\mathbb{R}$ is called an ***interval*** if

(i). it contains atleast 2 numbers; and

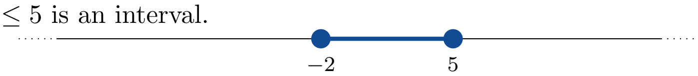
(ii). it doesn't have any holes in it.

Example 1.1. The set $\{x \mid x \text{ is a real number and } x > 6\}$ is an interval.



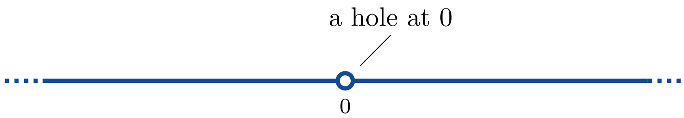
Because 6 is not in this set, we use ○ at 6.

Example 1.2. The set of all real numbers x such that $-2 \leq x \leq 5$ is an interval.



Because −2 and 5 are in this set, we use ● at −2 and 5.

Example 1.3. The set $\{x \mid x \in \mathbb{R} \text{ and } x \neq 0\}$ is not an interval.



A finite interval is

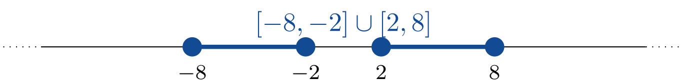
- ***closed*** if it contains both its endpoints;
- ***half-open*** if it contains one of its endpoints;
- ***open*** if it does not contain its endpoints;

as shown in table 1.1 on page 6. An infinite interval is

- ***closed*** if it contains a finite endpoint;
- ***open*** if it is not closed.

There is one exception to this rule: The whole real line is called both open and closed. See table 1.2 on page 6.

We can combine two (or more) intervals with the notation $\cup$. For example, $[-8,-2] \cup [2,8]$ is called the ***union*** of $[-8,-2]$ and $[2,8]$ and is shown below.



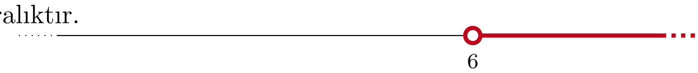
Intervals

$\mathbb{R}$ nin şu iki özelliğini sağlayan bir altkümesine ***aralık*** denir

(i). en az 2 sayı içeriyorsa; ve

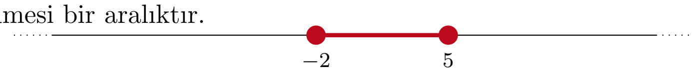
(ii). içerisinde hiç boşluk yoksa.

Örnek 1.1. The set $\{x \mid x \text{ reel sayı ve } x > 6\}$ kümesi bir aralıktır.



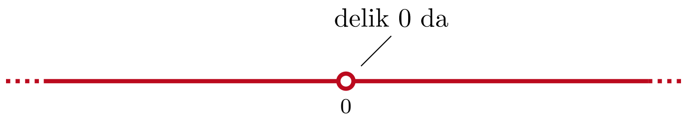
6 bu kümede olmadığından, 6 noktasında ○ olarak gösteririz.

Örnek 1.2. $-2 \leq x \leq 5$ olacak şekildeki tüm x reel sayılarının kümesi bir aralıktır.



−2 ve 5 bu kümede yer aldıklarından, −2 ve 5 noktalarında ● kullanırız.

Örnek 1.3. $\{x \mid x \in \mathbb{R} \text{ ve } x \neq 0\}$ kümesi bir aralık değildir.



Bir sonlu aralık

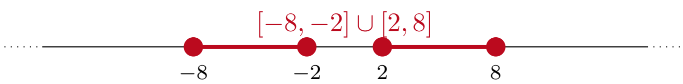
- uç noktalarının her ikisini de içeriyorsa ***kapalı***;
- uç noktalarının birisini içeriyorsa ***yarı-açık***;
- uç noktalarının hiçbirini içermiyorsa, ***açık*** olarak adlandırılır.

6 daki tablo 1.1 gösterilmektedir. Bir sonsuz aralık

- bir sonlu uç noktasını içeriyorsa ***kapalı***;
- kapalı değilse de ***açık*** adını alır.

Bu kuralın bir istisnası vardır: Tüm reel sayı doğrusu hem açık hem kapalıdır. Bakınız sayfa 6 tablo 1.2.

İki (veya daha fazla) aralığı, $\cup$ notasyonu ile birleştirebiliriz. Örneğin $[-8,-2] \cup [2,8]$ 'a $[-8,-2]$ ve $[2,8]$ in ***birleşimi*** denir ve aşağıdaki şekilde gösterilmiştir.



Notation Notasyon	Set Açıklama	Type Tip	Picture Resim
(a, b)	$\{x \mid a < x < b\}$	open / açık	
$[a, b]$	$\{x \mid a \leq x \leq b\}$	closed / kapalı	
$[a, b)$	$\{x \mid a \leq x < b\}$	half open / yarı-açık	
$(a, b]$	$\{x \mid a < x \leq b\}$	half open / yarı-açık	

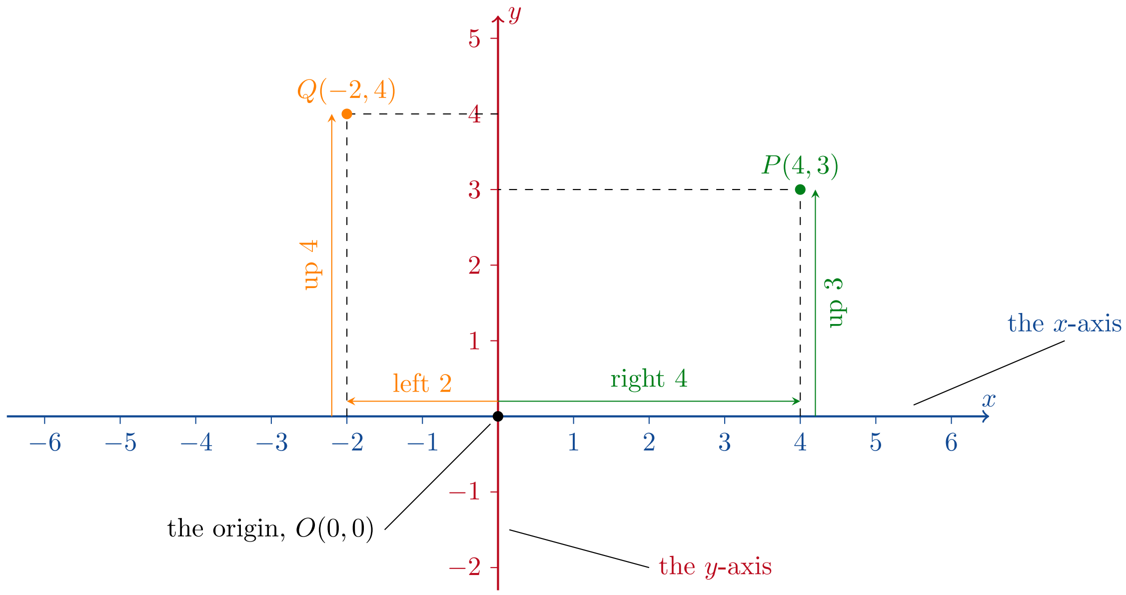
Table 1.1: Types of Finite Interval
Tablo 1.1: Sonlu Aralık Çeşitleri

Notation Notasyon	Set Açıklama	Type Tip	Picture Resim
(a, ∞)	$\{x \mid a < x\}$	open / açık	
$[a, \infty)$	$\{x \mid a \leq x\}$	closed / kapalı	
$(-\infty, b)$	$\{x \mid x < b\}$	open / açık	
$(-\infty, b]$	$\{x \mid x \leq b\}$	closed / kapalı	
$(-\infty, \infty)$	$\mathbb{R}$	both open and closed hem açık hem kapalı	

Table 1.2: Types of Infinite Interval
Tablo 1.2: Sonsuz Aralık Çeşitleri

2

Cartesian Coordinates Kartezyen Koordinatlar



Definition. The set

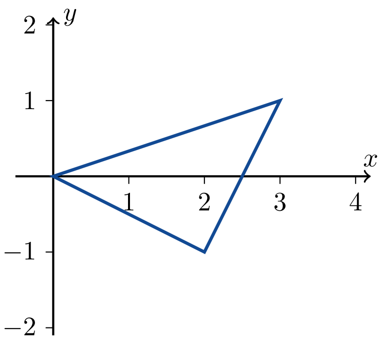
$$\{(x, y)|x, y \in \mathbb{R}\}$$

is denoted by $\mathbb{R}^2$.

Definition. The point $O(0, 0)$ is called the ***origin***.

Example 2.1. Let $A(2, -1)$ and $B(3, 1)$be points in $\mathbb{R}^2$. Draw the triangle OAB.

solution:



Tanım.

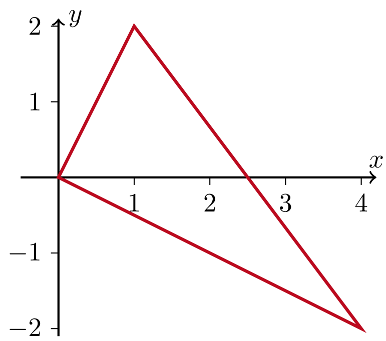
$$\{(x, y)|x, y \in \mathbb{R}\}$$

kümesini $\mathbb{R}^2$ ile gösteririz.

Tanım. $O(0, 0)$ noktası ***orijin*** olarak adlandırılır.

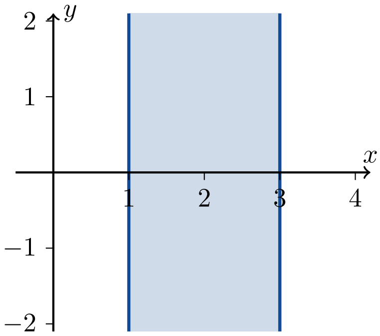
Örnek 2.2. $A(1, 2)$ ve $B(4, -2)$, $\mathbb{R}^2$ de noktalar olsun. OAB üçgenini çiziniz.

çözüm:



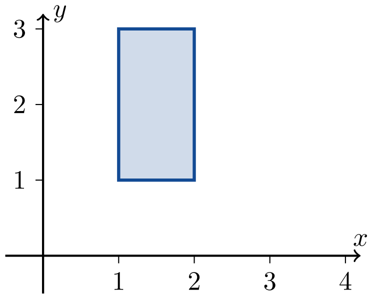
Example 2.3. Draw the region of points which satisfy $1 \leq x \leq 3$.

solution:



Example 2.5. Draw the region of points which satisfy $1 \leq x \leq 2$ and $1 \leq y \leq 3$.

solution:



Distance in $\mathbb{R}^2$.

Definition. The ***distance*** between $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ is

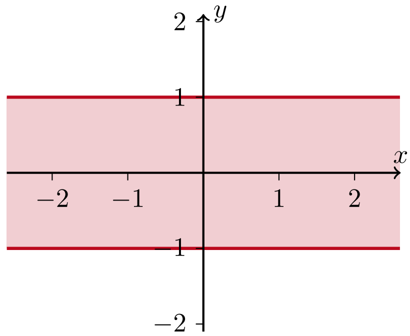
$$\|P_1P_2\| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

Example 2.7. The distance between $A(1, 3)$ and $B(4, -1)$ is

$$\|AB\| = \sqrt{(4-1)^2 + (-1-3)^2} = \sqrt{3^2 + (-4)^2} = \sqrt{25} = 5.$$

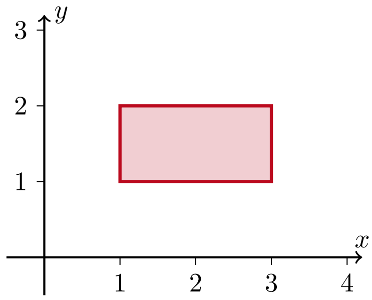
Örnek 2.4. $-1 \leq y \leq 1$ koşulunu sağlayan bölgeyi çiziniz.

çözüm:



Örnek 2.6. $1 \leq x \leq 3$ ve $1 \leq y \leq 2$ eşitsizliklerinin sağladığı bölgeyi çiziniz.

çözüm:



Distance in $\mathbb{R}^2$.

Tanım. $P_1(x_1, y_1)$ ve $P_2(x_2, y_2)$ arasındaki ***uzaklık***

$$\|P_1P_2\| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

Örnek 2.7. $A(1, 3)$ ve $B(4, -1)$ arasındaki uzaklık

$$\|AB\| = \sqrt{(4-1)^2 + (-1-3)^2} = \sqrt{3^2 + (-4)^2} = \sqrt{25} = 5.$$

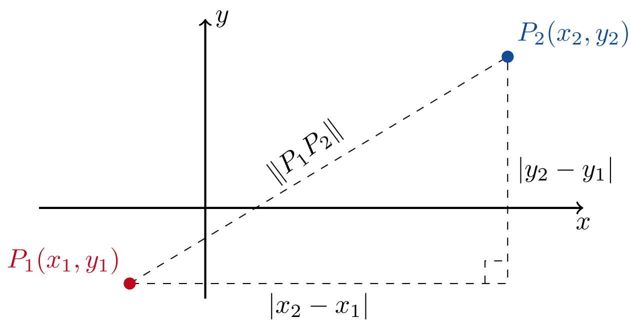


Figure 2.1: The distance between P_1 and P_2 is easy to calculate using Pythagoras.
Şekil 2.1: P_1 ve P_2 arasındaki uzaklık Pisagor bağıntısı kullanarak kolayca elde edilebilir.

3 Functions

Fonksiyonlar

dependent variable
bağımlı değişken

"y is equal to f of x"

$$y = f(x)$$

"y eşittir f x"

function
fonksiyon

independent variable
bağımsız değişken

Definition. A ***function*** from a set D to a set Y is a rule that assigns a unique element of Y to each element of D.

Definition. The set D of all possible values is called the ***domain*** of f.

Definition. The set Y is called the ***target*** of f.

Definition. The set of all possible values of $f(x)$ is called the ***range*** of f.

If f is a function with domain D and target Y, we can write

$$f : D \to Y$$

domain — target

Example 3.1. $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$.

Example 3.2. $f : (-\infty, \infty) \to [0, \infty)$, $f(x) = x^2$.

Tanım. D ve Y boş olmayan iki küme olmak üzere D nin her bir elemanını Y nin sadece bir elemanına eşleyen kurala ***fonksiyon*** denir.

Tanım. D kümesine f nin ***tanım kümesi*** denir.

Tanım. Y kümesine f nin ***değer kümesi*** denir.

Tanım. Bütün mümkün $f(x)$ değerlerinin kümesine f nin ***görüntü kümesi*** denir.

Eğer f tanım kümesi D ve değer kümesi Y olan bir fonksiyon ise, bunu şöyle gösteririz

$$f : D \to Y$$

tanım kümesi — değer kümesi

Örnek 3.1. $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$.

Örnek 3.2. $f : (-\infty, \infty) \to [0, \infty)$, $f(x) = x^2$.

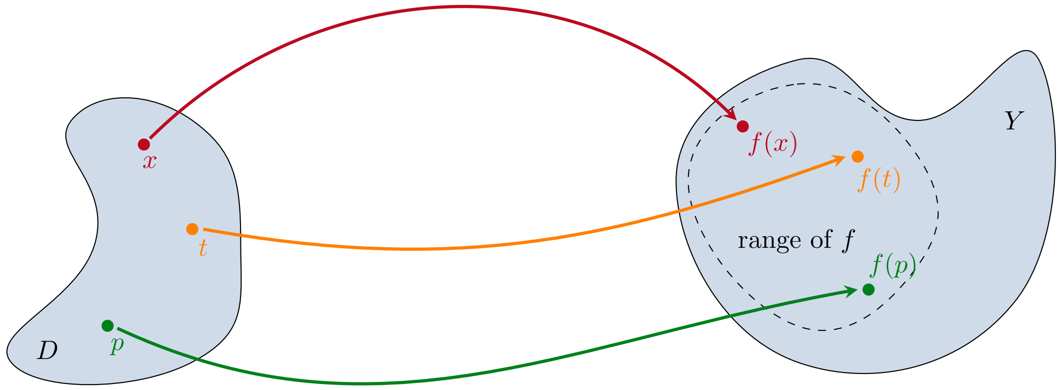


Figure 3.1: A function $f : D \to Y$.
Şekil 3.1: $f : D \to Y$ Bir Fonksiyon.

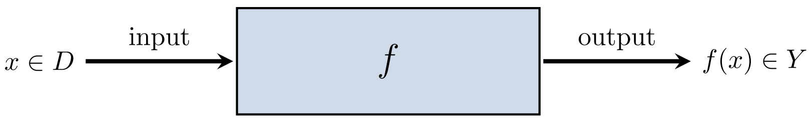


Figure 3.2: A function $f : D \to Y$.
Şekil 3.2: $f : D \to Y$ Bir Fonksiyon.

function fonksiyon	domain (x) tanım kümesi (x)	range (y) görüntü kümesi (y)
$y = x^2$	$(-\infty, \infty)$	$[0, \infty)$
$y = \frac{1}{x}$	$\{x \mid x \in \mathbb{R}, x \neq 0\}$	$\{x \mid x \in \mathbb{R}, x \neq 0\}$
$y = \sqrt{x}$	$[0, \infty)$	$[0, \infty)$
$y = \sqrt{4 - x}$	$(\infty, 4]$	$[0, \infty)$
$y = \sqrt{1 - x^2}$	$[-1, 1]$	$[0, 1]$
$y = x^2$	$[1, 2]$	$[1, 4]$
$y = x^2$	$[2, \infty)$	$[4, \infty)$
$y = x^2$	$(-\infty, -2]$	$[4, \infty)$
$y = 1 + x^2$	$[1, 3)$	$[2, 10)$
$y = 1 - \sqrt{x}$	$[0, \infty)$	$(-\infty, 1]$

Table 3.1: Domains and ranges of some fuctions.
Tablo 3.1: Bazı fonksiyonların tanım ve görüntü kümeleri.

Graphs of Functions

Definition. The ***graph*** of f is the set containing all the points (x, y) which satisty $y = f(x)$.

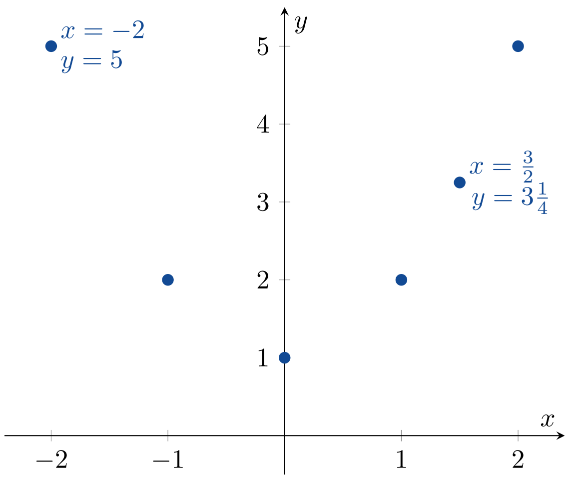
Example 3.3. Graph the function $y = 1+x^2$ over the interval $[-2, 2]$.

solution:

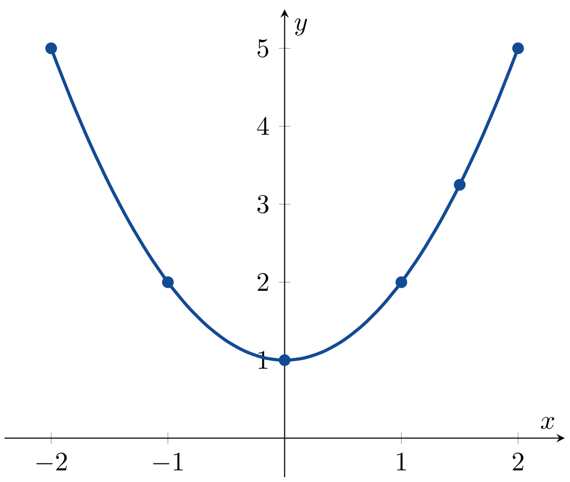
STEP 1. Make a table of (x, y) points which satisfy $y = 1+x^2$.

x	y
-2	5
-1	2
0	1
1	2
$\frac{3}{2}$	$\frac{13}{4} = 3\frac{1}{4}$
2	5

STEP 2. Plot these points.



STEP 3. Draw a smooth curve through these points.



Fonksiyonların Grafikleri

Tanım. $y = f(x)$ eşitliğini sağlayan (x, y) noktalarının kümesine f nin ***grafiği*** denir.

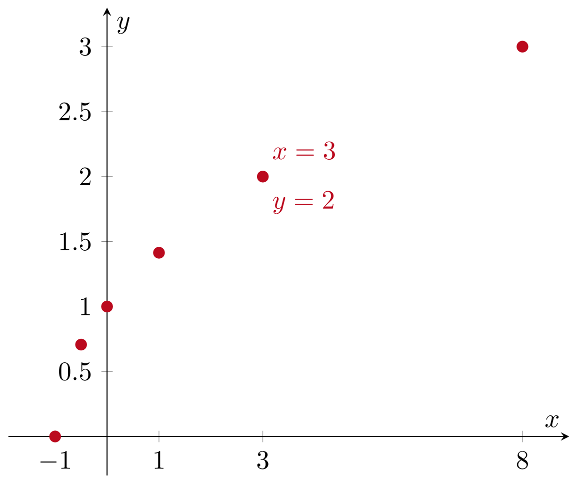
Örnek 3.4. $y = \sqrt{1+x}$ fonksiyonunun $[-1, 8]$ aralığındaki grafiğini çiziniz.

çözüm:

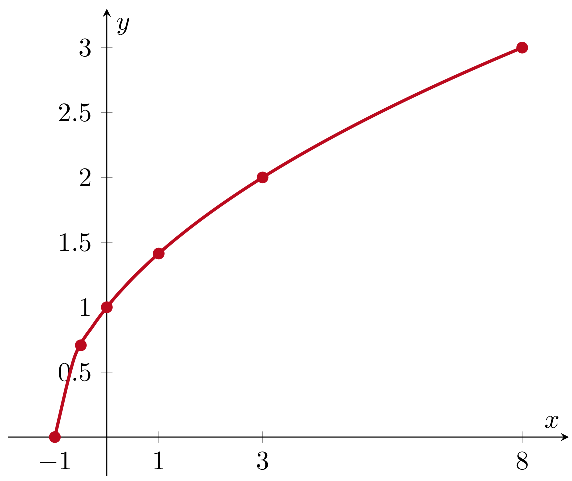
ADIM 1. $y = \sqrt{1+x}$ eşitliğini sağlayan (x, y) noktalarının bir tablosunu yapın.

x	y
-1	0
$-\frac{1}{2}$	≈ 0.707
0	1
1	≈ 1.414
3	2
8	3

ADIM 2. Bu noktaları koordinat sisteminde gösterin.



ADIM 3. Bu noktalardan geçen pürüzsüz bir eğri çiziniz.



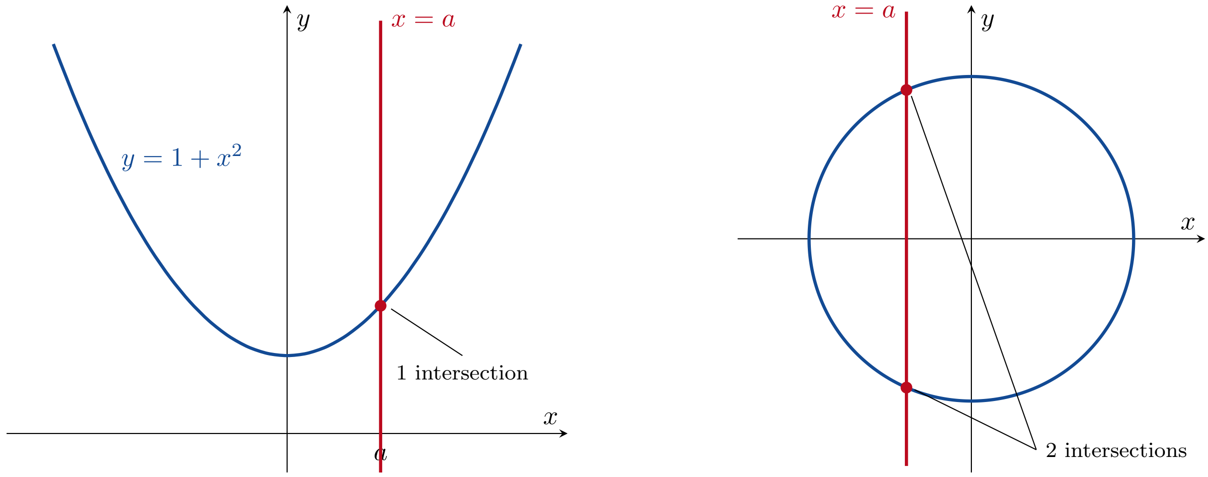


Figure 3.3: The Vertical Line Test.
Şekil 3.3: Dikey Doğru Testi

The Vertical Line Test

Not every curve that you draw is a graph of a function. A function can have only one value $f(x)$ for each $x \in D$. This means that a vertical line can intersect the graph of a function at most once.

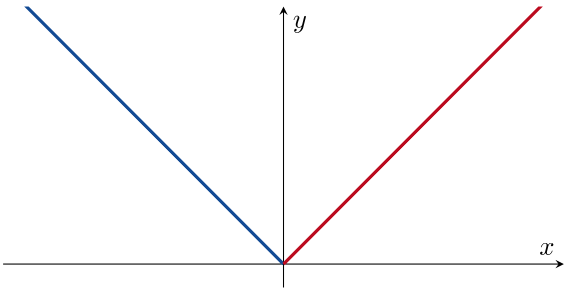
See figure 3.3. A circle can not be the graph of a function because some vertical lines intersect the circle at two points.

If $a \in D$, then the vertical line $x = a$ will intersect the graph of $f : D \to Y$ only at the point $(a, f(a))$.

Piecewise-Defined Functions

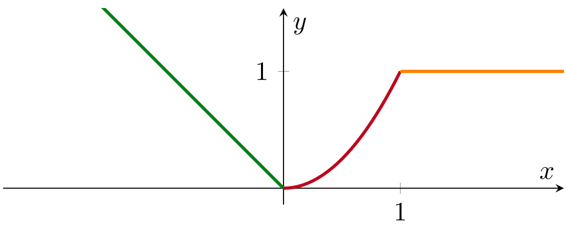
Example 3.5.

$$|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$$



Example 3.6.

$$f(x) = \begin{cases} -x & x < 0 \\ x^2 & 0 \leq x \leq 1 \\ 1 & x > 1 \end{cases}$$



Düşey Doğru Testi

Çizdiğiniz her eğri bir fonksiyonun grafiği değildir. Bir fonksiyon her $x \in D$ için yalnızca bir tane $f(x)$ değerine sahip olabilir. Bu, düşey her doğrunun, bir fonksiyonunun grafiğini en fazla bir kez kesebileceği anlamına gelir.

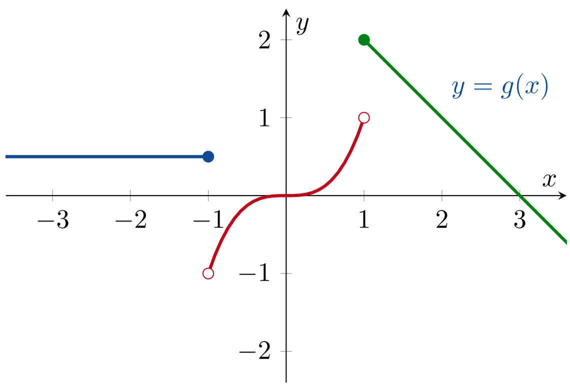
Bakınız şekil 3.3. Bir çember, bir fonksiyonun grafiği olamaz; çünkü bazı düşey doğrular çemberi iki noktada keser.

$a \in D$ ise, $x = a$ düşey doğrusu $f : D \to Y$'nin grafiğini $(a, f(a))$ noktasında kesecektir.

Parçalı Tanımlı Fonksiyonlar

Örnek 3.7.

$$g(x) = \begin{cases} \frac{1}{2} & x \leq -1 \\ x^3 & -1 < x < 1 \\ 3 - x & x \geq 1 \end{cases}$$



Increasing and Decreasing Functions

Definition. Let I be an interval. Let $f : I \to \mathbb{R}$ be a function.

(i). f is called ***increasing on I*** if

$$f(x_1) < f(x_2)$$

for all $x_1, x_2 \in I$ which satisfy $x_1 < x_2$;

(ii). f is called ***decreasing on I*** if

$$f(x_1) > f(x_2)$$

for all $x_1, x_2 \in I$ which satisfy $x_1 < x_2$.

Artan ve Azalan Fonksiyonlar

Tanım. I bir aralık ve $f : I \to \mathbb{R}$ bir fonksiyon olsun.

(i). her $x_1, x_2 \in I$ için $x_1 < x_2$ iken

$$f(x_1) < f(x_2)$$

oluyorsa f ye ***I da artan*** denir;

(ii). her $x_1, x_2 \in I$ için $x_1 < x_2$ iken

$$f(x_1) > f(x_2)$$

oluyorsa f ye ***I da azalan*** denir.

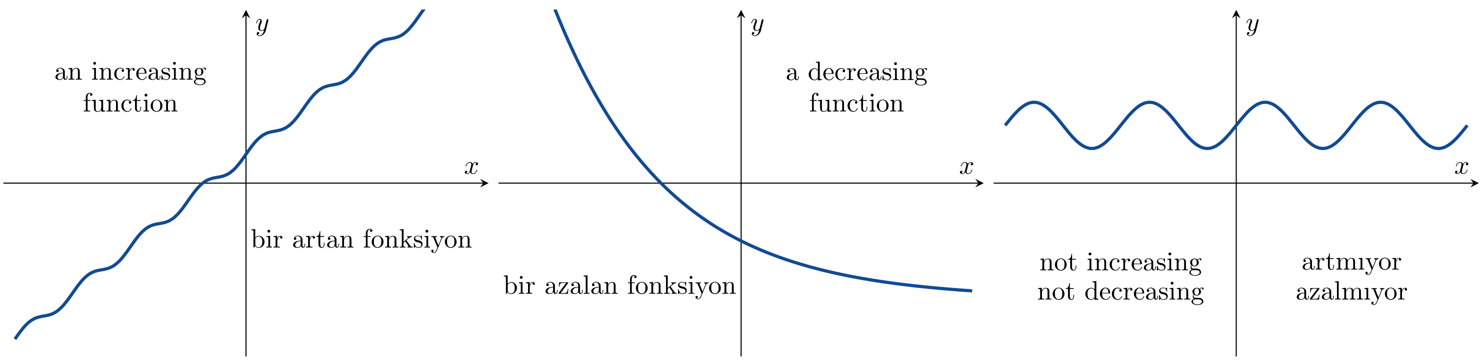


Figure 3.4: A increasing function, a decreasing function and a function which is neither increasing nor decreasing.
Şekil 3.4:

Even Functions and Odd Functions

Recall that

- $2, 4, 6, 8, 10, \ldots$ are even numbers; and
- $1, 3, 5, 7, 9, \ldots$ are odd numbers.

Definition.

(i). $f : D \to \mathbb{R}$ is an ***even function*** if $f(-x) = f(x)$ for all $x \in D$;

(ii). $f : D \to \mathbb{R}$ is an ***odd function*** if $f(-x) = -f(x)$ for all $x \in D$.

Example 3.8. $f(x) = x^2$ is an even function because

$$f(-x) = (-x)^2 = x^2 = f(x).$$

See figure 3.5.

Example 3.9. $f(x) = x^3$ is an odd function because

$$f(-x) = (-x)^3 = -x^3 = -f(x).$$

See figure 3.6.

Çift Fonksiyonlar ve Tek Fonsiyonlar

Hatırlayalım ki

- $2, 4, 6, 8, 10, \ldots$ sayıları çift; ve
- $1, 3, 5, 7, 9, \ldots$ sayıları da tek sayılardır.

Tanım.

(i). Bir $f : D \to \mathbb{R}$ fonksiyona her $x \in D$ için $f(-x) = f(x)$ oluyorsa ***çift fonksiyon*** denir ;

(ii). $f : D \to \mathbb{R}$ fonksiyonu her $x \in D$ için $f(-x) = -f(x)$ oluyorsa ***tek fonksiyon*** adını alır.

Örnek 3.8. $f(x) = x^2$ bir çift fonksiyondur çünkü

$$f(-x) = (-x)^2 = x^2 = f(x).$$

Bakınız şekil 3.5.

Örnek 3.9. $f(x) = x^3$ bir tek fonksiyondur çünkü

$$f(-x) = (-x)^3 = -x^3 = -f(x).$$

Bakınız şekil 3.6.

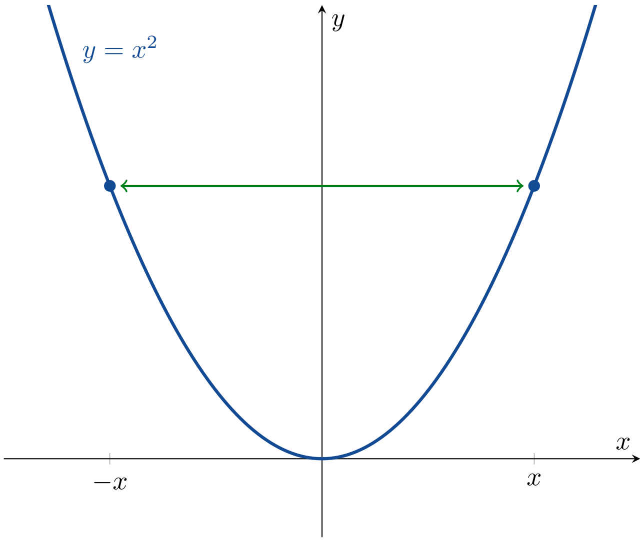


Figure 3.5: 2 is an even number and $f(x) = x^2$ is an even function.
Şekil 3.5: 2 bir çift sayıdır ve $f(x) = x^2$ bir çift fonksiyondur.

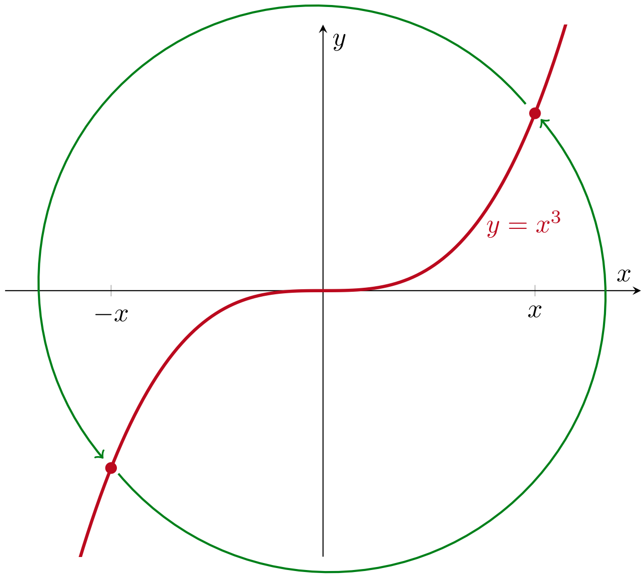


Figure 3.6: 3 is an odd number and $f(x) = x^3$ is an odd function.
Şekil 3.6: 3 bir tek sayıdır ve $f(x) = x^3$ bir tek fonksiyon.

Example 3.10. Is $f(x) = x^2 + 1$ even, odd or neither?

solution: Since

$$f(-x) = (-x)^2 + 1 = x^2 + 1 = f(x),$$

f is an even function.

Example 3.11. Is $g(x) = x + 1$ even, odd or neither?

solution: Since $g(-2) = -2 + 1 = -1$ and $g(2) = 3$, we have $g(-2) \neq g(2)$ and $g(-2) \neq -g(2)$. Hence g is neither even nor odd.

Örnek 3.10. $f(x) = x^2 + 1$ fonksiyonu çift, tek yoksa hiçbiri mi?

çözüm:

$$f(-x) = (-x)^2 + 1 = x^2 + 1 = f(x),$$

f olduğundan bir çift fonksiyondur.

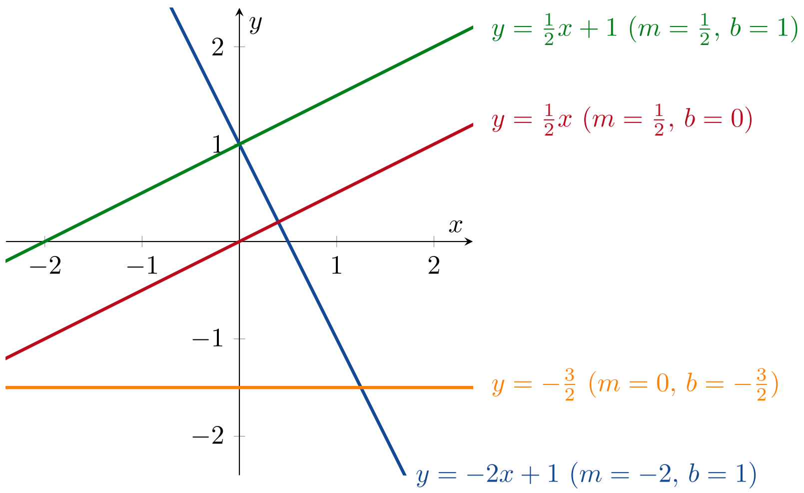
Örnek 3.11. $g(x) = x + 1$ fonksiyonu çift, tek veya hiçbirisi mi?

çözüm: $g(-2) = -2 + 1 = -1$ ve $g(2) = 3$ olduğundan, $g(-2) \neq g(2)$ ve $g(-2) \neq -g(2)$ olur. Böylece g fonksiyonu ne çift fonksiyondur ne de tek.

Linear Functions

Lineer Fonksiyonlar

$$f(x) = mx + b$$
$$(m, b \in \mathbb{R})$$

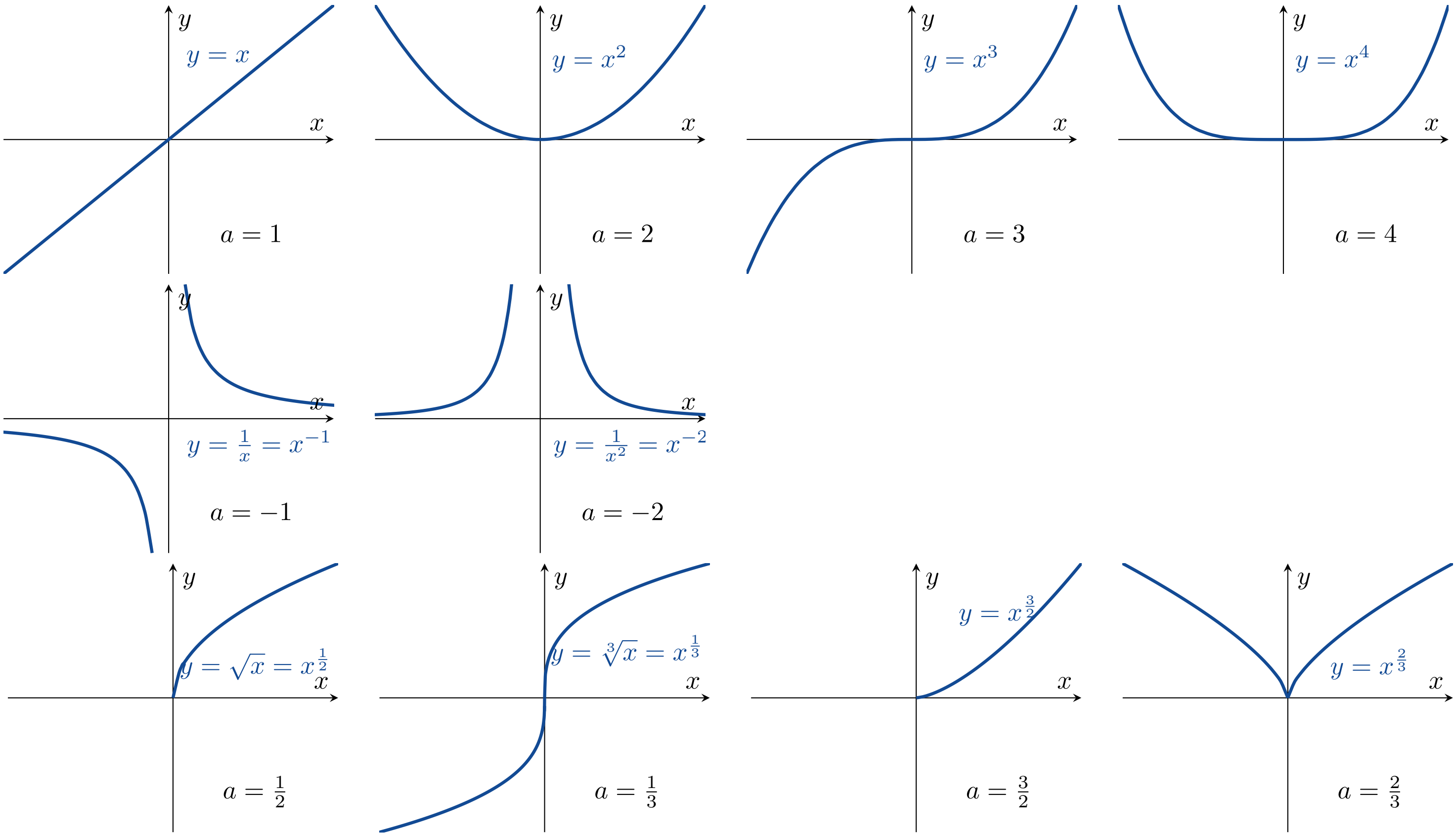


Power Functions

Kuvvet Fonksiyonları

"x to the power of a"

$$f(x) = x^a \qquad (a \in \mathbb{R})$$



Polynomials

Polinomlar

$$p(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0$$

$(n \in \mathbb{N} \cup \{0\},\ a_j \in \mathbb{R})$.

The domain of a polynomial is always $(-\infty, \infty)$. If $n > 0$ and $a_n \neq 0$, then n is called the ***degree*** of $p(x)$.

Bir polinomun tanım kümesi $(-\infty, \infty)$ dur. $n > 0$ ve $a_n \neq 0$ ise, n tamsayısına $p(x)$ in ***derecesi*** denir.

Rational Functions

Rasyonel Fonsiyonlar

$$f(x) = \frac{p(x)}{q(x)}$$

rational function / rasyonel fonksiyon: $f(x)$

polynomial / polinom fonksiyon: $p(x)$, $q(x)$

Example 3.12.

$$f(x) = \frac{2x^3 - 3}{7x + 4}$$

Örnek 3.13.

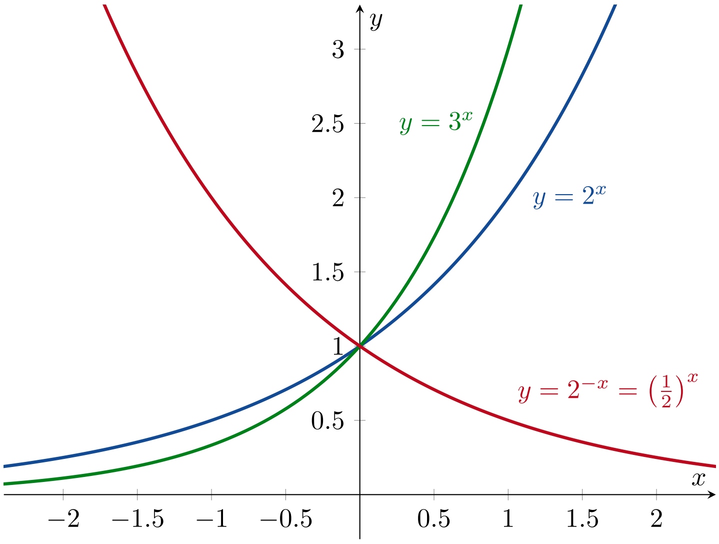
$$g(x) = \frac{5x^2 + 8x - 3}{3x^2 + 2}$$

Exponential Functions

$$f(x) = a^x$$
$$(a \in \mathbb{R},\ a > 0,\ a \neq 1)$$

The domain of an exponential function is $(-\infty, \infty)$.

Üstel Fonksiyonlar



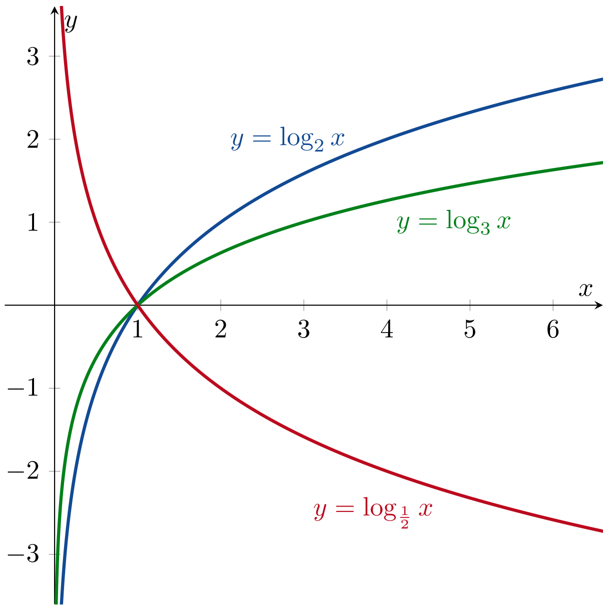
Üstel fonksiyonun tanım kümesi $(-\infty, \infty)$ dur.

Logarithmic Functions

$$y = \log_a x \iff x = a^y$$
$$(a \in \mathbb{R},\ a > 0,\ a \neq 1)$$

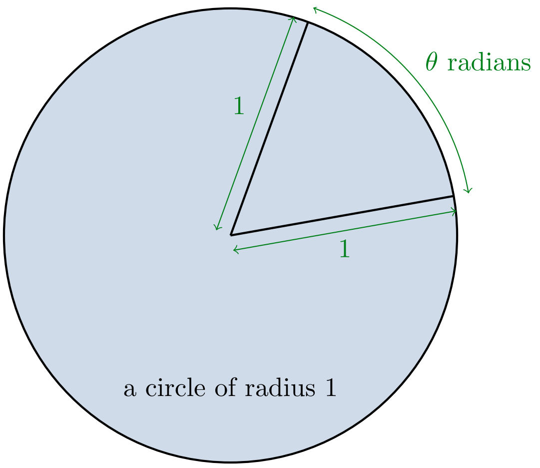
"log base a of x"

Logaritmik Fonksiyonlar



Angles

There are two ways to measure angles. Using degrees or using radians.



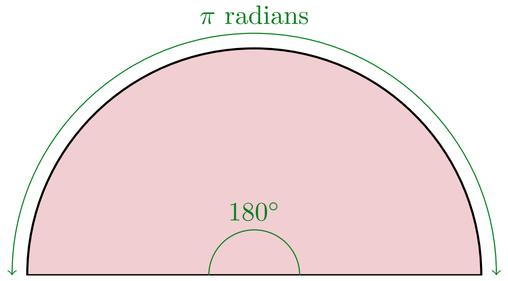
We have that

$$\pi \text{ radians} = 180 \text{ degrees}$$
$$1 \text{ radian} = \frac{180}{\pi} \text{ degrees}$$
$$1 \text{ degree} = \frac{\pi}{180} \text{ radians.}$$

Remark. In Calculus, we use radians!!!! If you see an angle in Part II of this course, it will be in radians. Calculus doesn't work with degrees!!

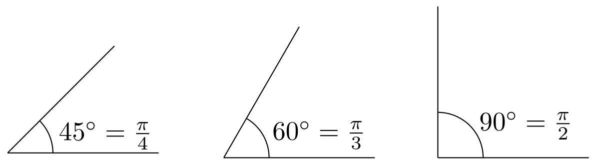
Açılar

Açı ölçmede iki yol vardır. Derece kullanarak veya radyan kullanarak



Şu bağıntılar mevcuttur.

$$\pi \text{ radyan} = 180 \text{ derece}$$
$$1 \text{ radyan} = \frac{180}{\pi} \text{ derece}$$
$$1 \text{ derece} = \frac{\pi}{180} \text{ radyan.}$$



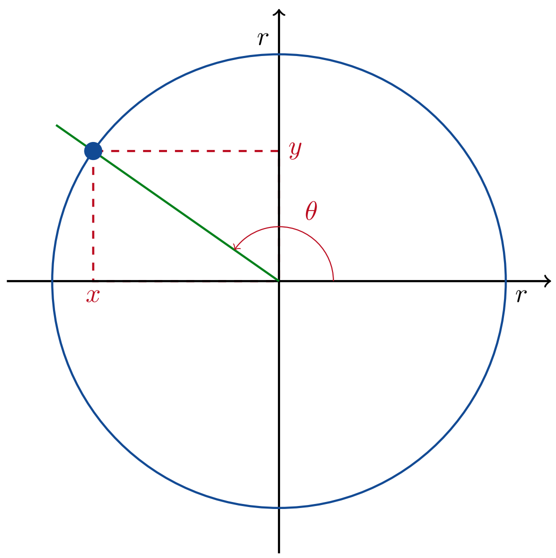
Not. Kalkülüste radyan kullanırız!!!! Bu dersin II kısmında bir açı görürseniz, o radyan cinsinden olacaktır. Kalkülüste derece kullanmayacağız!!

Trigonometric Functions

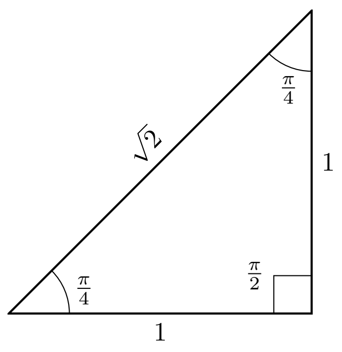
sine	$\sin\theta = \frac{y}{r}$	sinüs
cosine	$\cos\theta = \frac{x}{r}$	kosinüs
tangent	$\tan\theta = \frac{\sin\theta}{\cos\theta}$	tanjant
secant	$\sec\theta = \frac{1}{\cos\theta}$	sekant
cosecant	$\operatorname{cosec}\theta = \csc\theta = \frac{1}{\sin\theta}$	kosekant
cotangent	$\cot\theta = \frac{1}{\tan\theta}$	kotanjant

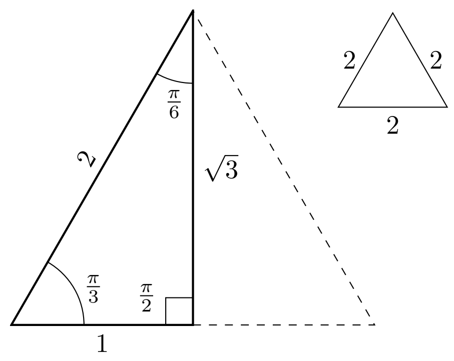
Remark. Note that $\tan\theta$ and $\sec\theta$ are only defined if $\cos\theta \neq 0$; and $\operatorname{cosec}\theta$ and $\cot\theta$ are only defined if $\sin\theta \neq 0$.

Trigonometrik Fonsiyonlar



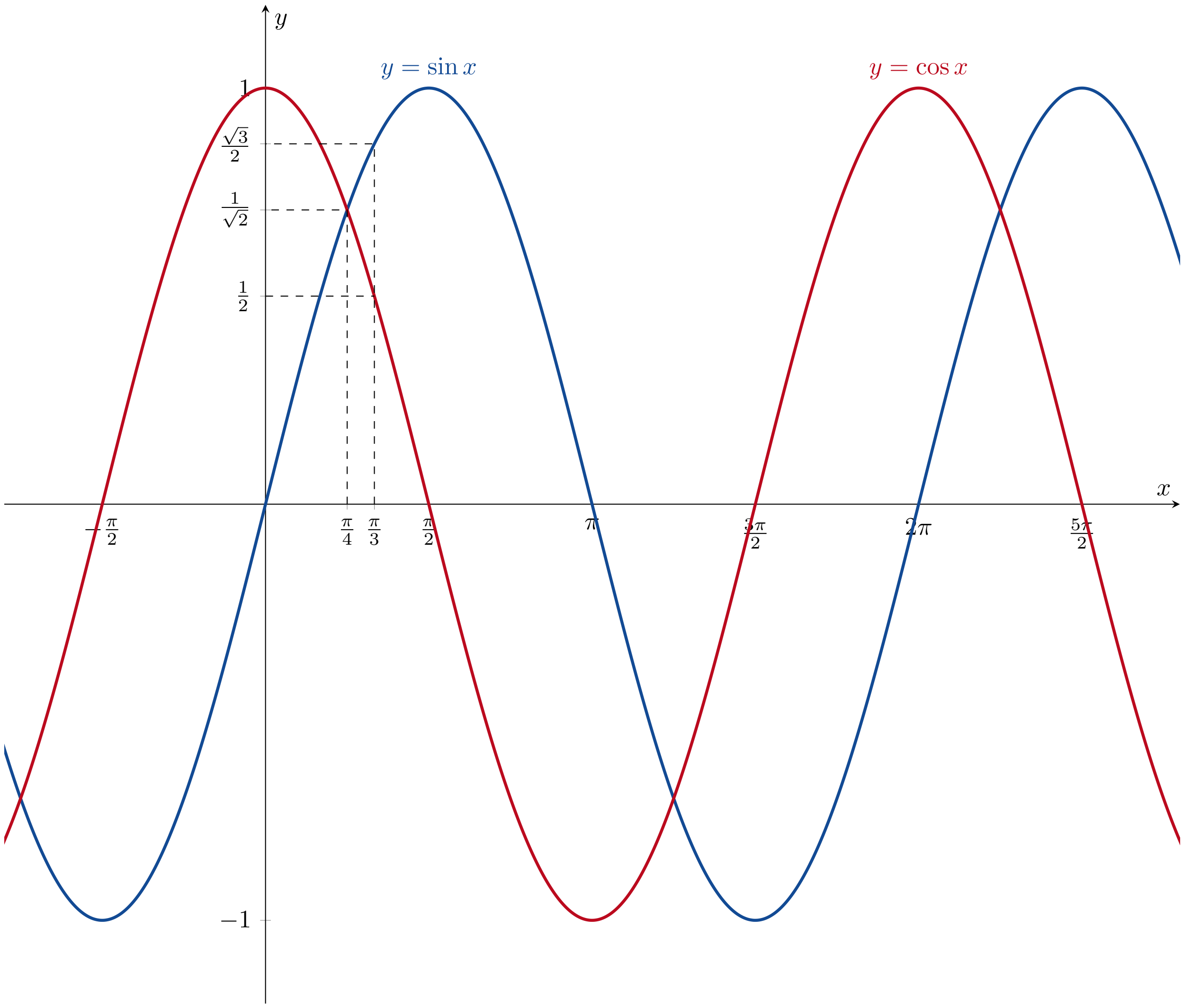
Not. $\tan\theta$ ve $\sec\theta$ nın sadece $\cos\theta \neq 0$ olduğunda; ve $\operatorname{cosec}\theta$ ve $\cot\theta$ nın da tam olarak $\sin\theta \neq 0$ ise tanımlı olduklarına dikkat edin.





$$\sin 45^\circ = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$$
$$\cos 45^\circ = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$$
$$\tan 45^\circ = \tan\frac{\pi}{4} = 1$$
$$\sec 45^\circ = \sec\frac{\pi}{4} = \sqrt{2}$$
$$\operatorname{cosec} 45^\circ = \operatorname{cosec}\frac{\pi}{4} = \sqrt{2}$$
$$\cot 45^\circ = \cot\frac{\pi}{4} = 1$$

$$\sin 60^\circ = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$$
$$\cos 60^\circ = \cos\frac{\pi}{3} = \frac{1}{2}$$
$$\tan 60^\circ = \tan\frac{\pi}{3} = \sqrt{3}$$
$$\sec 60^\circ = \sec\frac{\pi}{3} = \frac{2}{\sqrt{3}}$$
$$\operatorname{cosec} 60^\circ = \operatorname{cosec}\frac{\pi}{3} = 2$$
$$\cot 60^\circ = \cot\frac{\pi}{3} = \frac{1}{\sqrt{3}}$$



Problems

Problem 3.1 (Even and Odd Functions). State whether the following functions are even, odd or neither.

(a) $f(x) = 3$

(b) $f(x) = x^{77}$

(c) $f(x) = x^2 + 1$

(d) $f(x) = x^3 + x$

(e) $f(x) = x^3 + x$

(f) $f(x) = x^3 + 1$

(g) $f(x) = \frac{1}{x^2-1}$

(h) $f(x) = \frac{1}{x^2+1}$

(i) $f(x) = \frac{1}{x-1}$

(j) $f(x) = \sin x$

(k) $f(x) = 2x + 1$

(l) $f(x) = \cos x$

Problem 3.2 (Pointwise-Defined Functions). Graph the function $g : \mathbb{R} \to \mathbb{R}$ defined by

$$g(x) = \begin{cases} \frac{1}{x}, & x < 0 \\ x & x \geq 0. \end{cases}$$

Problem 3.3 (Rational Functions). Graph the following three functions on the same axes:

(i). $f : (0, \infty) \to \mathbb{R}$, $f(x) = x$;

(ii). $g : (0, \infty) \to \mathbb{R}$, $g(x) = \frac{1}{x}$;

(iii). $h : (0, \infty) \to \mathbb{R}$, $h(x) = x + \frac{1}{x}$.

Sorular

Soru 3.1 (Tek ve Çıft Fonksiyonlar). Aşağıdaki fonksiyonların çift, tek veya hiçbirisi olup olmadığını bulunuz.

(a) $f(x) = 3$

(b) $f(x) = x^{77}$

(c) $f(x) = x^2 + 1$

(d) $f(x) = x^3 + x$

(e) $f(x) = x^3 + x$

(f) $f(x) = x^3 + 1$

(g) $f(x) = \frac{1}{x^2-1}$

(h) $f(x) = \frac{1}{x^2+1}$

(i) $f(x) = \frac{1}{x-1}$

(j) $f(x) = \sin x$

(k) $f(x) = 2x + 1$

(l) $f(x) = \cos x$

Soru 3.2 (Parçalı-Tanımlı Fonksiyonlar).

$$g(x) = \begin{cases} \frac{1}{x}, & x < 0 \\ x & x \geq 0. \end{cases}$$

ile tanımlı $g : \mathbb{R} \to \mathbb{R}$ fonksiyonunun grafiğini çiziniz.

Soru 3.3 (Rayonel Fonksiyonlar). Aşağıdaki üç fonksiyonun grafiğini aynı koordinat düzleminde çiziniz:

(i). $f : (0, \infty) \to \mathbb{R}$, $f(x) = x$;

(ii). $g : (0, \infty) \to \mathbb{R}$, $g(x) = \frac{1}{x}$;

(iii). $h : (0, \infty) \to \mathbb{R}$, $h(x) = x + \frac{1}{x}$.

Part II

Calculus

4

Limits

Limit

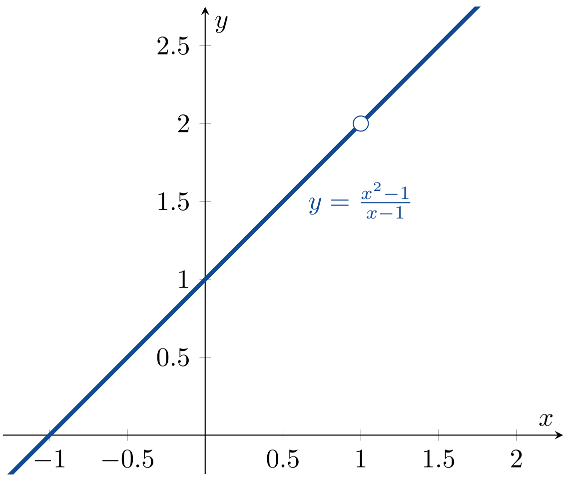


Figure 4.1: The function $f(x) = \frac{x^2-1}{x-1}$.
Şekil 4.1: $f(x) = \frac{x^2-1}{x-1}$ fonksiyonu.

x	$f(x)$
0.9	1.9
1.1	2.1
0.99	1.99
1.01	2.01
0.999	1.999
1.001	2.001

Table 4.1: Some values of $f(x) = \frac{x^2-1}{x-1}$.
Tablo 4.1: $f(x) = \frac{x^2-1}{x-1}$'nin bazı değerleri.

Consider the function $f : (-\infty, 1) \cup (1, \infty) \to \mathbb{R}$, $f(x) = \frac{x^2-1}{x-1}$ as shown in figure 4.1.

Question: How does f behave when x is close to 1?

We can see from table 4.1 that:

> "If x is close to 1, then $f(x)$ is close to 2."

Mathematically, we write this as

$$\lim_{x \to 1} f(x) = 2$$

and read it as "the limit, as x tends to 1, of $f(x)$ is equal to 2".

$f(x) = \frac{x^2-1}{x-1}$ ile tanımlı $f : (-\infty, 1) \cup (1, \infty) \to \mathbb{R}$ nın bazı değerleri şekil 4.1 de veriliyor.

Soru: x, 1'e yakın olduğunda f nasıl davranıyor?

Tablo 4.1 den şu gözlemi yapabiliriz:

> "x, 1'e yakınsa, $f(x)$ de, 2'ye yakın olur."

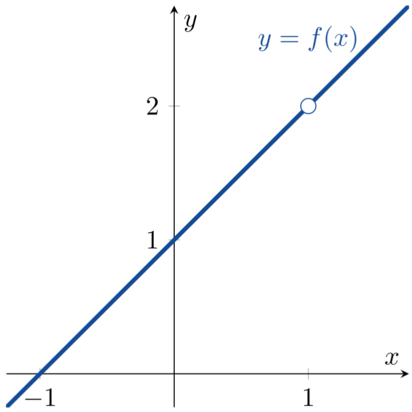
Matematiksel olarak, bunu

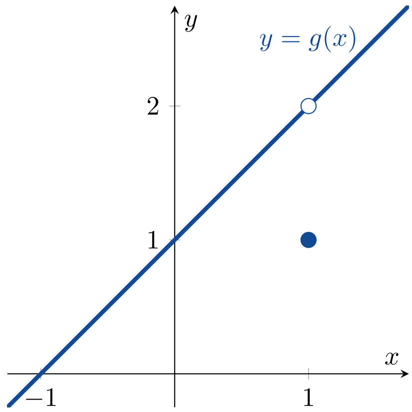
$$\lim_{x \to 1} f(x) = 2$$

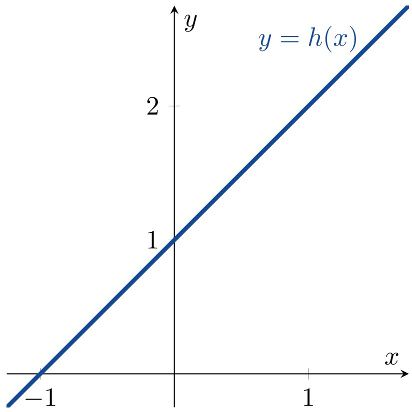
olarak yazarız ve x, 1 e yakaşırken, $f(x)$ in limiti 2'ye eşittir olarak okuruz .

Example 4.1. Consider the following three functions:

Örnek 4.1. Aşağıdaki üç fonksiyonu inceleyecek olursak:







$$f(x) = \frac{x^2 - 1}{x - 1}$$

$$g(x) = \begin{cases} \frac{x^2-1}{x-1} & x \neq 1 \\ 1 & x = 1 \end{cases}$$

$$h(x) = x + 1$$

Note that

- $\lim_{x \to 1} f(x) = 2$, but f is not defined at $x = 1$;
- $\lim_{x \to 1} g(x) = 2$, but $g(1) \neq 2$; and
- $\lim_{x \to 1} h(x) = 2$ and $h(1) = 2$.

- $\lim_{x \to 1} f(x) = 2$, fakat f, $x = 1$ de tanımlı değildir;
- $\lim_{x \to 1} g(x) = 2$, fakat $g(1) \neq 2$; ve
- $\lim_{x \to 1} h(x) = 2$ ve $h(1) = 2$.

olduğuna dikkat edelim.

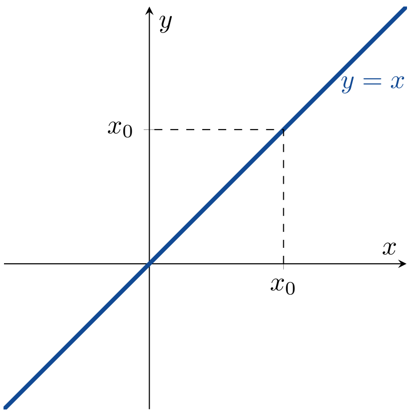


Figure 4.2: The Identity Function
Şekil 4.2: Özdeş fonksiyon.

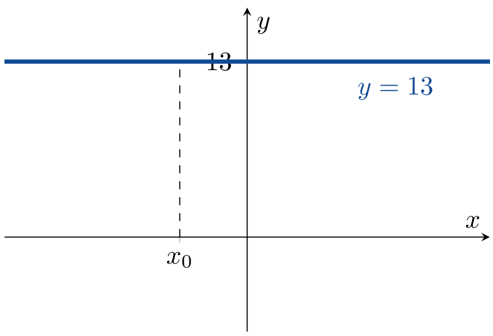


Figure 4.3: A Constant Function
Şekil 4.3: Sabit fonksiyon.

Example 4.2 (The Identity Function)**.** $f(x) = x$

$$\lim_{x \to x_0} f(x) = \lim_{x \to x_0} x = x_0$$

Example 4.3 (A Constant Function)**.** $f(x) = 13$

$$\lim_{x \to x_0} f(x) = \lim_{x \to x_0} 13 = 13$$

Örnek 4.2 (Birim Fonksiyon)**.** $f(x) = x$

$$\lim_{x \to x_0} f(x) = \lim_{x \to x_0} x = x_0$$

Örnek 4.3 (Sabit Fonksiyon)**.** $f(x) = 13$

$$\lim_{x \to x_0} f(x) = \lim_{x \to x_0} 13 = 13$$

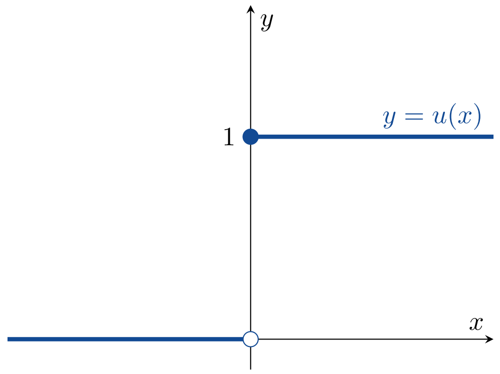


Figure 4.4: A graph of the function $u(x)$.
Şekil 4.4: $u(x)$ fonksiyonunun bir grafiği.

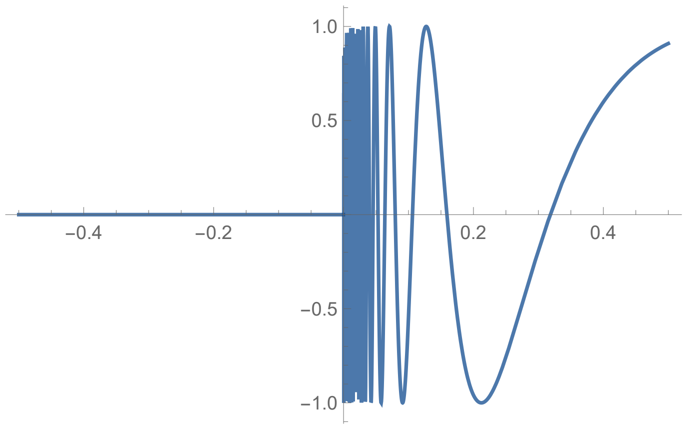


Figure 4.5: A graph of the function $v(x)$.
Şekil 4.5: $v(x)$ fonksiyonunun bir grafiği.

Example 4.4 (Sometimes Limits Do Not Exist)**.** Consider the functions

$$u(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases} \quad \text{and} \quad v(x) = \begin{cases} 0 & x \leq 0 \\ \sin \frac{1}{x} & x > 0 \end{cases}$$

as shown in figures 4.4 and 4.5.

Note that $\lim_{x \to 0} u(x)$ does not exist. To understand why, we consider x close to 0:

- If x is close to 0 and $x < 0$, then $u(x) = 0$.
- If x is close to 0 and $x > 0$, then $u(x) = 1$.

Because 0 is not close to 1, the limit as $x \to 0$ can not exist.

Moreover $\lim_{x \to 0} v(x)$ does not exist because $v(x)$ oscillates up and down too quickly if $x > 0$ and $x \to 0$.

Örnek 4.4 (Limit Her Zaman Mevcut Olmayabilir)**.** Şu fonksiyonları inceleyelim

$$u(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases} \quad \text{ve} \quad v(x) = \begin{cases} 0 & x \leq 0 \\ \sin \frac{1}{x} & x > 0 \end{cases}$$

bakınız şekil 4.4 ve 4.5.

$\lim_{x \to 0} u(x)$ limitinin mevcut olmadığına dikkat ediniz. Bunun neden mevcut olmadığını anlamak için, x 'in 0'a çok yakın olduğunu düşünelim:

- x, 0'a çok yakın ve $x < 0$ iken, $u(x) = 0$ dır.
- x, 0'a çok yakın ve $x > 0$ iken, $u(x) = 1$ olur.

0, 1'e çok yakın olmadığı için , $x \to 0$ iken limit mevcut değildir.

Ayrıca, $x > 0$ ve $x \to 0$ iken, $v(x)$, çok hızlı bir şekilde yukarıya ve aşağıya doğru salınır, çünkü $\lim_{x \to 0} v(x)$ mevcut değildir.

Theorem 4.1 (The Limit Laws). *Suppose that*

- $L, M, c, k \in \mathbb{R}$;
- f *and* g *are functions;*
- $\lim_{x\to c} f(x) = L$; *and*
- $\lim_{x\to c} g(x) = M$.

Then

(*i*). ***Sum Rule:***

$$\lim_{x\to c} \big(f(x) + g(x)\big) = L + M;$$

(*ii*). ***Difference Rule:***

$$\lim_{x\to c} \big(f(x) - g(x)\big) = L - M;$$

(*iii*). ***Constant Multiple Rule:***

$$\lim_{x\to c} \big(kf(x)\big) = kL;$$

(*iv*). ***Product Rule:***

$$\lim_{x\to c} \big(f(x)g(x)\big) = LM;$$

(*v*). ***Quotient Rule:*** *if* $M \neq 0$, *then*

$$\lim_{x\to c} \left(\frac{f(x)}{g(x)}\right) = \frac{L}{M};$$

(*vi*). ***Power Rule:*** *if* $n \in \mathbb{N}$, *then*

$$\lim_{x\to c} \big(f(x)\big)^n = L^n;$$

(*vii*). ***Root Rule:*** *if* $n \in \mathbb{N}$ *and* $\sqrt[n]{L}$ *exists, then*

$$\lim_{x\to c} \sqrt[n]{f(x)} = \sqrt[n]{L} = L^{\frac{1}{n}}.$$

Teorem 4.1 (Limit Kuralları). *Varsayalım ki*

- $L, M, c, k \in \mathbb{R}$;
- f *ve* g *iki fonksiyon;*
- $\lim_{x\to c} f(x) = L$; *ve*
- $\lim_{x\to c} g(x) = M$ *olsun.*

O halde

(*i*). ***Toplam Kuralı:***

$$\lim_{x\to c} \big(f(x) + g(x)\big) = L + M;$$

(*ii*). ***Fark Kuralı:***

$$\lim_{x\to c} \big(f(x) - g(x)\big) = L - M;$$

(*iii*). ***Sabitle Çarpım Kuralı:***

$$\lim_{x\to c} \big(kf(x)\big) = kL;$$

(*iv*). ***Çarpım Kuralı:***

$$\lim_{x\to c} \big(f(x)g(x)\big) = LM;$$

(*v*). ***Bölüm Kuralı:*** $M \neq 0$, *ise*

$$\lim_{x\to c} \left(\frac{f(x)}{g(x)}\right) = \frac{L}{M};$$

(*vi*). ***Kuvvet Kuralı:*** $n \in \mathbb{N}$, *ise*

$$\lim_{x\to c} \big(f(x)\big)^n = L^n;$$

(*vii*). ***Kök Kuralı:*** *if* $n \in \mathbb{N}$ *ve* $\sqrt[n]{L}$ *mevcutsa, then*

$$\lim_{x\to c} \sqrt[n]{f(x)} = \sqrt[n]{L} = L^{\frac{1}{n}}.$$

Example 4.5. Find $\lim_{x\to 2}(x^3 + 4x^2 - 3)$.

solution:

$$\begin{aligned}
\lim_{x\to 2}(x^3 + 4x^2 - 3) &= \big(\lim_{x\to 2} x^3\big) + \big(\lim_{x\to 2} 4x^2\big) - \big(\lim_{x\to 2} 3\big) \\
&\quad \text{(sum and difference rules)} \\
&= \big(\lim_{x\to 2} x\big)^3 + 4\big(\lim_{x\to 2} x\big)^2 - \big(\lim_{x\to 2} 3\big) \\
&\quad \text{(power and constant multiple rules)} \\
&= 2^3 + 4(2^2) - 3 = 21.
\end{aligned}$$

Örnek 4.6. $\lim_{x\to 6} 8(x-5)(x-7)$ limitini bulunuz.

çözüm:

$$\begin{aligned}
\lim_{x\to 6} 8(x-5)(x-7) &= 8\lim_{x\to 6}(x-5)(x-7) \\
&\quad \text{(sabitle çarpım kuralı)} \\
&= 8\big(\lim_{x\to 6}(x-5)\big)\big(\lim_{x\to 6}(x-7)\big) \\
&\quad \text{(çarpım kuralı)} \\
&= 8(1)(-1) = -8.
\end{aligned}$$

Example 4.7. Find $\lim_{x\to 5} \dfrac{x^4 + x^2 - 1}{x^2 + 5}$.

solution:

Örnek 4.8. $\lim_{x\to -5} \dfrac{x^2 + 3x - 11}{x + 6}$ limitini bulunuz.

çözüm:

$$\lim_{x\to 5}\frac{x^4+x^2-1}{x^2+5} = \frac{\lim_{x\to 5}(x^4+x^2-1)}{\lim_{x\to 5}(x^2+5)}$$

(quotient rule)

$$= \frac{\lim_{x\to 5}x^4+\lim_{x\to 5}x^2-\lim_{x\to 5}1}{\lim_{x\to 5}x^2+\lim_{x\to 5}5}$$

(sum and difference rules)

$$= \frac{5^4+5^2-1}{5^2+5}$$

(power rule)

$$= \frac{649}{30}.$$

$$\lim_{x\to -5}\frac{x^2+3x-11}{x+6} = \frac{\lim_{x\to -5}(x^2+3x-11)}{\lim_{x\to -5}(x+6)}$$

(bölüm kuralı)

$$= \frac{\lim_{x\to -5}x^2+\lim_{x\to -5}3x-\lim_{x\to -5}11}{\lim_{x\to -5}x+\lim_{x\to -5}6}$$

(toplam ve fark kuralı)

$$= \frac{(-5)^2-15-11}{-5+6}$$

(kuvvet kuralı)

$$= \frac{-1}{1} = -1.$$

Theorem 4.2 (Limits of Polynomial Functions). *If $P(x) = a_nx^2+a_{n-1}x^{n-1}+\ldots+a_1x+a_0$ is a polynomial function, then*

$$\lim_{x\to c}P(x) = P(c).$$

Teorem 4.2 (Polinomların Limitleri). *$P(x) = a_nx^2+a_{n-1}x^{n-1}+\ldots+a_1x+a_0$ bir polinom fonksiyonsa,*

$$\lim_{x\to c}P(x) = P(c).$$

Theorem 4.3 (Limits of Rational Functions). *If $P(x)$ and $Q(x)$ are polynomial functions and if $Q(c)\neq 0$, then*

$$\lim_{x\to c}\frac{P(x)}{Q(x)} = \frac{P(c)}{Q(c)}.$$

Teorem 4.3 (Rasyonel Fonksiyonların Limitlari). *$P(x)$ ve $Q(x)$ polinomlar ve $Q(c)\neq 0$ ise, o halde*

$$\lim_{x\to c}\frac{P(x)}{Q(x)} = \frac{P(c)}{Q(c)}.$$

Example 4.9.

$$\lim_{x\to -1}\frac{x^3+4x^2-3}{x^2+5} = \frac{(-1)^3+4(-1)^2-3}{(-1)^2+5} = \frac{0}{6} = 0.$$

Örnek 4.10.

$$\lim_{x\to 2}\frac{x^3+4x^2-3}{x^2+5} = \frac{(2)^3+4(2)^2-3}{(2)^2+5} = \frac{8+16-3}{4+5} = \frac{21}{9} = \frac{7}{3}.$$

Eliminating Zero Denominators Algebraically

$$\lim_{x\to c}\frac{P(x)}{Q(x)}$$

What can we do if $Q(c) = 0$?

Example 4.11.

$$\lim_{x\to 1}\frac{x^2+x-2}{x^2-x}$$

If we just put in $x = 1$, we would get "$\frac{0}{0}$" and we never never never want "$\frac{0}{0}$".

Instead, we try to factor x^2+x-2 and x^2-x. If $x\neq 1$, we have that

$$\frac{x^2+x-2}{x^2-x} = \frac{(x-1)(x+2)}{x(x-1)} = \frac{x+2}{x}.$$

So

$$\lim_{x\to 1}\frac{x^2+x-2}{x^2-x} = \lim_{x\to 1}\frac{x+2}{x} = \frac{1+2}{1} = 3.$$

Sıfır Paydaların Cebirsel Olarak Yok Edilmesi

$$\lim_{x\to c}\frac{P(x)}{Q(x)}$$

$Q(c) = 0$ ise ne yapılabilir?

Örnek 4.12.

$$\lim_{x\to -5}\frac{x^2+3x-10}{x^2+5x}$$

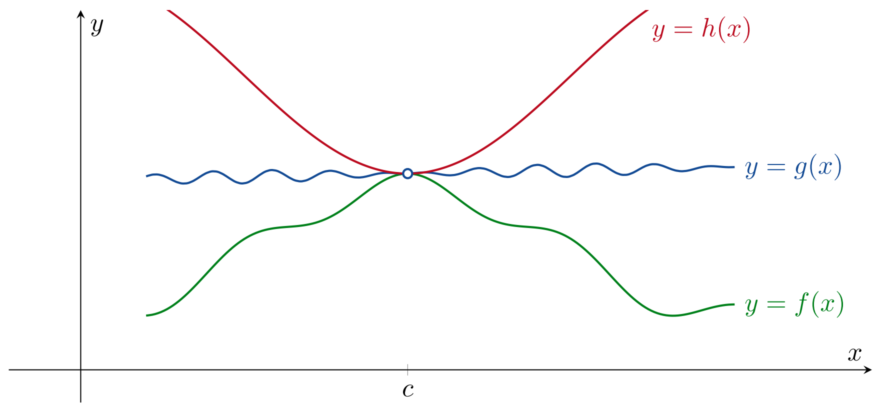
If we just put in $x = -5$ koyarsak, "$\frac{0}{0}$" buluruz ve unutmayın "$\frac{0}{0}$" asla ve asla istemediğimiz birşey.

Onun yerine, $x^2+3x-10$ ve x^2+5x yi çarpalarına ayırırız. $x\neq -5$ ise, şunu buluruz

$$\frac{x^2+3x-10}{x^2+5x} = \frac{(x+5)(x-2)}{x(x+5)} = \frac{x-2}{x}.$$

Yani

$$\lim_{x\to -5}\frac{x^2+3x-10}{x^2+5x} = \lim_{x\to -5}\frac{x-2}{x} = \frac{-5-2}{-5} = \frac{7}{5}.$$



$$f(x) \leq g(x) \leq h(x)$$

Figure 4.6: The Sandwich Theorem
Şekil 4.6: Sandöviç Teoremi

The Sandwich Theorem

See figure 4.6.

Theorem 4.4 (The Sandwich Theorem). *Suppose that*

- $f(x) \leq g(x) \leq h(x)$ *for all* x *"close" to* c $(x \neq c)$*; and*
- $\lim_{x \to c} f(x) = \lim_{x \to c} h(x) = L$.

Then

$$\lim_{x \to c} g(x) = L$$

also.

Example 4.13. The inequality

$$1 - \frac{x^2}{6} < \frac{x \sin x}{2 - 2\cos x} < 1$$

holds for all x close to 0 $(x \neq 0)$. Calculate $\lim_{x \to 0} \frac{x \sin x}{2 - 2\cos x}$.

solution: Since $\lim_{x \to 0} 1 - \frac{x^2}{6} = 1 - \frac{0}{6} = 1$ and $\lim_{x \to 0} 1 = 1$, it follows by the Sandwich Theorem that $\lim_{x \to 0} \frac{x \sin x}{2 - 2\cos x} = 1$.

Theorem 4.5. *If*

- $f(x) \leq g(x)$ *for all* x *close to* c $(x \neq c)$*;*
- $\lim_{x \to c} f(x)$ *exists; and*
- $\lim_{x \to c} g(x)$ *exists,*

then

$$\lim_{x \to c} f(x) \leq \lim_{x \to c} g(x).$$

Sandöviç Teoremi

See figure 4.6.

Teorem 4.4 (Sandöviç Teoremi). *Varsayalım ki*

- c $(x \neq c)$ *ye "çok yakın" bütün* x *ler için* $f(x) \leq g(x) \leq h(x)$ *ve*
- $\lim_{x \to c} f(x) = \lim_{x \to c} h(x) = L$ *olsun.*

O zaman

$$\lim_{x \to c} g(x) = L$$

ifadesi doğrudur.

Örnek 4.13.

$$1 - \frac{x^2}{6} < \frac{x \sin x}{2 - 2\cos x} < 1$$

eşitsizliği 0 a çok yakın bütün x ler $(x \neq 0)$ için doğrudur. $\lim_{x \to 0} \frac{x \sin x}{2 - 2\cos x}$ limitini bulunuz.

çözüm: $\lim_{x \to 0} 1 - \frac{x^2}{6} = 1 - \frac{0}{6} = 1$ ve $\lim_{x \to 0} 1 = 1$ olduğundan, Sandöviç Teoremi gereğince $\lim_{x \to 0} \frac{x \sin x}{2 - 2\cos x} = 1$ olarak bulunur.

Teorem 4.5. *Eğer*

- *Her* c *ye çok yakın (ama* $x \neq c$*) bütün* x *ler için* $f(x) \leq g(x)$ *ise ;*
- $\lim_{x \to c} f(x)$ *mevcutsa ve*
- $\lim_{x \to c} g(x)$ *mevcutsa,*

o vakit

$$\lim_{x \to c} f(x) \leq \lim_{x \to c} g(x)$$

doğru olur.

Problems

Problem 4.1. Find the following limits. For each one, state which limit laws or other theorems you are using.

(a) $\lim_{y\to -5} \frac{y^2}{y-5}$

(b) $\lim_{x\to \frac{2}{3}} 3x(2x-1)$

(c) $\lim_{y\to 2} \frac{y+2}{y^2+5y+6}$

(d) $\lim_{v\to 2} \frac{v^3-8}{v^4-16}$

Problem 4.2. If $2-x^2 \leq g(x) \leq 2\cos x$ for all x, find $\lim_{x\to 0} g(x)$. State which limit laws or other theorems you are using.

Problem 4.3. Suppose that $\lim_{x\to 4} f(x) = 0$ and $\lim_{x\to 4} g(x) = -3$. Find the following limits.

(a). $\lim_{x\to 4} \left(g(x)^2\right)$

(b). $\lim_{x\to 4} (g(x)+3)$

(c). $\lim_{x\to 4} xf(x)$

(d). $\lim_{x\to 4} \frac{g(x)}{f(x)-1}$

Sorular

Soru 4.1. Aşağıdaki limitleri bulunuz. her birinde, kullandığınız kural ve teoremleri yazınız.

(a) $\lim_{y\to -5} \frac{y^2}{y-5}$

(b) $\lim_{x\to \frac{2}{3}} 3x(2x-1)$

(c) $\lim_{y\to 2} \frac{y+2}{y^2+5y+6}$

(d) $\lim_{v\to 2} \frac{v^3-8}{v^4-16}$

Soru 4.2. Her x içın, $2-x^2 \leq g(x) \leq 2\cos x$ ise , $\lim_{x\to 0} g(x)$ limitini bulunuz. Kullandığınız kural ve teoremleri belirtiniz.

Soru 4.3. $\lim_{x\to 4} f(x) = 0$ ve $\lim_{x\to 4} g(x) = -3$ olsun. Aşağıdaki limitleri bulunuz.

(a). $\lim_{x\to 4} \left(g(x)^2\right)$

(b). $\lim_{x\to 4} (g(x)+3)$

(c). $\lim_{x\to 4} xf(x)$

(d). $\lim_{x\to 4} \frac{g(x)}{f(x)-1}$

5 Continuity / Süreklilik

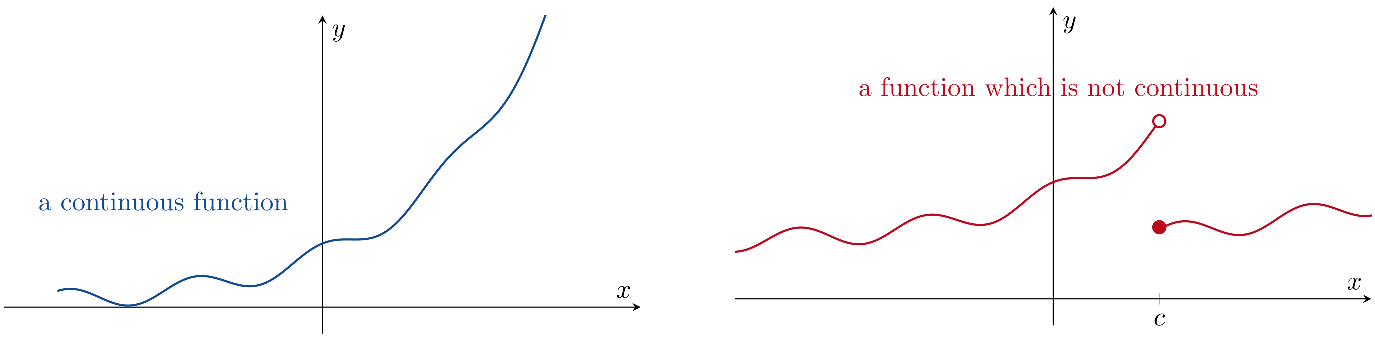


Figure 5.1: A continuous function and a function which is not continuous.
Şekil 5.1: Bir sürekli fonksiyon ve sürekli olmayan bir fonksiyon.

Definition. The function $f : D \to \mathbb{R}$ is ***continuous at*** $c \in D$ if

- $f(c)$ exists;
- $\lim_{x \to c} f(x)$ exists; and
- $\lim_{x \to c} f(x) = f(c)$.

Definition. If f is not continuous at c, we say that f is ***discontinuous*** at c – we say that c is a ***point of discontinuity*** of f.

Tanım. Şu üç koşulun hepsi sağlanırsa $f : D \to \mathbb{R}$ fonksiyonu bir $c \in D$ ***noktasında süreklidir*** denir.

- $f(c)$ tanımlı olacak;
- $\lim_{x \to c} f(x)$ mevcut olacak; ve
- $\lim_{x \to c} f(x) = f(c)$.

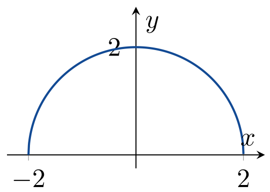
Tanım. Eğer f fonksiyonu c de sürekli değilse, f, c de ***süreksizdir*** denir – ve c'ye f'nin bir ***süreksizlik noktası*** denir.

Example 5.1. Consider the function $f : [0, 4] \to \mathbb{R}$ which has its graph shown in figure 5.2. Where is f continuous? Where is f discontinuous?

solution:

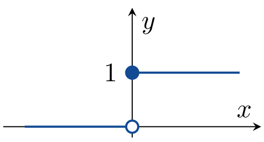
c	Is f continuous at c?	Why?
0	Yes	because $\lim_{x\to 0} f(x) = 1 = f(0)$
$(0,1)$	Yes	because $\lim_{x\to c} f(x) = f(c)$
1	No	because $\lim_{x\to 1} f(x)$ does not exist
$(1,2)$	Yes	because $\lim_{x\to c} f(x) = f(c)$
2	No	because $\lim_{x\to 2} f(x) = 1 \neq 2 = f(2)$
$(2,4)$	Yes	because $\lim_{x\to c} f(x) = f(c)$
4	No	because $\lim_{x\to 4} f(x) = 1 \neq \frac{1}{2} = f(4)$

Example 5.2. $f : [-2, 2] \to \mathbb{R}$, $f(x) = \sqrt{4 - x^2}$



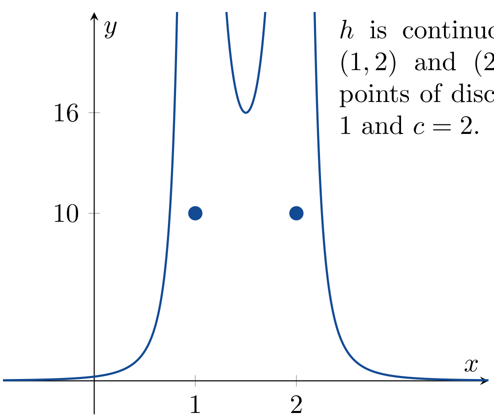
f is continuous at every $c \in [-2, 2]$.

Example 5.3. $g : \mathbb{R} \to \mathbb{R}$, $g(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases}$



g has a point of discontinuity at $c = 0$. g is continuous at every point $c \neq 0$.

Example 5.4. $h : \mathbb{R} \to \mathbb{R}$, $h(x) = \begin{cases} \frac{1}{(x-1)^2(x-2)^2} & x \neq 1 \text{ or } 2 \\ 10 & x = 1 \text{ or } 2 \end{cases}$



h is continuous on $(-\infty, 1)$, $(1, 2)$ and $(2, \infty)$. h has a points of discontinuity at $c = 1$ and $c = 2$.

Örnek 5.1. Grafiği şekil 5.2 deki $f : [0, 4] \to \mathbb{R}$ fonksiyonunu ele alalım. Bu f nerede süreklidir? Bu f nerede süreksizdir?

çözüm:

c	f fonksiyonu c de sürekli midir?	Neden?
0	Evet	çünkü $\lim_{x\to 0} f(x) = 1 = f(0)$
$(0,1)$	Evet	çünkü $\lim_{x\to c} f(x) = f(c)$
1	Hayır	çünkü $\lim_{x\to 1} f(x)$ does not exist
$(1,2)$	Evet	çünkü $\lim_{x\to c} f(x) = f(c)$
2	Hayır	çünkü $\lim_{x\to 2} f(x) = 1 \neq 2 = f(2)$
$(2,4)$	Evet	çünkü $\lim_{x\to c} f(x) = f(c)$
4	Hayır	çünkü $\lim_{x\to 4} f(x) = 1 \neq \frac{1}{2} = f(4)$

Örnek 5.2. $f : [-2, 2] \to \mathbb{R}$, $f(x) = \sqrt{4 - x^2}$
f fonksiyonu her $c \in [-2, 2]$ noktasında süreklidir.

Örnek 5.3. $g : \mathbb{R} \to \mathbb{R}$, $g(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases}$

g nin $c = 0$. g da bir süreksizlik noktası var ve fonksiyon her $c \neq 0$ için süreklidir.

Örnek 5.4. $h : \mathbb{R} \to \mathbb{R}$, $h(x) = \begin{cases} \frac{1}{(x-1)^2(x-2)^2} & x \neq 1 \text{ or } 2 \\ 10 & x = 1 \text{ or } 2 \end{cases}$

h fonksiyonu $(-\infty, 1)$, $(1, 2)$ ve $(2, \infty)$ aralıklarında süreklidir. h nin $c = 1$ ve $c = 2$ de süreksizlikleri mevcuttur.

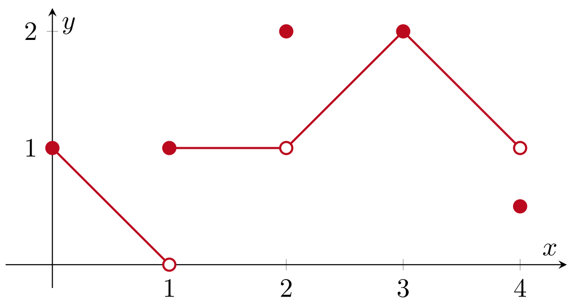


Figure 5.2: The function considered in example 5.1.
Şekil 5.2: Örnek 5.1 deki ele alınan fonksiyon.

Continuous Functions

Definition. $f : D \to \mathbb{R}$ is a ***continuous function*** if it is continuous at every $c \in D$.

> **Theorem 5.1.** *If f and g are continuous at c, then $f+g$, $f-g$, kf $(k \in \mathbb{R})$, fg, $\frac{f}{g}$ (if $g(c) \neq 0$) and f^n $(n \in \mathbb{N})$ are all continuous at c. If $\sqrt[n]{f}$ is defined on $(c-\delta, c+\delta)$, then $\sqrt[n]{f}$ is also continuous at c $(n \in \mathbb{N})$.*

Example 5.5. Every polynominal

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_2x^2 + a_1x + a_0$$

is continuous.

Example 5.6. If

- P and Q are polynomials; and
- $Q(c) \neq 0$,

then $\frac{P(x)}{Q(x)}$ is continuous at c.

Example 5.7. $\sin x$ and $\cos x$ are continuous.

Sürekli Fonksiyonlar

Tanım. Her $c \in D$ noktasında sürekli olan bir $f : D \to \mathbb{R}$ fonksiyonuna ***sürekli fonksiyon*** denir .

> **Teorem 5.1.** *Eğer f ve g fonksiyonları c'de sürekli iseler, o zaman $f+g$, $f-g$, kf $(k \in \mathbb{R})$, fg, $\frac{f}{g}$ $(g(c) \neq 0$ iken) ve f^n $(n \in \mathbb{N})$ fonksiyonlarının hepsi c'de süreklidir. Eğer $\sqrt[n]{f}$ fonksiyonu $(c-\delta, c+\delta)$ aralığında tanımlı ise, $\sqrt[n]{f}$ fonksiyonu da c'de süreklidir $(n \in \mathbb{N})$.*

Örnek 5.5. Her

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_2x^2 + a_1x + a_0$$

polinomu da süreklidir.

Örnek 5.6. If

- P ve Q polinomlar ve
- $Q(c) \neq 0$ ise,

o vakit $\frac{P(x)}{Q(x)}$ rasyonel fonksiyonu c'de süreklidir.

Örnek 5.7. $\sin x$ ve $\cos x$ sürekli fonksiyonlardır.

Composites

Bileşkeler

$$g \circ f(x)$$

$g \circ f(x)$ means $g(f(x))$.

> **Theorem 5.2.** *If*
>
> - *f is continuous at c; and*
> - *g is continuous at $f(c)$,*
>
> *then $g \circ f$ is continuous at c.*

Example 5.8. Show that $h(x) = \sqrt{x^2 - 2x - 5}$ is continuous on its domain.

solution: The function $g(t) = \sqrt{t}$ is continuous by Theorem 5.1. The function $f(x) = x^2 - 2x - 5$ is continuous because all polynomials are continuous. Therefore $h(x) = g \circ f(x)$ is continuous.

Example 5.9. Show that $\frac{x^{\frac{2}{3}}}{1+x^4}$ is continuous.

solution: $x^{\frac{2}{3}}$ and $1 + x^4$ are continuous. Because $1 + x^4 \neq 0$ for all x, we have that $\frac{x^{\frac{2}{3}}}{1+x^4}$ is continuous.

$g \circ f(x)$ demek $g(f(x))$ anlamındadır.

> **Teorem 5.2.** *Eğer*
>
> - *f fonksiyonu c'de sürekli ve*
> - *g fonksiyonu da $f(c)$'de sürekli ise,*
>
> *bu durumda $g \circ f$ fonksiyonu da c'de süreklidir.*

Örnek 5.8. $h(x) = \sqrt{x^2 - 2x - 5}$ fonksiyonunun tanım kümesinde sürekli olduğunu gösteriniz.

çözüm: Teorem 5.1 den $g(t) = \sqrt{t}$ fonksiyonu süreklidir. $f(x) = x^2 - 2x - 5$ fonksiyonu da süreklidir çünkü bütün polinomlar süreklidir. Bundan ötürü $h(x) = g \circ f(x)$ sürekli olur.

Örnek 5.9. Gösteriniz ki $\frac{x^{\frac{2}{3}}}{1+x^4}$ süreklidir.

çözüm: $x^{\frac{2}{3}}$ ve $1 + x^4$ süreklidir. Her x için, $1 + x^4 \neq 0$ olduğundan , $\frac{x^{\frac{2}{3}}}{1+x^4}$ süreklidir.

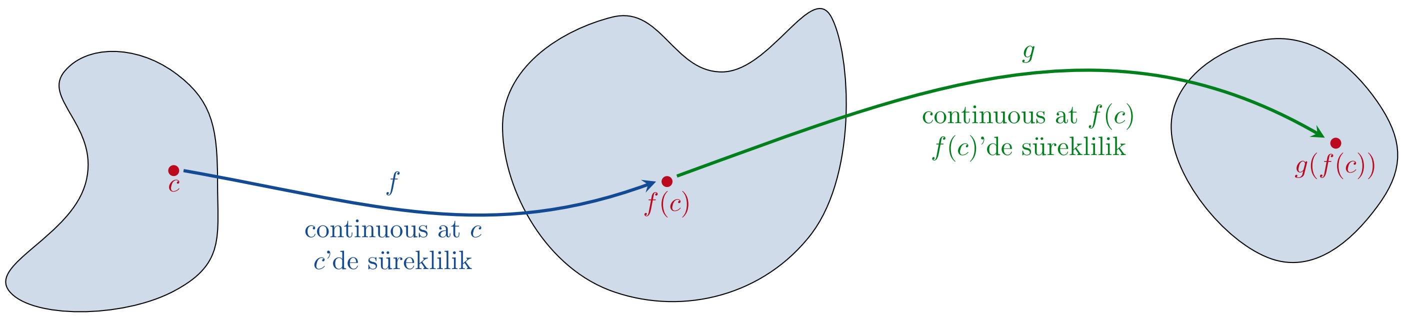


Figure 5.3: Composites of continuous functions are continuous.
Şekil 5.3: Sürekli fonsiyonların bileşkesi de süreklidir.

Theorem 5.3. *If*

- $g(x)$ *is continuous at* $x = b$*; and*
- $\lim_{x\to c} f(x) = b$,

then

$$\lim_{x\to c} g(f(x)) = g\Big(\lim_{x\to c} f(x)\Big).$$

Example 5.10. By Theorem 5.3,

$$\begin{aligned}
&\lim_{x\to\frac{\pi}{2}} \cos\left[2x + \sin\left(\frac{3\pi}{2} + x\right)\right] \\
&\quad = \cos\left[\lim_{x\to\frac{\pi}{2}} \left(2x + \sin\left(\frac{3\pi}{2} + x\right)\right)\right] \\
&\quad = \cos\left[\lim_{x\to\frac{\pi}{2}} (2x) + \lim_{x\to\frac{\pi}{2}} \left(\sin\left(\frac{3\pi}{2} + x\right)\right)\right] \\
&\quad = \cos\left[\pi + \sin\left(\lim_{x\to\frac{\pi}{2}} \left(\frac{3\pi}{2} + x\right)\right)\right] \\
&\quad = \cos\left[\pi + \sin 2\pi\right] = \cos\left[\pi + 0\right] = -1.
\end{aligned}$$

Teorem 5.3. *Eğer*

- $g(x)$ *fonksiyonu* $x = b$ *de sürekli ve*
- $\lim_{x\to c} f(x) = b$ *ise,*

o halde

$$\lim_{x\to c} g(f(x)) = g\Big(\lim_{x\to c} f(x)\Big).$$

Örnek 5.11. Teorem 5.3'den,

$$\begin{aligned}
&\lim_{x\to\frac{\pi}{2}} \tan\left[\frac{5x}{2} - \pi\cos\left(\frac{\pi}{2} - x\right)\right] \\
&\quad = \tan\left[\lim_{x\to\frac{\pi}{2}} \left(\frac{5x}{2} - \pi\cos\left(\frac{\pi}{2} - x\right)\right)\right] \\
&\quad = \tan\left[\lim_{x\to\frac{\pi}{2}} \left(\frac{5x}{2}\right) - \pi\lim_{x\to\frac{\pi}{2}} \left(\cos\left(\frac{\pi}{2} - x\right)\right)\right] \\
&\quad = \tan\left[\frac{5\pi}{4} - \pi\cos\left(\lim_{x\to\frac{\pi}{2}} \left(\frac{\pi}{2} - x\right)\right)\right] \\
&\quad = \tan\left[\frac{5\pi}{4} - \pi\cos 0\right] = \tan\left[\frac{5\pi}{4} - \pi\right] = \tan\frac{\pi}{4} = 1.
\end{aligned}$$

Problems

Problem 5.1. For what value(s) of b is

$$f(x) = \begin{cases} x & x < -2 \\ bx^2 & x \geq -2. \end{cases}$$

continuous at every x? Why?

Problem 5.2. Let

$$f(x) = \begin{cases} \frac{x^3-8}{x^2-4} & x \neq 2, x \neq -2 \\ 3 & x = 2 \\ 4 & x = -2. \end{cases}$$

(a). Show that f is continuous on $(-\infty, -2)$, on $(-2, 2)$ and on $(2, \infty)$.

(b). Show that f is continuous at $x = 2$.

(c). Show that f is discontinuous at $x = -2$.

Problem 5.3. Calculate $\lim_{t \to 0} \tan\left(\frac{\pi}{4} \cos\left(\sin t^{\frac{1}{3}}\right)\right)$.

Sorular

Soru 5.1. b'nin hangi değer(ler)i için,

$$f(x) = \begin{cases} x & x < -2 \\ bx^2 & x \geq -2. \end{cases}$$

her x noktasında süreklidir? Neden?

Soru 5.2. Farzedelim ki

$$f(x) = \begin{cases} \frac{x^3-8}{x^2-4} & x \neq 2, x \neq -2 \\ 3 & x = 2 \\ 4 & x = -2. \end{cases}$$

(a). f'nin $(-\infty, -2)$ de, $(-2, 2)$ de ve $(2, \infty)$ da sürekli olduğunu gösteriniz.

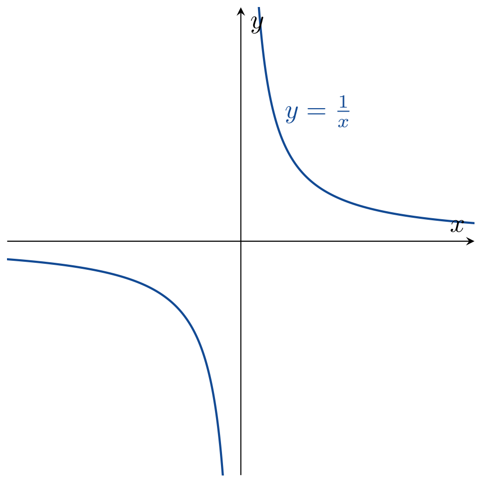
(b). f'nin $x = 2$'de sürekli olduğunu gösteriniz.

(c). f'nin $x = -2$'de süreksiz olduğunu gösteriniz.

Soru 5.3. $\lim_{t \to 0} \tan\left(\frac{\pi}{4} \cos\left(\sin t^{\frac{1}{3}}\right)\right)$ limitini bulunuz.

Limits Involving Infinity

Finite Limits as $x \to \pm\infty$



Question: If $x > 0$ and x gets bigger and bigger and bigger, what happens to $\frac{1}{x}$?

Answer: $\frac{1}{x}$ gets closer and closer and closer to 0. We write this as

$$\lim_{x\to\infty} \frac{1}{x} = 0.$$

Similarly we have that

$$\lim_{x\to-\infty} \frac{1}{x} = 0.$$

Theorem 6.1. *All of the limit laws (sum rule, difference rule, constant multiple rule, ...) are also true for* $\lim_{x\to\infty}$ *and* $\lim_{x\to-\infty}$.

Example 6.1.

$$\lim_{x\to\infty}\left(5+\frac{1}{x}\right) = \lim_{x\to\infty} 5 + \lim_{x\to\infty}\frac{1}{x}$$

(sum rule)

$$= 5 + 0 = 5.$$

Sonsuz Limitler

$x \to \pm\infty$ iken Sonlu Limitler

Soru: $x > 0$ ve x keyfi olarak büyüdüğünde, $\frac{1}{x}$ nasıl davranır?

Cevap: $\frac{1}{x}$ istenildiği kadar 0'a yakın olur. Bunu şöyle yazarız

$$\lim_{x\to\infty} \frac{1}{x} = 0.$$

Benzer şekilde

$$\lim_{x\to-\infty} \frac{1}{x} = 0$$

olarak yazacağız.

Teorem 6.1. *Limit kurallarının tümü (toplam kuralı, fark kuralı, sabitle çarpım kuralı, ...)* $\lim_{x\to\infty}$ *ve* $\lim_{x\to-\infty}$ *için de geçerlidir.*

Örnek 6.1.

$$\lim_{x\to\infty}\left(5+\frac{1}{x}\right) = \lim_{x\to\infty} 5 + \lim_{x\to\infty}\frac{1}{x}$$

(toplam kuralı)

$$= 5 + 0 = 5.$$

Örnek 6.2.

$$\lim_{x\to-\infty}\frac{\pi\sqrt{3}}{x^2} = \lim_{x\to-\infty}\left(\pi\sqrt{3}\frac{1}{x}\frac{1}{x}\right)$$

$$= \left(\lim_{x\to-\infty}\pi\sqrt{3}\right)\left(\lim_{x\to-\infty}\frac{1}{x}\right)\left(\lim_{x\to-\infty}\frac{1}{x}\right)$$

(çarpım küralı)

$$= \pi\sqrt{3} \times 0 \times 0 = 0.$$

Örnek 6.3 (Rasyonel Fonksiyonların Sonsuzdaki Limitleri)**.** $\lim_{x\to\infty}\frac{5x^2+8x-3}{3x^2+2}$ limitini bulunuz.

çözüm: Unutmayın ki cevap "$\frac{\infty}{\infty}$" değildir. "$\frac{\infty}{\infty}$". yazarsanız sınavda sıfır puan almanız beklenebilir.

Bunun yerine şöyle bir çözüm verebiliriz

$$\lim_{x\to\infty}\frac{5x^2+8x-3}{3x^2+2} = \lim_{x\to\infty}\frac{5+\frac{8}{x}-\frac{3}{x^2}}{3+\frac{2}{x^2}} = \frac{5+0-0}{3+0} = \frac{5}{3}.$$

Bkz. şekil 6.1.

Example 6.2.

$$\lim_{x\to-\infty}\frac{\pi\sqrt{3}}{x^2} = \lim_{x\to-\infty}\left(\pi\sqrt{3}\frac{1}{x}\frac{1}{x}\right)$$
$$= \left(\lim_{x\to-\infty}\pi\sqrt{3}\right)\left(\lim_{x\to-\infty}\frac{1}{x}\right)\left(\lim_{x\to-\infty}\frac{1}{x}\right)$$
(product rule)
$$= \pi\sqrt{3}\times 0\times 0 = 0.$$

Example 6.3 (Limits at Infinity of Rational Functions)**.** Find $\lim_{x\to\infty}\frac{5x^2+8x-3}{3x^2+2}$.

solution: Please note that the answer is not "$\frac{\infty}{\infty}$". You can expect to receive zero points in the exam if you write "$\frac{\infty}{\infty}$".

Instead we calculate that

$$\lim_{x\to\infty}\frac{5x^2+8x-3}{3x^2+2} = \lim_{x\to\infty}\frac{5+\frac{8}{x}-\frac{3}{x^2}}{3+\frac{2}{x^2}} = \frac{5+0-0}{3+0} = \frac{5}{3}.$$

See figure 6.1.

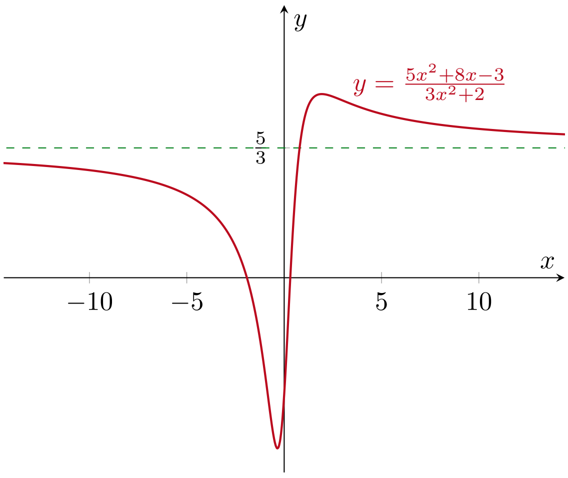


Figure 6.1: The graph of $y = \frac{5x^2+8x-3}{3x^2+2}$.
Şekil 6.1: $y = \frac{5x^2+8x-3}{3x^2+2}$'nin grafiği.

Example 6.4.

$$\lim_{x\to-\infty}\frac{11x+2}{2x^3-1} = \lim_{x\to-\infty}\frac{\frac{11}{x^2}+\frac{2}{x^3}}{2-\frac{1}{x^3}} = \frac{0+0}{2-0} = 0.$$

Example 6.5. Find $\lim_{x\to\infty}\frac{x^3-2}{|x|^3+1}$ and $\lim_{x\to-\infty}\frac{x^3-2}{|x|^3+1}$.

solution: If $x>0$, then

$$\lim_{x\to\infty}\frac{x^3-2}{|x|^3+1} = \lim_{x\to\infty}\frac{x^3-2}{x^3+1}$$
$$= \lim_{x\to\infty}\frac{1-\frac{2}{x^3}}{1+\frac{1}{x^3}} = \frac{1-0}{1+0} = 1$$

and if $x<0$ then

$$\lim_{x\to-\infty}\frac{x^3-2}{|x|^3+1} = \lim_{x\to-\infty}\frac{x^3-2}{(-x)^3+1}$$
$$= \lim_{x\to-\infty}\frac{1-\frac{2}{x^3}}{-1+\frac{1}{x^3}}$$
$$= \frac{1-0}{-1+0} = -1.$$

See figure 6.3.

Example 6.6. Use the Sandwich Theorem to calculate

$$\lim_{x\to\infty}\left(2+\frac{\sin x}{x}\right).$$

solution: Since $-1\leq\sin x\leq 1$, we have that

$$0\leq\left|\frac{\sin x}{x}\right|\leq\left|\frac{1}{x}\right|.$$

Because $\lim_{x\to\infty}\left|\frac{1}{x}\right| = 0$, it follows by the Sandwich Theorem that $\lim_{x\to\infty}\frac{\sin x}{x} = 0$. Therefore

$$\lim_{x\to\infty}\left(2+\frac{\sin x}{x}\right) = 2+0 = 2.$$

See figure 6.2.

Örnek 6.4.

$$\lim_{x\to-\infty}\frac{11x+2}{2x^3-1} = \lim_{x\to-\infty}\frac{\frac{11}{x^2}+\frac{2}{x^3}}{2-\frac{1}{x^3}} = \frac{0+0}{2-0} = 0.$$

Örnek 6.5. $\lim_{x\to\infty}\frac{x^3-2}{|x|^3+1}$ ve $\lim_{x\to-\infty}\frac{x^3-2}{|x|^3+1}$ limitlerini bulunuz.

çözüm: $x>0$ ise, o halde

$$\lim_{x\to\infty}\frac{x^3-2}{|x|^3+1} = \lim_{x\to\infty}\frac{x^3-2}{x^3+1}$$
$$= \lim_{x\to\infty}\frac{1-\frac{2}{x^3}}{1+\frac{1}{x^3}} = \frac{1-0}{1+0} = 1$$

ve eğer $x<0$ ise, o halde

$$\lim_{x\to-\infty}\frac{x^3-2}{|x|^3+1} = \lim_{x\to-\infty}\frac{x^3-2}{(-x)^3+1}$$
$$= \lim_{x\to-\infty}\frac{1-\frac{2}{x^3}}{-1+\frac{1}{x^3}}$$
$$= \frac{1-0}{-1+0} = -1.$$

Bkz. şekil 6.3.

Örnek 6.6. Sandwich Teoremi kullanarak

$$\lim_{x\to\infty}\left(2+\frac{\sin x}{x}\right)$$

limitini bulunuz.
çözüm: $-1\leq\sin x\leq 1$ olduğundan, şunu elde ederiz

$$0\leq\left|\frac{\sin x}{x}\right|\leq\left|\frac{1}{x}\right|.$$

$\lim_{x\to\infty}\left|\frac{1}{x}\right| = 0$ olduğu için, Sandwich Teoremi'nden $\lim_{x\to\infty}\frac{\sin x}{x} = 0$ buluruz. Buradan

$$\lim_{x\to\infty}\left(2+\frac{\sin x}{x}\right) = 2+0 = 2.$$

Bkz. şekil 6.2.

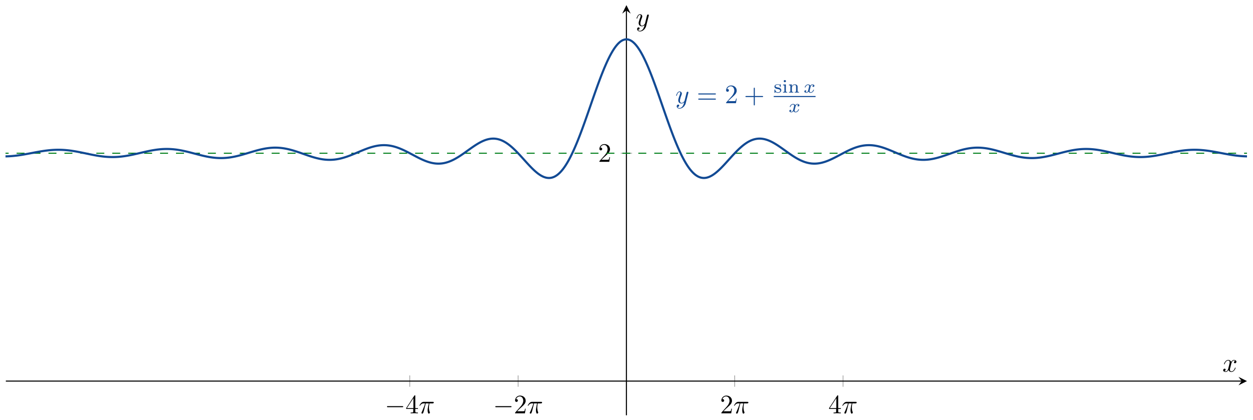


Figure 6.2: The graph of $y = 2 + \frac{\sin x}{x}$.
Şekil 6.2: $y = 2 + \frac{\sin x}{x}$'in grafiği.

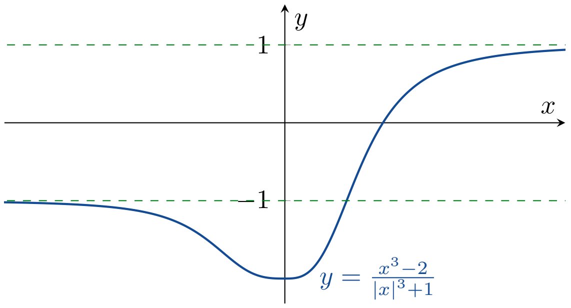


Figure 6.3: The graph of $y = \frac{x^3-2}{|x|^3+1}$.
Şekil 6.3: $y = \frac{x^3-2}{|x|^3+1}$'in grafiği.

Remark. There is one more trick for limits. Because $(a-b)(a+b) = a^2 - b^2$, it follows that

$$a - b = \frac{a^2 - b^2}{a+b}.$$

This can be useful if the limit contains a $\sqrt{\ }$.

Example 6.7. Calculate $\lim_{x\to\infty}\left(x - \sqrt{x^2+16}\right)$.

solution:

$$\begin{aligned}
&\lim_{x\to\infty}\left(x - \sqrt{x^2+16}\right) \\
&\quad= \lim_{x\to\infty}\left(\left(x - \sqrt{x^2+16}\right)\frac{x+\sqrt{x^2+16}}{x+\sqrt{x^2+16}}\right) \\
&\quad= \lim_{x\to\infty}\frac{x^2-(x^2+16)}{x+\sqrt{x^2+16}} \\
&\quad= \lim_{x\to\infty}\frac{-16}{x+\sqrt{x^2+16}} \\
&\quad= \lim_{x\to\infty}\frac{\frac{-16}{x}}{1+\sqrt{1+\frac{16}{x^2}}} \\
&\quad= \frac{0}{1+\sqrt{1+0}}.
\end{aligned}$$

Not. Limit hesaplamalarında bir yol daha var. $(a-b)(a+b) = a^2 - b^2$ olduğundan, bunun ardından

$$a - b = \frac{a^2 - b^2}{a+b}.$$

Limit bir $\sqrt{\ }$ içeriyorsa, bu işe yarayabiliyor..

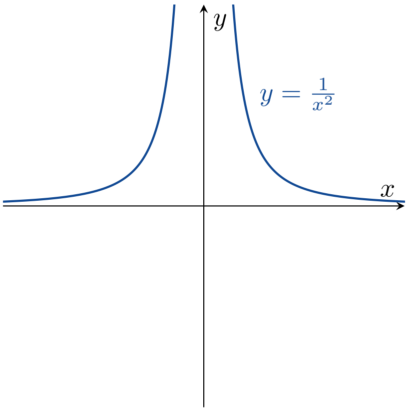
Örnek 6.7. $\lim_{x\to\infty}\left(x - \sqrt{x^2+16}\right)$ limitini bulunuz.

çözüm:

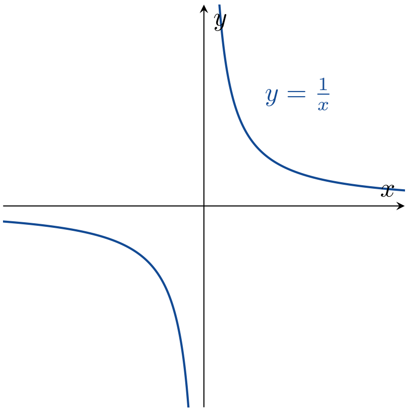
$$\begin{aligned}
&\lim_{x\to\infty}\left(x - \sqrt{x^2+16}\right) \\
&\quad= \lim_{x\to\infty}\left(\left(x - \sqrt{x^2+16}\right)\frac{x+\sqrt{x^2+16}}{x+\sqrt{x^2+16}}\right) \\
&\quad= \lim_{x\to\infty}\frac{x^2-(x^2+16)}{x+\sqrt{x^2+16}} \\
&\quad= \lim_{x\to\infty}\frac{-16}{x+\sqrt{x^2+16}} \\
&\quad= \lim_{x\to\infty}\frac{\frac{-16}{x}}{1+\sqrt{1+\frac{16}{x^2}}} \\
&\quad= \frac{0}{1+\sqrt{1+0}}.
\end{aligned}$$

Infinite Limits

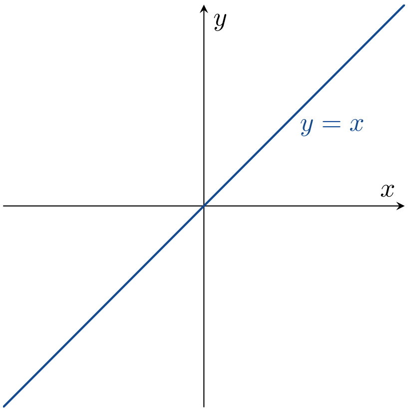
Sonsuz Limitler



$$\lim_{x\to 0} \frac{1}{x^2} = \infty$$



$$\lim_{x\to 0} \frac{1}{x} \text{ does not exist.}$$



$$\lim_{x\to \infty} x = \infty$$

Example 6.8. Find $\lim_{x\to 2} \frac{2-x}{(x-2)^3}$ or explain why it doesn't exist.

solution:

$$\lim_{x\to 2} \frac{2-x}{(x-2)^3} = \lim_{x\to 2} \frac{-(x-2)}{(x-2)^3} = \lim_{x\to 2} \frac{-1}{(x-2)^2} = -\infty.$$

Example 6.9. Find $\lim_{x\to 2} \frac{x-3}{x^2-4}$ or explain why it doesn't exist.

solution:

$$\lim_{x\to 2} \frac{x-3}{x^2-4} = \lim_{x\to 2} \frac{x-3}{(x-2)(x+2)} = \lim_{x\to 2} \left[\left(\frac{x-3}{x+2}\right)\left(\frac{1}{x-2}\right)\right]$$

does not exist. To understand why, note that

- if $2 < x < 2.01$, then $(x-2) > 0$ and $\frac{1}{x-2} > 100$; but
- if $1.99 < x < 2$, then $(x-2) < 0$ and $\frac{1}{x-2} < -100$.

See figure 6.4.

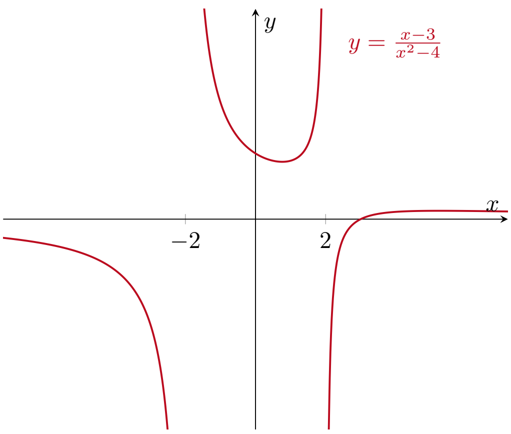


Figure 6.4: The graph of $y = \frac{x-3}{x^2-4}$.
Şekil 6.4: $y = \frac{x-3}{x^2-4}$'ün grafiği.

Örnek 6.8. $\lim_{x\to 2} \frac{2-x}{(x-2)^3}$ limitini bulunuz veya mevcut değilse neden olmadığını açıklayınız.

çözüm:

$$\lim_{x\to 2} \frac{2-x}{(x-2)^3} = \lim_{x\to 2} \frac{-(x-2)}{(x-2)^3} = \lim_{x\to 2} \frac{-1}{(x-2)^2} = -\infty.$$

Örnek 6.9. Mevcutsa, $\lim_{x\to 2} \frac{x-3}{x^2-4}$ limitini bulunuz veya mevcut değilse açıklayınız.

çözüm:

$$\lim_{x\to 2} \frac{x-3}{x^2-4} = \lim_{x\to 2} \frac{x-3}{(x-2)(x+2)} = \lim_{x\to 2} \left[\left(\frac{x-3}{x+2}\right)\left(\frac{1}{x-2}\right)\right]$$

mevcut değildir. Neden olmadığını görebilmek için, şunlara dikkat edelim

- $2 < x < 2.01$ ise, bu durumda $(x-2) > 0$ ve $\frac{1}{x-2} > 100$ olur; fakat
- $1.99 < x < 2$ iken, $(x-2) < 0$ and $\frac{1}{x-2} < -100$ olur.

See figure 6.4.

Problems

Problem 6.1. Find the following limits.

(a) $\lim_{x\to\infty} \frac{2+\sqrt{x}}{2-\sqrt{x}}$

(b) $\lim_{t\to-\infty} \frac{\sqrt{t^2+1}}{t+1}$

(c) $\lim_{p\to 0} \frac{-1}{p^2(p+1)}$

Sorular

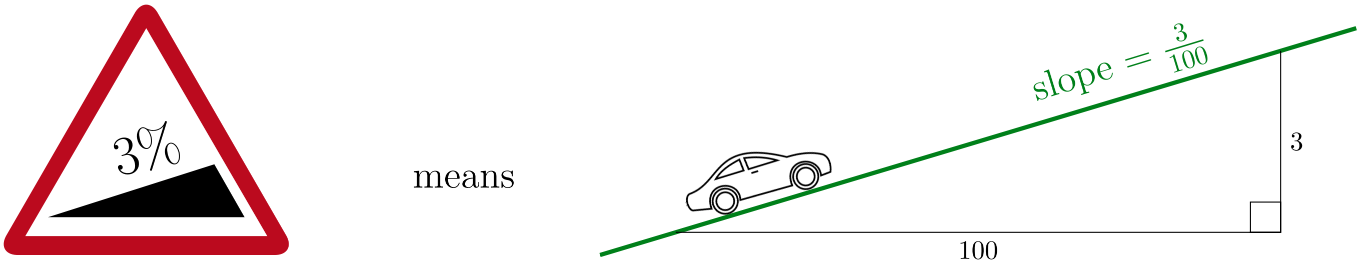
Soru 6.1. Aşağıdaki limitleri bulunuz.

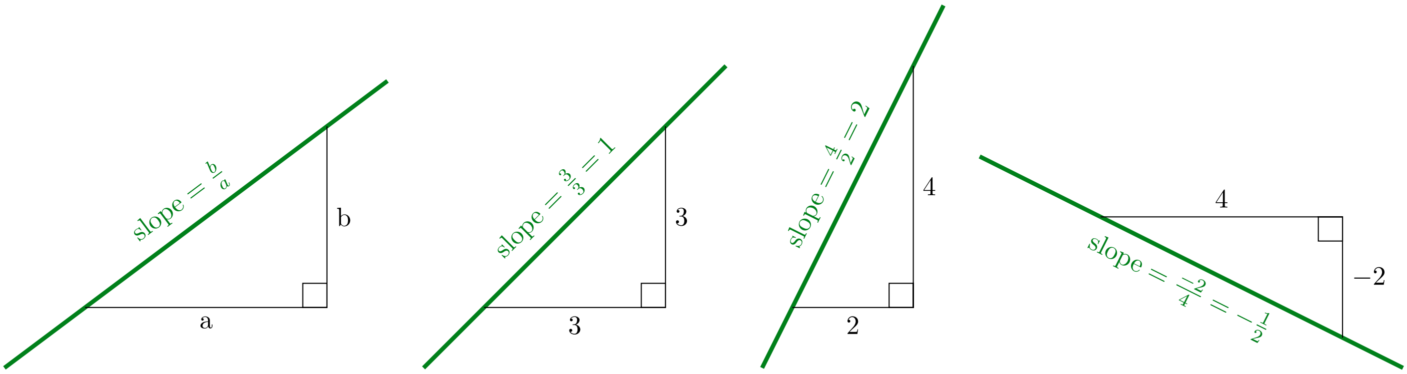
(a) $\lim_{x\to\infty} \frac{2+\sqrt{x}}{2-\sqrt{x}}$

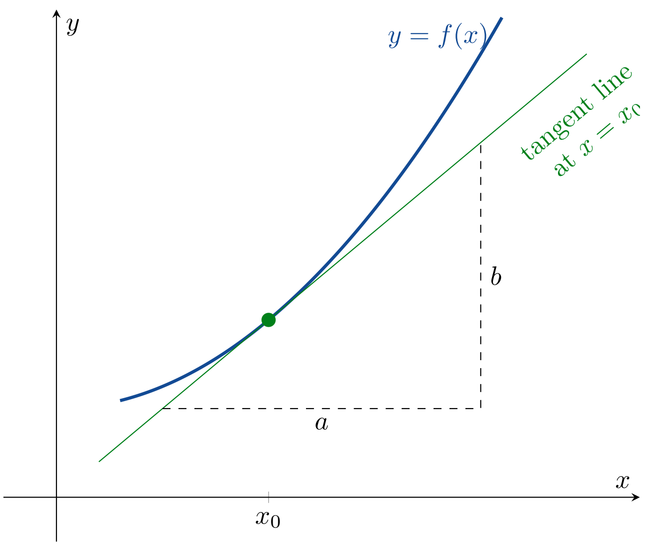
(b) $\lim_{t\to-\infty} \frac{\sqrt{t^2+1}}{t+1}$

(c) $\lim_{p\to 0} \frac{-1}{p^2(p+1)}$

7 Differentiation / Türev



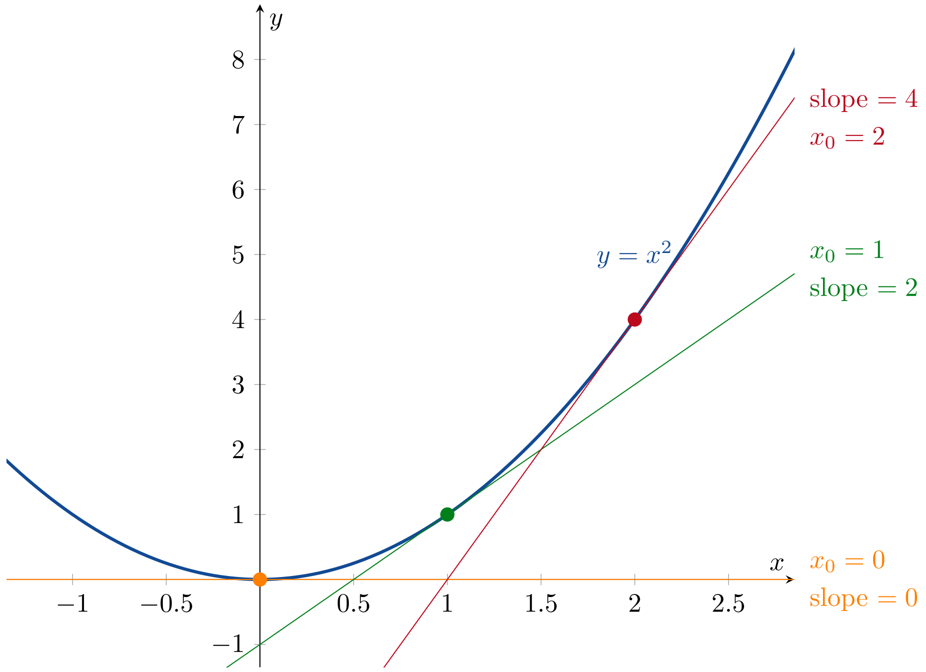




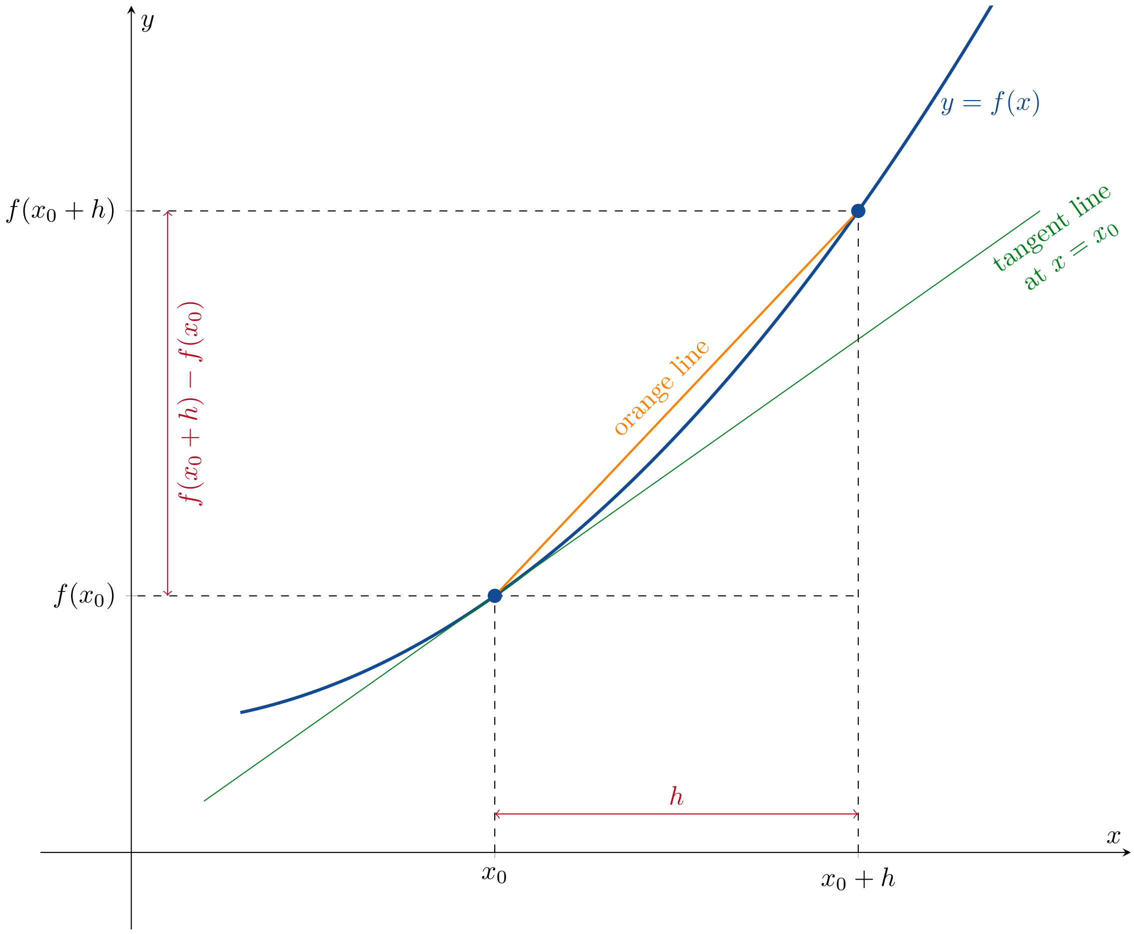
We can say that

$$\left(\begin{array}{c} \text{slope of } y = f(x) \\ \text{at } x = x_0 \end{array} \right) = \left(\begin{array}{c} \text{slope of the tangent} \\ \text{line at } x = x_0 \end{array} \right)$$

Example 7.1. **Örnek 7.1.**



The slope of $y = x^2$ at $x_0 = 0$ is 0.
The slope of $y = x^2$ at $x_0 = 1$ is 2.
The slope of $y = x^2$ at $x_0 = 2$ is 4.
How do we know this?



If h is very very small, then

$$\begin{pmatrix} \text{slope of the} \\ \text{tangent line} \end{pmatrix} \approx \begin{pmatrix} \text{slope of the} \\ \text{orange line} \end{pmatrix} = \frac{f(x_0+h) - f(x_0)}{h}$$

h çok ama çok küçükse, o zaman

$$\begin{pmatrix} \text{slope of the} \\ \text{tangent line} \end{pmatrix} \approx \begin{pmatrix} \text{slope of the} \\ \text{orange line} \end{pmatrix} = \frac{f(x_0+h) - f(x_0)}{h}$$

The Derivative of f

Definition. The ***derivative of a function*** f ***at a point*** x_0 is

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h}$$

if the limit exists.

(f' is pronounced "f prime")

Example 7.2. Consider the function $g(x) = \frac{1}{x}$, $x \neq 0$.

If $x_0 \neq 0$, then

$$\begin{aligned} g'(x_0) &= \lim_{h \to 0} \frac{g(x_0+h) - g(x_0)}{h} \\ &= \lim_{h \to 0} \frac{\frac{1}{x_0+h} - \frac{1}{x_0}}{h} \\ &= \lim_{h \to 0} \frac{\left(\frac{x_0}{x_0(x_0+h)} - \frac{x_0+h}{x_0(x_0+h)}\right)}{h} \\ &= \lim_{h \to 0} \frac{x_0 - x_0 - h}{hx_0(x_0+h)} \\ &= \lim_{h \to 0} \frac{-1}{x_0(x_0+h)} \\ &= -\frac{1}{x_0^2}. \end{aligned}$$

See figure 7.1.

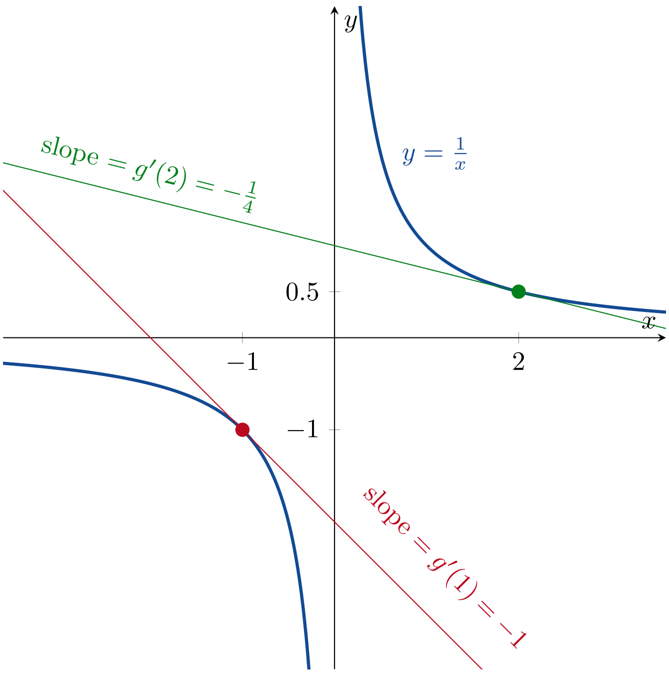


Figure 7.1: The graph of $g(x) = \frac{1}{x}$, $x \neq 0$ and two tangents to this graph.
Şekil 7.1:

Definition. If $f'(x_0)$ exists, we say that ***f is differentiable at x_0***.

Definition. Let $f : D \to \mathbb{R}$ be a function. If f is differentiable at every $x_0 \in D$, we say that f is ***differentiable***.

The Derivative of f

Tanım. ***Bir f fonksiyonunun x_0 noktasındaki türevi limitin mevcut olması koşuyla***

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h}$$

olarak tanımlanır.

(f' sembolü "f üssü" olarak okunur)

Örnek 7.3. $g(x) = \frac{1}{x}$, $x \neq 0$ fonksiyonunu ele alalım.

$x_0 \neq 0$ ise,

$$\begin{aligned} g'(x_0) &= \lim_{h \to 0} \frac{g(x_0+h) - g(x_0)}{h} \\ &= \lim_{h \to 0} \frac{\frac{1}{x_0+h} - \frac{1}{x_0}}{h} \\ &= \lim_{h \to 0} \frac{\left(\frac{x_0}{x_0(x_0+h)} - \frac{x_0+h}{x_0(x_0+h)}\right)}{h} \\ &= \lim_{h \to 0} \frac{x_0 - x_0 - h}{hx_0(x_0+h)} \\ &= \lim_{h \to 0} \frac{-1}{x_0(x_0+h)} \\ &= -\frac{1}{x_0^2}. \end{aligned}$$

Bkz. şekil 7.1.

Tanım. $f'(x_0)$ mevcutsa, ***f fonksiyonu x_0'da türevlenebilirdir*** deriz.

Tanım. $f : D \to \mathbb{R}$ bir fonksiyon olsun. f her $x_0 \in D$ noktasında türevlenebilir ise, f bir ***türevlenebilir*** fonksiyondur deriz.

$f : D \to \mathbb{R}$ türevlenebilir ise, elimizde yeni bir $f' : D \to \mathbb{R}$ fonksiyonu olur.

Tanım. f' fonksiyonuna f'nin ***türevi*** denir.

Örnek 7.4. $f(x) = \frac{x}{x-1}$'nin türevini bulunuz.

çözüm: İlk olarak $f(x+h) = \frac{x+h}{x+h-1}$. Buradan

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{\frac{x+h}{x+h-1} - \frac{x}{x-1}}{h} \\ &= \lim_{h \to 0} \frac{1}{h}\left(\frac{(x+h)(x-1) - x(x+h-1)}{(x-1)(x+h-1)}\right) \\ &= \lim_{h \to 0} \frac{1}{h}\left(\frac{-h}{(x-1)(x+h-1)}\right) \\ &= \lim_{h \to 0} \frac{-1}{(x-1)(x+h-1)} \\ &= \frac{-1}{(x-1)(x+0-1)} \\ &= \frac{-1}{(x-1)^2} \end{aligned}$$

buluruz.

If $f : D \to \mathbb{R}$ is differentiable, then we have a new function $f' : D \to \mathbb{R}$.

Definition. f' is called the ***derivative*** of f.

Example 7.3. Differentiate $f(x) = \frac{x}{x-1}$.

solution: First note that $f(x+h) = \frac{x+h}{x+h-1}$. Therefore

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{\frac{x+h}{x+h-1} - \frac{x}{x-1}}{h} \\ &= \lim_{h\to 0} \frac{1}{h}\left(\frac{(x+h)(x-1) - x(x+h-1)}{(x-1)(x+h-1)}\right) \\ &= \lim_{h\to 0} \frac{1}{h}\left(\frac{-h}{(x-1)(x+h-1)}\right) \\ &= \lim_{h\to 0} \frac{-1}{(x-1)(x+h-1)} \\ &= \frac{-1}{(x-1)(x+0-1)} \\ &= \frac{-1}{(x-1)^2}. \end{aligned}$$

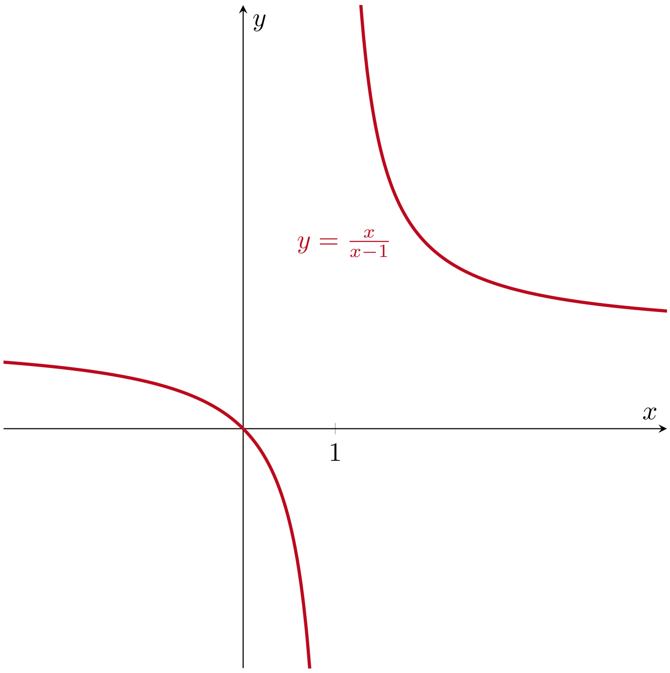


Figure 7.2: The graph of $y = \frac{x}{x-1}$.
Şekil 7.2: $y = \frac{x}{x-1}$'in grafiği

Notations

There are many ways to write the derivative of $y = f(x)$.

$$f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx}f(x) = \dot{y} = \dot{f}(x)$$

"the derivative of y with respect to x"

Calculus was started by two men who hated each other: Sir Isaac Newton (UK, 1642-1726) used $\dot{f}$ and $\dot{y}$. Gottfried Leibniz (GER, 1646-1716) used $\frac{df}{dx}$ and $\frac{dy}{dx}$. The f' and y' notation came later from Joseph-Louis Lagrange (ITA, 1736-1813).

If we want the derivative of $y = f(x)$ at the point $x = x_0$, we can write

$$f'(x_0) = \left.\frac{dy}{dx}\right|_{x=x_0} = \left.\frac{df}{dx}\right|_{x=x_0} = \left.\frac{d}{dx}f(x)\right|_{x=x_0}$$

"the derivative of y with respect to x at $x = x_0$"

For example, if $u(x) = \frac{1}{x}$, then

$$u'(4) = \frac{d}{dx}\left(\frac{1}{x}\right)\bigg|_{x=4} = \frac{-1}{x^2}\bigg|_{x=4} = \frac{-1}{4^2} = \frac{-1}{16}.$$

Notations

$y = f(x)$'nin türevini yazmanın birçok yolu vardır.

$$f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx}f(x) = \dot{y} = \dot{f}(x)$$

"y nin x'e göre türevi"

Calculus birbirinden nefret eden iki kişi tarafından başladı: Sir Isaac Newton (İngiltere, 1642-1726) $\dot{f}$ ve $\dot{y}$ kullandı. Gottfried Leibniz (Almanya, 1646-1716) $\frac{df}{dx}$ ve $\frac{dy}{dx}$ sembollerini kullandı. F' ve y' gösterimi daha sonra Joseph-Louis Lagrange'den (ITA, 1736-1813) tarafından ilk kullanıldı.

$y = f(x)$'nin $x = x_0$'daki türevini bulmak için, şöyle yazarız

$$f'(x_0) = \left.\frac{dy}{dx}\right|_{x=x_0} = \left.\frac{df}{dx}\right|_{x=x_0} = \left.\frac{d}{dx}f(x)\right|_{x=x_0}$$

"y'nin x'e göre $x = x_0$'daki türevi"

Örneğin, $u(x) = \frac{1}{x}$ ise, o zaman

$$u'(4) = \frac{d}{dx}\left(\frac{1}{x}\right)\bigg|_{x=4} = \frac{-1}{x^2}\bigg|_{x=4} = \frac{-1}{4^2} = \frac{-1}{16}.$$

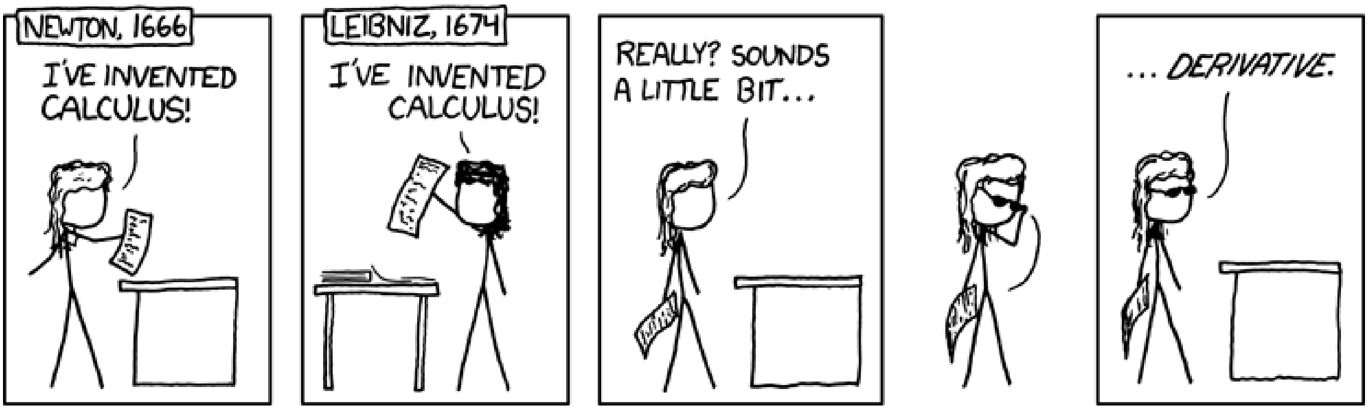


Figure 7.3: A web comic taken from `https://xkcd.com/626/`
Şekil 7.3: `https://xkcd.com/626/`

Example 7.4. Show that $f(x) = |x|$ is differentiable on $(-\infty, 0)$ and on $(0, \infty)$, but is not differentiable at $x = 0$.

solution: If $x > 0$ then

$$\frac{df}{dx} = \frac{d}{dx}(|x|) = \frac{d}{dx}(x) = \lim_{h\to 0} \frac{(x+h)-x}{h} = \lim_{h\to 0} 1 = 1.$$

Similarly, if $x < 0$ then

$$\frac{df}{dx} = \frac{d}{dx}(|x|) = \frac{d}{dx}(-x) = \lim_{h\to 0} \frac{(-x-h)-(-x)}{h} = \lim_{h\to 0} -1 = -1.$$

Therefore f is differentiable on $(-\infty, 0)$ and on $(0, \infty)$.

Since $\lim_{h\to 0} \frac{|0+h|-|0|}{h} = \lim_{h\to 0} \frac{|h|}{h} = \lim_{h\to 0}(\pm 1)$ does not exist, f is not differentiable at 0.

See figure 7.4.

Örnek 7.5. $f(x) = |x|$'nin $(-\infty, 0)$ ve $(0, \infty)$ aralıklarında türevlenebilir ama $x = 0$'da türevlenebilir olmadığını gösteriniz.

çözüm: $x > 0$ ise o vakit

$$\frac{df}{dx} = \frac{d}{dx}(|x|) = \frac{d}{dx}(x) = \lim_{h\to 0} \frac{(x+h)-x}{h} = \lim_{h\to 0} 1 = 1.$$

Benzer olarak, $x < 0$ ise o halde

$$\frac{df}{dx} = \frac{d}{dx}(|x|) = \frac{d}{dx}(-x) = \lim_{h\to 0} \frac{(-x-h)-(-x)}{h} = \lim_{h\to 0} -1 = -1.$$

Yani f foksiyonu $(-\infty, 0)$ ve $(0, \infty)$'da türevlenebilirdir.

$\lim_{h\to 0} \frac{|0+h|-|0|}{h} = \lim_{h\to 0} \frac{|h|}{h} = \lim_{h\to 0}(\pm 1)$ mevcut olmadığından, f 0'da türevlenenemez.

Bkz. şekil 7.4.

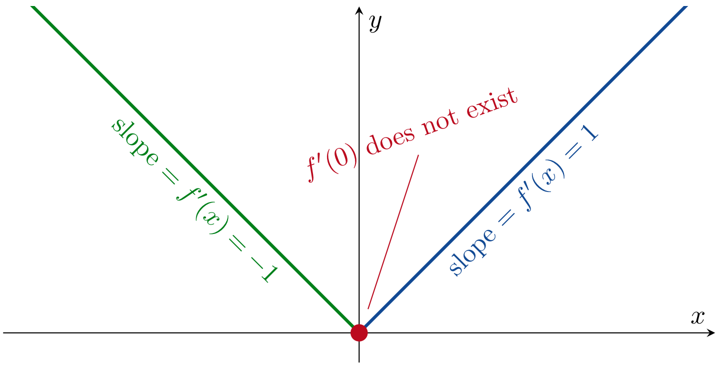
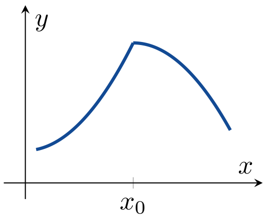


Figure 7.4: The graph of $y = |x|$.
Şekil 7.4: $y = |x|$'in grafiği.

When Does a Function Not Have a Derivative at a Point?

Hangi Durumlarda Bir Fonksiyonun Türevi Yoktur?

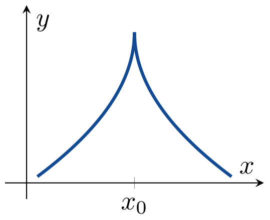


a corner

$f'(x_0)$ does not exist

köşe durumu

$f'(x_0)$ mevcut değil

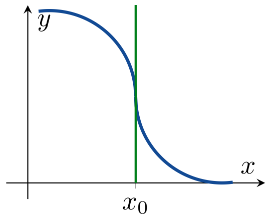


a cusp

$f'(x_0)$ does not exist

içten bükülme

$f'(x_0)$ mevcut değil

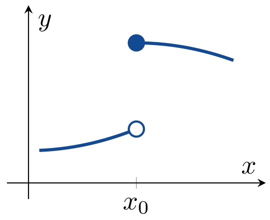


a vertical tangent

$f'(x_0)$ does not exist

dikey teğet

$f'(x_0)$ mevcut değil

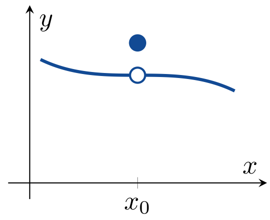


a discontinuity

$f'(x_0)$ does not exist

süreksizlik

$f'(x_0)$ mevcut değil



a discontinuity

$f'(x_0)$ does not exist

süreksizlik

$f'(x_0)$ mevcut değil

Theorem 7.1.

$$\begin{pmatrix} f \text{ has a derivative} \\ \text{at } x = x_0 \end{pmatrix} \implies \begin{pmatrix} f \text{ is continuous} \\ \text{at } x = x_0 \end{pmatrix}$$

Teorem 7.1.

$$\begin{pmatrix} f\text{'nin at } x = x_0 \text{ da} \\ \text{türevi mevcut} \end{pmatrix} \implies \begin{pmatrix} f,\ x = x_0\text{'da} \\ \text{sürekli} \end{pmatrix}$$

8 Differentiation Rules

Constant Function

If $k \in \mathbb{R}$, then

$$\boxed{\frac{d}{dx}(k) = 0.}$$

Power Function

If $n \in \mathbb{R}$, then

$$\boxed{\frac{d}{dx}\left(x^n\right) = nx^{n-1}.}$$

Example 8.1.

$$\frac{d}{dx}\left(x^3\right) = 3x^{3-1} = 3x^2$$

Example 8.2.

$$\frac{d}{dx}\left(\sqrt{x}\right) = \frac{d}{dx}\left(x^{\frac{1}{2}}\right) = \frac{1}{2}x^{\frac{1}{2}-1} = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$$

Example 8.3.

$$\frac{d}{dx}\left(\frac{1}{x^4}\right) = \frac{d}{dx}\left(x^{-4}\right) = -4x^{-4-1} = -4x^{-5} = -\frac{4}{x^5}$$

The Constant Multiple Rule

If $u(x)$ is differentiable and $k \in \mathbb{R}$, then

$$\boxed{\frac{d}{dx}(ku) = k\frac{du}{dx}.}$$

Proof.

$$\begin{aligned}\frac{d}{dx}(ku) &= \lim_{h\to 0}\frac{ku(x+h)-ku(x)}{h}\\ &= k\lim_{h\to 0}\frac{u(x+h)-u(x)}{h} = k\frac{du}{dx}\end{aligned}$$

□

Example 8.4.

$$\frac{d}{dx}\left(3x^2\right) = 3\frac{d}{dx}\left(x^2\right) = 3\times 2x = 6x$$

Example 8.5.

$$\frac{d}{dx}(-u) = \frac{d}{dx}(-1\times u) = -1\times\frac{du}{dx} = -\frac{du}{dx}$$

Türev Kuralları

Sabit Fonksiyon

$k \in \mathbb{R}$ ise, o halde

$$\boxed{\frac{d}{dx}(k) = 0.}$$

Kuvvet Fonksiyonu

$n \in \mathbb{R}$ ise, bu durumda

$$\boxed{\frac{d}{dx}\left(x^n\right) = nx^{n-1}.}$$

Örnek 8.1.

$$\frac{d}{dx}\left(x^3\right) = 3x^{3-1} = 3x^2$$

Örnek 8.2.

$$\frac{d}{dx}\left(\sqrt{x}\right) = \frac{d}{dx}\left(x^{\frac{1}{2}}\right) = \frac{1}{2}x^{\frac{1}{2}-1} = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$$

Örnek 8.3.

$$\frac{d}{dx}\left(\frac{1}{x^4}\right) = \frac{d}{dx}\left(x^{-4}\right) = -4x^{-4-1} = -4x^{-5} = -\frac{4}{x^5}$$

Sabitle Çarpım Kuralı

$u(x)$ türevlenebilir ve $k \in \mathbb{R}$ ise,

$$\boxed{\frac{d}{dx}(ku) = k\frac{du}{dx}.}$$

Kanıt.

$$\begin{aligned}\frac{d}{dx}(ku) &= \lim_{h\to 0}\frac{ku(x+h)-ku(x)}{h}\\ &= k\lim_{h\to 0}\frac{u(x+h)-u(x)}{h} = k\frac{du}{dx}\end{aligned}$$

□

Örnek 8.4.

$$\frac{d}{dx}\left(3x^2\right) = 3\frac{d}{dx}\left(x^2\right) = 3\times 2x = 6x$$

Örnek 8.5.

$$\frac{d}{dx}(-u) = \frac{d}{dx}(-1\times u) = -1\times\frac{du}{dx} = -\frac{du}{dx}$$

The Sum Rule

If $u(x)$ and $v(x)$ are differentiable at x_0, then $u+v$ is also differentiable at x_0 and

$$\boxed{\frac{d}{dx}(u+v) = \frac{du}{dx} + \frac{dv}{dx}.}$$

Example 8.6. Differentiate $y = x^3 + \frac{4}{3}x^2 - 5x + 1$.

solution:

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}\left(x^3 + \frac{4}{3}x^2 - 5x + 1\right) \\ &= \frac{d}{dx}(x^3) + \frac{d}{dx}\left(\frac{4}{3}x^2\right) - \frac{d}{dx}(5x) + \frac{d}{dx}(1) \\ &= 3x^2 + \frac{8}{3}x - 5 + 0\end{aligned}$$

Example 8.7. Does the curve $y = x^4 - 2x^2 + 2$ have any points where $\frac{dy}{dx} = 0$? If so, where?

solution: Since

$$\frac{dy}{dx} = 4x^3 - 4x = 4x(x^2-1) = 4x(x-1)(x+1),$$

we can see that $\frac{dy}{dx} = 0$ if and only if $x = -1$, 0 or 1. See figure 8.1.

Toplam Kuralı

$u(x)$ ve $v(x)$ fonksiyonları x_0'da türevlenebilirlerse, $u+v$'de x_0 türevlenebilirdir ve

$$\boxed{\frac{d}{dx}(u+v) = \frac{du}{dx} + \frac{dv}{dx}.}$$

Örnek 8.6. $y = x^3 + \frac{4}{3}x^2 - 5x + 1$ fonksiyonunun türevini bulunuz.

çözüm:

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}\left(x^3 + \frac{4}{3}x^2 - 5x + 1\right) \\ &= \frac{d}{dx}(x^3) + \frac{d}{dx}\left(\frac{4}{3}x^2\right) - \frac{d}{dx}(5x) + \frac{d}{dx}(1) \\ &= 3x^2 + \frac{8}{3}x - 5 + 0\end{aligned}$$

Örnek 8.7. $y = x^4 - 2x^2 + 2$ eğrisi üzerinde $\frac{dy}{dx} = 0$ olan nokta(lar) var mıdır? Varsa, nelerdir?

çözüm:

$$\frac{dy}{dx} = 4x^3 - 4x = 4x(x^2-1) = 4x(x-1)(x+1),$$

olduğundan şunu gözlemleyebiliriz $\frac{dy}{dx} = 0$ ancak ve ancak $x = -1$, 0 veya 1 olur. Bkz. şekil 8.1.

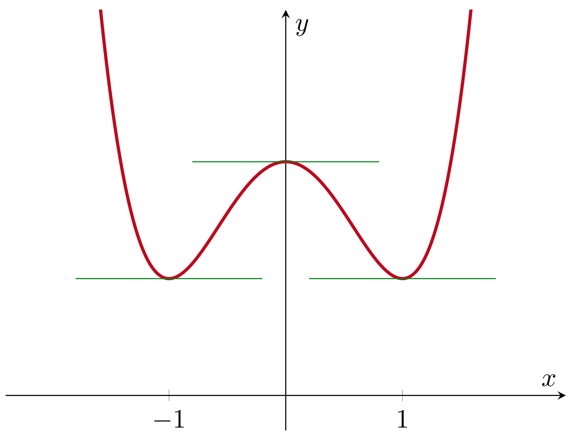


Figure 8.1: The graph of $y = x^4 - 2x^2 + 2$.
Şekil 8.1: $y = x^4 - 2x^2 + 2$'nin grafiği.

The Product Rule

If $u(x)$ and $v(x)$ are differentiable at x_0, then $u(x)v(x)$ is also differentiable at x_0 and

$$\boxed{\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}.}$$

Using prime notation, the product rule is

$$\boxed{(uv)' = u'v + uv'.}$$

Example 8.8. Differentiate $y = (x^2+1)(x^3+3)$.

solution 1: We have $y = uv$ with $u = x^2+1$ and $v = x^3+3$. So

$$\begin{aligned}\frac{dy}{dx} &= (x^2+1)'(x^3+3) + (x^2+1)(x^3+3)' \\ &= (2x+0)(x^3+3) + (x^2+1)(3x^2+0) \\ &= 2x^4+6x+3x^4+3x^2 \\ &= 5x^4+3x^2+6x.\end{aligned}$$

solution 2: Since

$$y = (x^2+1)(x^3+3) = x^5+x^3+3x^2+3,$$

we have that

$$\frac{dy}{dx} = 5x^4+3x^2+6x+0.$$

The Quotient Rule

If $u(x)$ and $v(x)$ are differentiable at x_0 and if $v(x_0) \neq 0$, then $\frac{u}{v}$ is also differentiable at x_0 and

$$\boxed{\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{u'v-uv'}{v^2}.}$$

Example 8.9. Differentiate $y = \frac{t^2-1}{t^3+1}$.

solution: We have $y = \frac{u}{v}$ with $u = t^2-1$ and $v = t^3+1$. Therefore

$$\begin{aligned}\frac{dy}{dt} &= \frac{u'v-uv'}{v^2} \\ &= \frac{(t^2-1)'(t^3+1)-(t^2-1)(t^3+1)'}{(t^3+1)^2} \\ &= \frac{(2t)(t^3+1)-(t^2-1)(3t^2)}{(t^3+1)^2} \\ &= \frac{2t^4+2t-3t^4+3t^2}{(t^3+1)^2} \\ &= \frac{-t^4+3t^2+2t}{(t^3+1)^2}.\end{aligned}$$

Example 8.10. Differentiate $f(s) = \frac{\sqrt{s}-1}{\sqrt{s}+1}$.

solution: We have $f(s) = \frac{u}{v}$ with $u = \sqrt{s}-1$ and $v = \sqrt{s}+1$.

Çarpım Kuralı

$u(x)$ ve $v(x)$ fonksiyonlarla x_0'da türevlenebilirlerse, $u(x)v(x)$ fonksiyonu da x_0 türevlenebilirdir ve

$$\boxed{\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}.}$$

Üs notasyonu kullanarak, çarpım kuralı da

$$\boxed{(uv)' = u'v + uv'.}$$

Örnek 8.8. $y = (x^2+1)(x^3+3)$ fonksiyonunun türevini bulunuz.

çözüm 1: Elimizde şunlar var: $y = uv$ ile $u = x^2+1$ ve $v = x^3+3$. Yani

$$\begin{aligned}\frac{dy}{dx} &= (x^2+1)'(x^3+3) + (x^2+1)(x^3+3)' \\ &= (2x+0)(x^3+3) + (x^2+1)(3x^2+0) \\ &= 2x^4+6x+3x^4+3x^2 \\ &= 5x^4+3x^2+6x.\end{aligned}$$

çözüm 2:

$$y = (x^2+1)(x^3+3) = x^5+x^3+3x^2+3$$

olduğundan,

$$\frac{dy}{dx} = 5x^4+3x^2+6x+0$$

buluruz.

Bölüm Kuralı

Eğer $u(x)$ ve $v(x)$ fonksiyonları x_0'da türevlenebilirlerse ve $v(x_0) \neq 0$ ise, o zaman $\frac{u}{v}$ fonksiyonu da x_0'da türevlenebilirdir ve türevi de şyledir:

$$\boxed{\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{u'v-uv'}{v^2}.}$$

Örnek 8.9. $y = \frac{t^2-1}{t^3+1}$ fonksiyonunun türevini alınız.

çözüm: $u = t^2-1$ ve $v = t^3+1$ olmak üzere $y = \frac{u}{v}$ olsun. Buradan

$$\begin{aligned}\frac{dy}{dt} &= \frac{u'v-uv'}{v^2} \\ &= \frac{(t^2-1)'(t^3+1)-(t^2-1)(t^3+1)'}{(t^3+1)^2} \\ &= \frac{(2t)(t^3+1)-(t^2-1)(3t^2)}{(t^3+1)^2} \\ &= \frac{2t^4+2t-3t^4+3t^2}{(t^3+1)^2} \\ &= \frac{-t^4+3t^2+2t}{(t^3+1)^2}\end{aligned}$$

buluruz.

Remember that $\frac{d}{ds}\left(\sqrt{s}\right) = \frac{1}{2\sqrt{s}}$. Therefore

$$\begin{aligned}
\frac{df}{ds} &= \frac{u'v - uv'}{v^2} \\
&= \frac{(\sqrt{s}-1)'(\sqrt{s}+1) - (\sqrt{s}-1)(\sqrt{s}+1)'}{\left(\sqrt{s}+1\right)^2} \\
&= \frac{\left(\frac{1}{2\sqrt{s}}\right)(\sqrt{s}+1) - (\sqrt{s}-1)\left(\frac{1}{2\sqrt{s}}\right)}{\left(\sqrt{s}+1\right)^2} \\
&= \frac{\frac{1}{2} + \frac{1}{2\sqrt{s}} - \frac{1}{2} + \frac{1}{2\sqrt{s}}}{\left(\sqrt{s}+1\right)^2} \\
&= \frac{1}{\sqrt{s}\left(\sqrt{s}+1\right)^2}.
\end{aligned}$$

Second Order Derivatives

If $y = f(x)$ is a differentiable function, then $f'(x)$ is also a function. If $f'(x)$ is also differentiable, then we can differentiate to find a new function called f'' ("f double prime"). f'' is called the ***second derivative*** of f. We can write

$$f''(x) = \frac{d}{dx}f'(x) = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2} = \frac{dy'}{dx} = y''$$

"d squared y, dx squared"

Example 8.11. If $y = x^6$, then $y' = \frac{d}{dx}\left(x^6\right) = 6x^5$ and $y'' = \frac{d}{dx}\left(y'\right) = \frac{d}{dx}\left(6x^5\right) = 30x^4$. Equivalently, we can write

$$\frac{d^2}{dx^2}\left(x^6\right) = \frac{d}{dx}\left(\frac{d}{dx}\left(x^6\right)\right) = \frac{d}{dx}\left(6x^5\right) = 30x^4.$$

Higher Order Derivatives

If f'' is differentiable, then its derivative $f''' = \frac{d^3f}{dx^3}$ is the ***third derivative*** of f.
If f''' is differentiable, then its derivative $f^{(4)} = \frac{d^4f}{dx^4}$ is the ***fourth derivative*** of f.
If $f^{(4)}$ is differentiable, then its derivative $f^{(5)} = \frac{d^5f}{dx^5}$ is the ***fifth derivative*** of f.

$$\vdots$$

If $f^{(n-1)}$ is differentiable, then its derivative $f^{(n)} = \frac{d^nf}{dx^n}$ is the ***nth derivative*** of f.

Example 8.12. Find the first four derivatives of $y = x^3 - 3x^2 + 2$.

solution:
First derivative: $y' = 3x^2 - 6x$
Second derivative: $y'' = 6x - 6$
Third derivative: $y''' = 6$
Fourth derivative: $y^{(4)} = 0$.

(Note that since $\frac{d}{dx}(0) = 0$, if $n \geq 4$ then $y^{(n)} = 0$ also.)

Örnek 8.10. $f(s) = \frac{\sqrt{s}-1}{\sqrt{s}+1}$ fonksiyonunun türevini bulunuz.

çözüm: $f(s) = \frac{u}{v}$ olsun burada $u = \sqrt{s} - 1$ ve $v = \sqrt{s} + 1$. Unutmayınız ki $\frac{d}{ds}\left(\sqrt{s}\right) = \frac{1}{2\sqrt{s}}$. Dolayısıyla

$$\begin{aligned}
\frac{df}{ds} &= \frac{u'v - uv'}{v^2} \\
&= \frac{(\sqrt{s}-1)'(\sqrt{s}+1) - (\sqrt{s}-1)(\sqrt{s}+1)'}{\left(\sqrt{s}+1\right)^2} \\
&= \frac{\left(\frac{1}{2\sqrt{s}}\right)(\sqrt{s}+1) - (\sqrt{s}-1)\left(\frac{1}{2\sqrt{s}}\right)}{\left(\sqrt{s}+1\right)^2} \\
&= \frac{\frac{1}{2} + \frac{1}{2\sqrt{s}} - \frac{1}{2} + \frac{1}{2\sqrt{s}}}{\left(\sqrt{s}+1\right)^2} \\
&= \frac{1}{\sqrt{s}\left(\sqrt{s}+1\right)^2}
\end{aligned}$$

buluruz.

İkinci Mertebeden Türevler

$y = f(x)$ türevlenebilir bir fonksiyon ise, o zaman $f'(x)$ de bir fonksiyondur. $f'(x)$ de türevlenebilir ise, bu durumda yine türev alır ve yeni bir f'' ("f iki üssü") fonksiyonu buluruz. f'' fonksiyonuna f'nin ***ikinci türevi*** denir. Şöyle dösteririz

$$f''(x) = \frac{d}{dx}f'(x) = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2} = \frac{dy'}{dx} = y''$$

"d kare y bölü dx kare"

Örnek 8.11. $y = x^6$ ise, $y' = \frac{d}{dx}\left(x^6\right) = 6x^5$ ve $y'' = \frac{d}{dx}\left(y'\right) = \frac{d}{dx}\left(6x^5\right) = 30x^4$. Buna eşdeğer olarak,

$$\frac{d^2}{dx^2}\left(x^6\right) = \frac{d}{dx}\left(\frac{d}{dx}\left(x^6\right)\right) = \frac{d}{dx}\left(6x^5\right) = 30x^4$$

yazabiliriz.

Yüksek Mertebeden Türevler

f'' türevlenebilir ise, türevi olan $f''' = \frac{d^3f}{dx^3}$ fonksiyona f'nin ***üçüncü türevi*** denir.
f''' türevlenebilir ise, türevi olan $f^{(4)} = \frac{d^4f}{dx^4}$ fonksiyonuna f'nin ***dördüncü türevi*** denir.
$f^{(4)}$ türevlenebilir ise, türevi olan $f^{(5)} = \frac{d^5f}{dx^5}$ fonksiyonuna f'nin ***beşinci türevi***.

$$\vdots$$

$f^{(n-1)}$ türevlenebilir ise, türevi olan $f^{(n)} = \frac{d^nf}{dx^n}$ fonksiyonuna f'nin ***n inci türevi*** denir.

Örnek 8.12. $y = x^3 - 3x^2 + 2$ ise, ilk dört mertebeden türevlerini bulunuz.

çözüm:
Birinci mertebeden türev: $y' = 3x^2 - 6x$
İkinci mertebeden türev: $y'' = 6x - 6$

Üçüncü mertebeden türev: $y''' = 6$
Dördüncü inci mertebeden türev: $y^{(4)} = 0$.

($\frac{d}{dx}(0) = 0$ olsuğundan, $n \geq 4$ ise $y^{(n)} = 0$ olduğunu unutmayınız.)

Problems

Problem 8.1.

(a). Find $\dfrac{ds}{dt}$ if $s = -2t^{-1} + \dfrac{4}{t^2}$.

(b). Find w'' if $w = (z+1)(z-1)(z^2+1)$.

(c). Find $\dfrac{dy}{dx}$ if $y = (2x+3)(x^4 + \frac{1}{3}x^3 + 11)$.

Problem 8.2. Find $\dfrac{db}{dx}$ if $b = \dfrac{x^2-1}{x^2+x-2}$.

Sorular

Soru 8.1.

(a). $s = -2t^{-1} + \dfrac{4}{t^2}$ ise $\dfrac{ds}{dt}$ yi bulunuz.

(b). $w = (z+1)(z-1)(z^2+1)$ ise w'''yü bulunuz.

(c). $y = (2x+3)(x^4 + \frac{1}{3}x^3 + 11)$ ise $\dfrac{dy}{dx}$'i bulunuz.

Soru 8.2. $b = \dfrac{x^2-1}{x^2+x-2}$ ise $\dfrac{db}{dx}$'i bulunuz.

Derivatives of Trigonometric Functions

Trigonometrik Fonksiyonların Türevleri

9

Sine and Cosine

$$\boxed{\frac{d}{dx}(\sin x) = \cos x}$$

Example 9.1. Differentiate $y = x^2 - \sin x$.

solution:

$$\frac{dy}{dx} = \frac{d}{dx}\left(x^2\right) - \frac{d}{dx}(\sin x) = 2x - \cos x.$$

Example 9.2. Differentiate $y = x^2 \sin x$.

solution: We will use the product rule ($(uv)' = u'v + uv'$) with $u = x^2$ and $v = \sin x$.

$$y' = (x^2)'(\sin x) + (x^2)(\sin x)' = 2x \sin x + x^2 \cos x.$$

Example 9.3. Differentiate $y = \frac{\sin x}{x}$.

solution: This time we use the quotient rule ($\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$) with $u = \sin x$ and $v = x$.

$$y' = \frac{(\sin x)'x - (\sin x)(x)'}{x^2} = \frac{x\cos x - \sin x}{x^2}.$$

Example 9.4. Differentiate $y = 5x + \cos x$.

solution:

$$\frac{dy}{dx} = \frac{d}{dx}(5x) + \frac{d}{dx}(\cos x) = 5 - \sin x.$$

Example 9.5. Differentiate $y = \sin x \cos x$.

solution: By the product rule, we have that

$$\frac{dy}{dx} = \frac{d}{dx}(\sin x)\cos x + \sin x \frac{d}{dx}(\cos x) = \cos^2 x - \sin^2 x.$$

Example 9.6. Differentiate $y = \frac{\cos x}{1 - \sin x}$.

solution: By the quotient rule, we have that

Sinüs ve Kosinüs

$$\boxed{\frac{d}{dx}(\cos x) = -\sin x}$$

Örnek 9.1. $y = x^2 - \sin x$ fonksiyonunun türevini alınız.

çözüm:

$$\frac{dy}{dx} = \frac{d}{dx}\left(x^2\right) - \frac{d}{dx}(\sin x) = 2x - \cos x.$$

Örnek 9.2. $y = x^2 \sin x$ fonksiyonunun türevini alınız.

çözüm: Çarpım kuralı kullanırsak ($(uv)' = u'v + uv'$) burada $u = x^2$ ve $v = \sin x$ oluyor.

$$y' = (x^2)'(\sin x) + (x^2)(\sin x)' = 2x \sin x + x^2 \cos x.$$

Örnek 9.3. $y = \frac{\sin x}{x}$ fonksiyonunun türevini alınız.

çözüm: Bu sefer de bölüm kuralı kullanırsak ($\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$) burada $u = \sin x$ ve $v = x$ oluyor.

$$y' = \frac{(\sin x)'x - (\sin x)(x)'}{x^2} = \frac{x\cos x - \sin x}{x^2}.$$

Örnek 9.4. $y = 5x + \cos x$ fonksiyonunun türevini alınız.

çözüm:

$$\frac{dy}{dx} = \frac{d}{dx}(5x) + \frac{d}{dx}(\cos x) = 5 - \sin x.$$

Örnek 9.5. $y = \sin x \cos x$ fonksiyonunun türevini alınız.

çözüm: Çarpım kuralı gereğince,

$$\frac{dy}{dx} = \frac{d}{dx}(\sin x)\cos x + \sin x \frac{d}{dx}(\cos x) = \cos^2 x - \sin^2 x.$$

Örnek 9.6. $y = \frac{\cos x}{1 - \sin x}$ fonksiyonunun türevini alınız.

$$\begin{aligned}\frac{dy}{dx} &= \frac{\frac{d}{dx}(\cos x)(1-\sin x) - (\cos x)\frac{d}{dx}(1-\sin x)}{(1-\sin x)^2} \\ &= \frac{-\sin x(1-\sin x) - \cos x(0-\cos x)}{(1-\sin x)^2} \\ &= \frac{-\sin x + \sin^2 x + \cos^2 x}{(1-\sin x)^2} \\ &= \frac{-\sin x + 1}{(1-\sin x)^2} = \frac{1-\sin x}{(1-\sin x)^2} \\ &= \frac{1}{1-\sin x}.\end{aligned}$$

çözüm: Bölüm kuralından,

$$\begin{aligned}\frac{dy}{dx} &= \frac{\frac{d}{dx}(\cos x)(1-\sin x) - (\cos x)\frac{d}{dx}(1-\sin x)}{(1-\sin x)^2} \\ &= \frac{-\sin x(1-\sin x) - \cos x(0-\cos x)}{(1-\sin x)^2} \\ &= \frac{-\sin x + \sin^2 x + \cos^2 x}{(1-\sin x)^2} \\ &= \frac{-\sin x + 1}{(1-\sin x)^2} = \frac{1-\sin x}{(1-\sin x)^2} \\ &= \frac{1}{1-\sin x}.\end{aligned}$$

The Tangent Function

Tanjant Fonksiyonu

$$\boxed{\frac{d}{dx}(\tan x) = \sec^2 x}$$

Proof. Using the quotient rule, we can calculate that

$$\begin{aligned}\frac{d}{dx}(\tan x) &= \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) \\ &= \frac{\frac{d}{dx}(\sin x)(\cos x) - (\sin x)\frac{d}{dx}(\cos x)}{\cos^2 x} \\ &= \frac{(\cos x)(\cos x) - (\sin x)(-\sin x)}{\cos^2 x} \\ &= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} \\ &= \frac{1}{\cos^2 x} \\ &= \sec^2 x.\end{aligned}$$

□

Kanıt. Bölüm türevinden,

$$\begin{aligned}\frac{d}{dx}(\tan x) &= \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) \\ &= \frac{\frac{d}{dx}(\sin x)(\cos x) - (\sin x)\frac{d}{dx}(\cos x)}{\cos^2 x} \\ &= \frac{(\cos x)(\cos x) - (\sin x)(-\sin x)}{\cos^2 x} \\ &= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} \\ &= \frac{1}{\cos^2 x} \\ &= \sec^2 x.\end{aligned}$$

□

The Other Three

Diğer Üç Fonksiyon

$$\boxed{\frac{d}{dx}(\sec x) = \sec x \tan x} \qquad \boxed{\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x} \qquad \boxed{\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x}$$

You can use the quotient rule to prove these three rules. We may ask you to prove one of them in an exam.

Example 9.7. Find y'' if $y = \sec x$.

solution: Since $y' = \sec x \tan x$, we have that

$$\begin{aligned}y'' &= \frac{d}{dx}(y') = \frac{d}{dx}(\sec x \tan x) \\ &= \frac{d}{dx}(\sec x)\tan x + \sec x \frac{d}{dx}(\tan x) \\ &= (\sec x \tan x)(\tan x) + (\sec x)(\sec^2 x) \\ &= \sec x \tan^2 x + \sec^3 x.\end{aligned}$$

Bu üç kuralın kanıtlanması için bölüm kuralını kullanabilirsiniz. Bunlardan birisini sınavda kanıtlamanızı isteyebiliriz.

Örnek 9.7. $y = \sec x$ ise y''nü bulunuz.

çözüm: $y' = \sec x \tan x$ olduğundan,

$$\begin{aligned}y'' &= \frac{d}{dx}(y') = \frac{d}{dx}(\sec x \tan x) \\ &= \frac{d}{dx}(\sec x)\tan x + \sec x \frac{d}{dx}(\tan x) \\ &= (\sec x \tan x)(\tan x) + (\sec x)(\sec^2 x) \\ &= \sec x \tan^2 x + \sec^3 x\end{aligned}$$

buluruz.

Problems

Problem 9.1.

(a). Find $\dfrac{ds}{dx}$ if $s = (\sin x + \cos x)\sec x$.

(b). Find $\dfrac{dr}{d\theta}$ if $r = \theta \sin\theta + \cos\theta$.

Problem 9.2. Use the quotient rule to prove that the following are true:

(a). $\dfrac{d}{dx}(\sec x) = \sec x \tan x$.

(b). $\dfrac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$.

(c). $\dfrac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$.

Sorular

Soru 9.1.

(a). $s = (\sin x + \cos x)\sec x$ ise $\dfrac{ds}{dx}$'i bulunuz.

(b). $r = \theta \sin\theta + \cos\theta$ ise $\dfrac{dr}{d\theta}$'yı bulunuz.

Soru 9.2. Bölüm kuralı kullanarak, aşağıdakileri kanıtlayınız:

(a). $\dfrac{d}{dx}(\sec x) = \sec x \tan x$.

(b). $\dfrac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$.

(c). $\dfrac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$.

10 The Chain Rule

How do we differentiate $F(x) = \sin(x^2 - 4)$?

Theorem 10.1 (The Chain Rule). *Suppose that*

- *$y = f(u)$ is differentiable at the point $u = g(x)$; and*
- *$g(x)$ is differentiable at x.*

Then $f \circ g$ is differentiable at x and

$$\boxed{(f \circ g)'(x) = f'\big(g(x)\big)g'(x).}$$

The Chain Rule is easier to remember if we use Leibniz's notation:

$$\boxed{\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}}$$

Example 10.1. Differentiate $y = \sin(x^2 - 4)$.

solution: We have $y = \sin u$ with $u = x^2 - 4$. Now $\frac{dy}{du} = \cos u$ and $\frac{du}{dx} = 2x$. Therefore

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du}\frac{du}{dx} = (\cos u)(2x)\\ &= 2x\cos u = 2x\cos(x^2 - 4)\end{aligned}$$

by the Chain Rule.

Example 10.2. Differentiate $\sin(x^2 + x)$.

solution: Let $u = x^2 + x$. Then

$$\begin{aligned}\frac{d}{dx}\big(\sin(x^2 + x)\big) &= \frac{d}{du}(\sin u)\frac{du}{dx}\\ &= (\cos u)(2x + 1)\\ &= (2x + 1)\cos(x^2 + x)\end{aligned}$$

by the Chain Rule.

Example 10.3 (Using the Chain Rule Two Times). Differentiate $g(t) = \tan(5 - \sin 2t)$.

solution: Let $u = 5 - \sin 2t$. Then $g(t) = \tan u$. Hence

$$\frac{dg}{dt} = \frac{dg}{du}\frac{du}{dt} = (\sec^2 u)\frac{d}{dt}(5 - \sin 2t).$$

Zincir Kuralı

$F(x) = \sin(x^2 - 4)$ fonksiyonunun türevini nasıl alırız?

Teorem 10.1 (The Chain Rule). *Varsayalım ki*

- *$y = f(u)$ fonksiyonu $u = g(x)$ notasında türevlenebilir ve*
- *$g(x)$ fonksiyonu da x'de türevlenebilir olsun.*

Bu durumda $f \circ g$ fonksiyonu da x noktasında türevlenebilirdir ve türevi de

$$\boxed{(f \circ g)'(x) = f'\big(g(x)\big)g'(x).}$$

Zincir Kuralı'nı Leibniz notasyonu kullanarak kolayca hatırlanab

$$\boxed{\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}}$$

Örnek 10.1. $y = \sin(x^2 - 4)$ fonksiyonunun türevini alınız.

çözüm: $u = x^2 - 4$ olsun ve $y = \sin u$ olur. Böylece $\frac{dy}{du} = \cos u$ ve $\frac{du}{dx} = 2x$ olur. Yani

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du}\frac{du}{dx} = (\cos u)(2x)\\ &= 2x\cos u = 2x\cos(x^2 - 4)\end{aligned}$$

Zincir Kuralı kullanarak buluruz.

Örnek 10.2. $\sin(x^2 + x)$ fonksiyonunun türevini alınız.

çözüm: $u = x^2 + x$ diyelim. Buradan Zincir Kuralı yardımıyla,

$$\begin{aligned}\frac{d}{dx}\big(\sin(x^2 + x)\big) &= \frac{d}{du}(\sin u)\frac{du}{dx}\\ &= (\cos u)(2x + 1)\\ &= (2x + 1)\cos(x^2 + x)\end{aligned}$$

bulunur.

Örnek 10.3 (İki kez Zincir Kuralı).
$g(t) = \tan(5 - \sin 2t)$ fonksiyonunun türevini alınız.

çözüm: Let $u = 5 - \sin 2t$. Then $g(t) = \tan u$. Hence

We need to use the Chain Rule a second time: Let $w = 2t$. Then

$$\begin{aligned}\frac{dg}{dt} &= (\sec^2 u)\frac{d}{dt}(5-\sin 2t)\\ &= (\sec^2 u)\frac{d}{dw}(5-\sin w)\frac{d2}{dt}\\ &= (\sec^2 u)(-\cos w)(2)\\ &= -2\cos 2t\sec^2(5-\sin 2t).\end{aligned}$$

(Note: Your final answer should not have u or w in it.)

$$\frac{dg}{dt} = \frac{dg}{du}\frac{du}{dt} = (\sec^2 u)\frac{d}{dt}(5-\sin 2t).$$

We need to use the Chain Rule a second time: Let $w = 2t$. Then

$$\begin{aligned}\frac{dg}{dt} &= (\sec^2 u)\frac{d}{dt}(5-\sin 2t)\\ &= (\sec^2 u)\frac{d}{dw}(5-\sin w)\frac{d2}{dt}\\ &= (\sec^2 u)(-\cos w)(2)\\ &= -2\cos 2t\sec^2(5-\sin 2t).\end{aligned}$$

(Not: Cevabınız u or w içermemelidir.)

Powers of a Function

If

- f is a differentiable function of u;
- u is a differentiable function of x; and
- $y = f(u)$,

then the Chain Rule $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$ is the same as

$$\frac{d}{dx}f(u) = f'(u)\frac{du}{dx}.$$

Now suppose that $n \in \mathbb{R}$ and $f(u) = u^n$. Then $f'(u) = nu^{n-1}$. So

$$\boxed{\frac{d}{dx}\left(u^n\right) = nu^{n-1}\frac{du}{dx}.}$$

Example 10.4.

$$\begin{aligned}\frac{d}{dx}\left(5x^3-x^4\right)^7 &= 7\left(5x^3-x^4\right)^6\frac{d}{dx}\left(5x^3-x^4\right)\\ &= 7\left(5x^3-x^4\right)^6\left(15x^-4x^3\right).\end{aligned}$$

Example 10.5.

$$\begin{aligned}\frac{d}{dx}\left(\frac{1}{3x-2}\right) &= \frac{d}{dx}(3x-2)^{-1} = -1(3x-2)^{-2}\frac{d}{dx}(3x-2)\\ &= -\left(\frac{1}{(3x-2)^2}\right)(2) = \frac{-3}{(3x-2)^2}.\end{aligned}$$

Example 10.6.

$$\frac{d}{dx}\left(\sin^5 x\right) = 5\sin^4 x\frac{d}{dx}(\sin x) = 5\sin^4 x\cos x.$$

Example 10.7. Differentiate $|x|$.

solution: Since $|x| = \sqrt{x^2}$, we can calculate that if $x \neq 0$ then

$$\begin{aligned}\frac{d}{dx}|x| &= \frac{d}{dx}\left(\sqrt{x^2}\right) = \frac{d}{du}\left(\sqrt{u}\right)\frac{d}{dx}\left(x^2\right)\\ &= \frac{1}{2\sqrt{u}}2x = \frac{x}{\sqrt{x^2}} = \frac{x}{|x|}.\end{aligned}$$

Example 10.8. Let $y = \frac{1}{(1-2x)^3}$ for $x \neq \frac{1}{2}$. Show that $\frac{dy}{dx} > 0$.

solution: First we calculate that

Kuvvet Fonksiyonları

Eğer

- f, u'ya bağlı türevlenebilir fonksiyon;
- u, x'e bağlı türevlenebilir fonksiyon ve
- $y = f(u)$ ise,

Zincir Kuralı gereğince $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$ ile

$$\frac{d}{dx}f(u) = f'(u)\frac{du}{dx}$$

ifadesi aynıdır.

Şimdi $n \in \mathbb{R}$ ve $f(u) = u^n$ olsun. O halde $f'(u) = nu^{n-1}$ olur. Böylece

$$\boxed{\frac{d}{dx}\left(u^n\right) = nu^{n-1}\frac{du}{dx}.}$$

Örnek 10.4.

$$\begin{aligned}\frac{d}{dx}\left(5x^3-x^4\right)^7 &= 7\left(5x^3-x^4\right)^6\frac{d}{dx}\left(5x^3-x^4\right)\\ &= 7\left(5x^3-x^4\right)^6\left(15x^-4x^3\right).\end{aligned}$$

Örnek 10.5.

$$\begin{aligned}\frac{d}{dx}\left(\frac{1}{3x-2}\right) &= \frac{d}{dx}(3x-2)^{-1} = -1(3x-2)^{-2}\frac{d}{dx}(3x-2)\\ &= -\left(\frac{1}{(3x-2)^2}\right)(2) = \frac{-3}{(3x-2)^2}.\end{aligned}$$

Örnek 10.6.

$$\frac{d}{dx}\left(\sin^5 x\right) = 5\sin^4 x\frac{d}{dx}(\sin x) = 5\sin^4 x\cos x.$$

Örnek 10.7. $|x|$ fonksiyonunun türevini alınız.

çözüm: $|x| = \sqrt{x^2}$ olduğundan, $x \neq 0$ ise

$$\begin{aligned}\frac{d}{dx}|x| &= \frac{d}{dx}\left(\sqrt{x^2}\right) = \frac{d}{du}\left(\sqrt{u}\right)\frac{d}{dx}\left(x^2\right)\\ &= \frac{1}{2\sqrt{u}}2x = \frac{x}{\sqrt{x^2}} = \frac{x}{|x|}\end{aligned}$$

buluruz.

$$\frac{dy}{dx} = \frac{d}{dx}(1-2x)^{-3} = -3(1-2x)^{-4}\frac{d}{dx}(1-2x)$$
$$= -3(1-2x)^{-4}(-2) = \frac{6}{(1-2x)^4}$$

if $x \neq \frac{1}{2}$. Since $(1-2x)^4 > 0$ if $x \neq \frac{1}{2}$ and $6 > 0$, we have that $\frac{dy}{dx} > 0$ if $x \neq \frac{1}{2}$.

Örnek 10.8. $y = \frac{1}{(1-2x)^3}$ for $x \neq \frac{1}{2}$ olsun. $\frac{dy}{dx} > 0$ olduğunu gösteriniz.

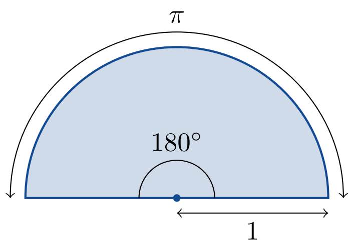
çözüm: Öncelikle, $x \neq \frac{1}{2}$ ise

$$\frac{dy}{dx} = \frac{d}{dx}(1-2x)^{-3} = -3(1-2x)^{-4}\frac{d}{dx}(1-2x)$$
$$= -3(1-2x)^{-4}(-2) = \frac{6}{(1-2x)^4}$$

buluruz. Eğer $x \neq \frac{1}{2}$ ise $(1-2x)^4 > 0$ olur ve $6 > 0$ bulunur, buradan $\frac{dy}{dx} > 0$ if $x \neq \frac{1}{2}$ elde edilir.

Example 10.9 (Why Do We Use Radians in Calculus?)**.** Remember that $\frac{d}{dx}\sin x = \cos x$ is true *only if we use radians*. What happens if we use degrees?

Örnek 10.9 (Kalkülüste Neden Radyan Kullanırız?)**.** Unutmayınız ki $\frac{d}{dx}\sin x = \cos x$ doğrudur *tabii radyan kullanırsak*. Derece kullansaydık ne olurdu?



Remember that

$$180 \text{ degrees} = \pi \text{ radians}$$
$$180^\circ = \pi$$
$$1^\circ = \frac{\pi}{180}$$
$$x^\circ = \frac{\pi x}{180}.$$

So

$$\frac{d}{dx}\sin x^\circ = \frac{d}{dx}\sin\left(\frac{\pi x}{180}\right) = \frac{\pi}{180}\cos\left(\frac{\pi x}{180}\right) = \frac{\pi}{180}\cos x^\circ.$$

Therefore we have

$\frac{d}{dx}\sin x = \cos x$	and	$\frac{d}{dx}\sin x^\circ = \frac{\pi}{180}\cos x^\circ$
a nice formula		not nice

This is why we use radians in Calculus.

Hatırlayacak olursak,

$$180 \text{ derece} = \pi \text{ radyan}$$
$$180^\circ = \pi$$
$$1^\circ = \frac{\pi}{180}$$
$$x^\circ = \frac{\pi x}{180}.$$

Yani

$$\frac{d}{dx}\sin x^\circ = \frac{d}{dx}\sin\left(\frac{\pi x}{180}\right) = \frac{\pi}{180}\cos\left(\frac{\pi x}{180}\right) = \frac{\pi}{180}\cos x^\circ.$$

Elimize geçen

$\frac{d}{dx}\sin x = \cos x$	and	$\frac{d}{dx}\sin x^\circ = \frac{\pi}{180}\cos x^\circ$
güzel bir formül		hiç güzel olmayan formül

Bu yüzden Kalkülüste radyan kullanıyoruz.

Problems

Problem 10.1.

(a). Find $\frac{ds}{dt}$ if $s = \left(\frac{t}{2} - 1\right)^{-10}$.

(b). Find $\frac{dy}{dt}$ if $y = \cos\left(5\sin\left(\frac{t}{3}\right)\right)$.

Sorular

Soru 10.1.

(a). $s = \left(\frac{t}{2} - 1\right)^{-10}$ ise $\frac{ds}{dt}$'yi bulunnuz .

(b). $y = \cos\left(5\sin\left(\frac{t}{3}\right)\right)$ ise $\frac{dy}{dt}$'yi bulunuz.

e^x and $\ln$

e^x ve $\ln$

$$\boxed{\frac{d}{dx}\left(e^x\right) = e^x} \qquad \boxed{\frac{d}{dx}\left(\ln|x|\right) = \frac{1}{x}}$$

Example 11.1.

$$\frac{d}{dx}\left(e^x \sin x\right) = \frac{d}{dx}\left(e^x\right)\sin x + e^x \frac{d}{dx}\left(\sin x\right) = e^x \sin x + e^x \cos x.$$

Example 11.2. Differentiate 2^x.

solution: Remember that $e^{\ln z} = z$. Therefore $2^x = e^{\ln 2^x} = e^{x\ln 2}$. Hence

$$\frac{d}{dx}\left(2^x\right) = \frac{d}{dx}\left(e^{x\ln 2}\right) = (\ln 2)e^{x\ln 2} = (\ln 2)2^x.$$

Example 11.3. Differentiate $y = \log_{10}|x|$.

solution: First note that

$$\begin{aligned} |x| &= 10^y \\ \ln|x| &= \ln 10^y = y\ln 10 \\ \frac{\ln|x|}{\ln 10} &= y. \end{aligned}$$

Therefore

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{\ln|x|}{\ln 10}\right) = \frac{1}{\ln 10}\frac{d}{dx}\left(\ln|x|\right) = \frac{1}{x\ln 10}.$$

Örnek 11.1.

$$\frac{d}{dx}\left(e^x \sin x\right) = \frac{d}{dx}\left(e^x\right)\sin x + e^x \frac{d}{dx}\left(\sin x\right) = e^x \sin x + e^x \cos x.$$

Örnek 11.2. 2^x fonksiyonunun türevini alınız.

çözüm: Hatırlarsak $e^{\ln z} = z$. Yani $2^x = e^{\ln 2^x} = e^{x\ln 2}$. Böylece

$$\frac{d}{dx}\left(2^x\right) = \frac{d}{dx}\left(e^{x\ln 2}\right) = (\ln 2)e^{x\ln 2} = (\ln 2)2^x.$$

Örnek 11.3. $y = \log_{10}|x|$ fonksiyonunun türevini alınız.

çözüm: İlk olarak

$$\begin{aligned} |x| &= 10^y \\ \ln|x| &= \ln 10^y = y\ln 10 \\ \frac{\ln|x|}{\ln 10} &= y. \end{aligned}$$

Buradan

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{\ln|x|}{\ln 10}\right) = \frac{1}{\ln 10}\frac{d}{dx}\left(\ln|x|\right) = \frac{1}{x\ln 10}.$$

Problems

Problem 11.1. Differentiate the following functions.

(a). $y = e^{(x^2)}$

(b). $y = (e^x)^2$

(c). $f(t) = \ln|2t|$

(d). $g(t) = \sin\left(e^{2t}\right)$

(e). $y = 3^x$

(f). $h(z) = e^{3z}\cos(-2z)$

Problem 11.2. Calculate $\frac{d^2}{dx^2}\left(\frac{e^x+e^{-x}}{2}\right)$.

Sorular

Soru 11.1. Aşağıdaki fonksiyonların türevlerini bulunuz.

(a). $y = e^{(x^2)}$

(b). $y = (e^x)^2$

(c). $f(t) = \ln|2t|$

(d). $g(t) = \sin\left(e^{2t}\right)$

(e). $y = 3^x$

(f). $h(z) = e^{3z}\cos(-2z)$

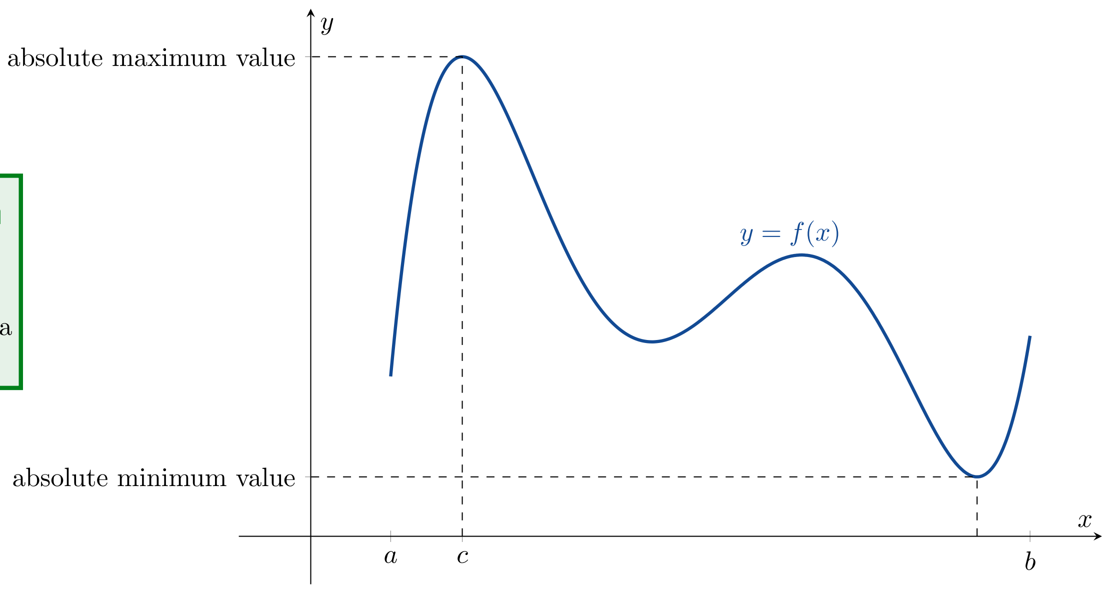
Soru 11.2. $\frac{d^2}{dx^2}\left(\frac{e^x+e^{-x}}{2}\right)$ ifadesini hesaplayınız.

12

Extreme Values of Functions

Fonksiyonların Ekstremum Değerleri

Mini English Lesson	
1 dog	2 dogs
1 man	2 men
1 extremum	2 extrema



Definition. Let $f : D \to \mathbb{R}$ be a function.

- f has an ***absolute maximum value*** on D at a point c if
$$f(x) \leq f(c)$$
for all $x \in D$.

- f has an ***absolute minimum value*** on D at a point c if
$$f(x) \geq f(c)$$
for all $x \in D$.

Maximum and minimum values are called ***extrema***/***extreme values***.

Tanım. $f : D \to \mathbb{R}$ bir fonksiyon olsun.

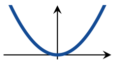
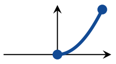
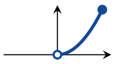
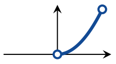
- Eğer her $x \in D$ için
$$f(x) \leq f(c)$$
doğru oluyorsa, f fonksiyonunun D üzerindeki bir c noktasında ***mutlak maksimum değeri*** vardır.

- Eğer her $x \in D$ için
$$f(x) \geq f(c)$$
doğru oluyorsa, f fonksiyonunun D üzerindeki bir c noktasında ***mutlak minimum değeri*** vardır.

Maksimum ve minimum değerlere ***ekstrema***/***extremum değerler*** denir.

Example 12.1.

Örnek 12.1.

function fonksiyon	domain, D tanın kümesi	graph graf	absolute extrema on D	D üzerinde mutlak ekstremum
$y = x^2$	$(-\infty, \infty)$		No absolute maximum. Absolute minimum of 0 at $x = 0$.	Mutlak maksimum yok. $x = 0$'da mutlak minimum 0.
$y = x^2$	$[0, 2]$		Absolute maximum of 4 at $x = 2$. Absolute minimum of 0 at $x = 0$.	$x = 2$'de mutlak maksimum 4. $x = 0$'da mutlak minimum 0.
$y = x^2$	$(0, 2]$		Absolute maximum of 4 at $x = 2$. No absolute minimum.	$x = 2$'de mutlak maksimum 4. Mutlak minimum yok.
$y = x^2$	$(0, 2)$		No absolute extrema.	Mutlak ekstremum yok.

Theorem 12.1. *Suppose that*

- *$f : D \to \mathbb{R}$ is continuous; and*
- *$D = [a, b]$ is a closed interval.*

Then f attains both an absolute maximum value M and an absolute minimum value m in $[a, b]$.

Teorem 12.1. *Varsayalım ki*

- *$f : D \to \mathbb{R}$ sürekli ve*
- *$D = [a, b]$ bir kapalı aralık olsun.*

Bu durumda $[a, b]$ üzerinde f, hem bir M mutlak maksimum hem de bir m mutlak minimum değerlerine sahiptir.

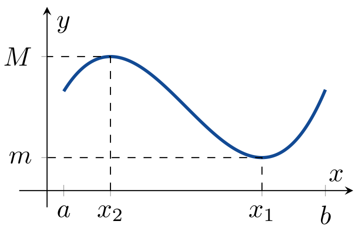
Remark. Theorem 12.1 says that there are numbers $x_1, x_2 \in [a, b]$ such that

- $f(x_1) = m$;
- $f(x_2) = M$; and
- $m \le f(x) \le M$ for all $x \in [a, b]$.

Not. Önceki Teorem 12.1 der ki

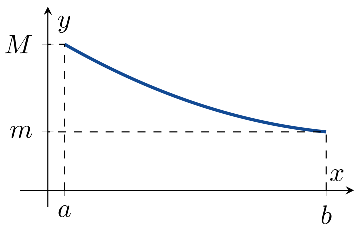
- $f(x_1) = m$;
- $f(x_2) = M$ ve
- her $x \in [a, b]$ için$m \le f(x) \le M$.

olacak şekilde $x_1, x_2 \in [a, b]$ sayıları bulmak mümkündür.



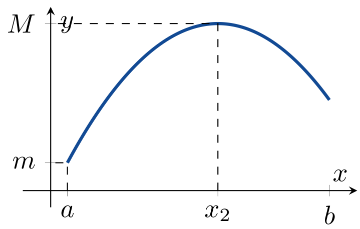
The absolute maximum and absolute minimum are at interior points.

İç noktalarda maksimum ve minimum.



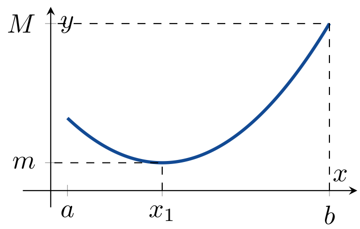
The absolute maximum and absolute minimum are at endpoints.

Uç noktalarda maksimum ve minimum.



The absolute maximum is at an interior point. The absolute minimum is at an endpoint.

İç noktalarda maksimum, uç noktalarda minimum.



The absolute maximum is at an endpoint. The absolute minimum is at an interior point.

Uç noktalarda maksimum, iç noktalarda minimum.

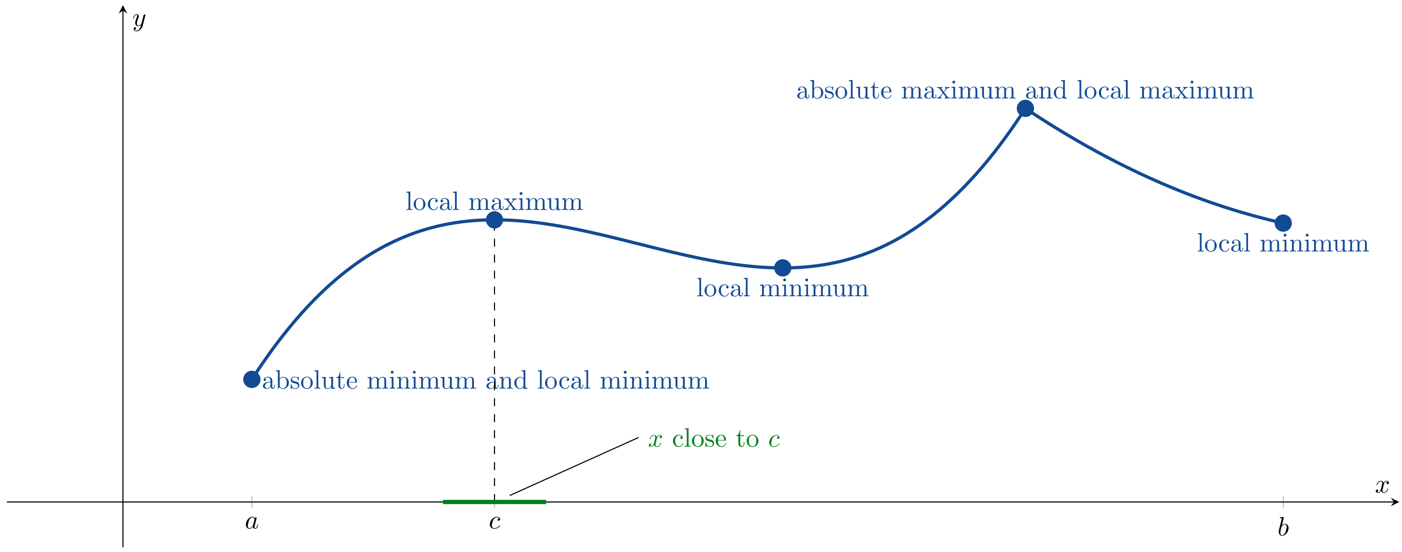


Figure 12.1: Local Extrema.
Şekil 12.1: Yerel Ekstrema.

Local Extreme Values

Definition. Let $f : D \to \mathbb{R}$ be a function.

- f has a ***local maximum value*** at a point $c \in D$ if
$$f(x) \leq f(c)$$
for all x close to c.
- f has a ***local minimum value*** at a point $c \in D$ if
$$f(x) \geq f(c)$$
for all x close to c.

See figure 12.1. An absolute maximum is always a local maximum too. An absolute minimum is alway a local minimum too.

Theorem 12.2 (The First Derivative Test)**.** *Suppose that*

- *f has a local maximum/minimum value at an interior point $c \in D$; and*
- *$f'(c)$ exists.*

Then $f'(c) = 0$.

Remark. The First Derivative Test tells us that the only places where $f : D \to \mathbb{R}$ can have an extreme value are

- interior points where $f'(c) = 0$;
- interior points where $f'(c)$ does not exist; and
- endpoints of D.

Definition. An interior point of D where either

- $f' = 0$; or
- f' does not exist,

is called a ***critical point*** of f.

Yerel Ekstremum Değerler

Tanım. $f : D \to \mathbb{R}$ bir fonksiyon olsun.

- Eğer c'ye çok yakın bütün x'ler için
$$f(x) \leq f(c)$$
oluyorsa, f'nin $c \in D$ noktasında bir ***yerel maksimum değeri*** vardır.
- c'ye çok yakın bütün x'ler için
$$f(x) \geq f(c)$$
oluyorsa, f'nin $c \in D$ noktasında bir ***yerel minimum değeri*** vardır.

Bkz. şekil 12.1. Her mutlak maksimum aynı zamanda bir yerel maksimumdur. Her mutlak minimum da aynı zamanda bir yerel minimumdur.

Teorem 12.2 (Birinci Türev Testi)**.** *Varsayalım ki*

- *f'nin bir $c \in D$ iç noktasında yerel maksimum/minimum değeri olsun ve*
- *$f'(c)$ mevcut.*

Bu durumda $f'(c) = 0$ olur.

Not. Birinci Türev Testi bize bir $f : D \to \mathbb{R}$ fonksiyonunun ekstrem değerlere sahip olabileceği yerlerin şunlardan birisi olduğunu söyler

- $f'(c) = 0$ olduğu iç noktalar;
- $f'(c)$ mevcut olmadığı iç noktalar ve
- D'nin uç noktaları.

Tanım. Eğer D'nin bir iç noktasında

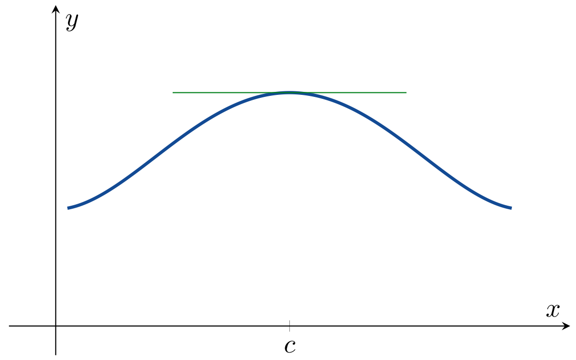


Figure 12.2: The First Derivative Test.
Şekil 12.2: Birinci Türev Testi.

How to find the absolute extrema of a continuous function $f : [a, b] \to \mathbb{R}$

STEP 1. Find the critical points of f.

STEP 2. Calculate $f(x)$ at all of the critical points.

STEP 3. Calculate $f(a)$ and $f(b)$.

STEP 4. Take the largest and smallest values.

Example 12.2. Find the absolute maximum and absolute minimum values of $f(x) = x^2$ on $[-2, 1]$.

solution:

1. We know that $f(x) = x^2$ is differentiable on $[-2, 1]$. So $f'(x)$ exists for all interior points $x \in (-2, 1)$. The only critical point is
$$0 = f'(x) = 2x \implies x = 0.$$
2. $f(0) = 0$.
3. $f(-2) = 4$ and $f(1) = 1$.
4. The largest and smallest numbers in $\{0, 1, 4\}$ are 4 and 0. Therefore the absolute maximum value of $f(x) = x^2$ on $[-2, 1]$ is 4 and the absolute minimum value of f on $[-2, 1]$ is 0. We can write
$$\max_{x\in[-2,1]} x^2 = 4 \quad \text{and} \quad \min_{x\in[-2,1]} x^2 = 0.$$

Example 12.3. Find the absolute maximum and absolute minimum values of $g(t) = 8t - t^4$ on $[-2, 1]$.

solution:

1. $g'(t)$ exists for all $t \in (-2, 1)$. Since
$$0 = g'(t) = 8 - 4t^3 \implies t^3 = \frac{8}{4} = 2 \implies t = \sqrt[3]{2} > 1,$$
g does not have any critical points in $[-2, 1]$.
2.
3. $g(-2) = -32$ and $g(1) = 7$.

- $f' = 0$ veya
- f' mevcut değilse,

o iç nokktaya f'nin bir ***kritik noktası*** denir.

Sürekli bir $f : [a, b] \to \mathbb{R}$ fonksiyonunun mutlak ekstremum değerleri nasıl bulunur

ADIM 1. f'nin kritik noktaları bulunur.

ADIM 2. $f(x)$'in bütün kritik noktalardaki değerleri bulunur.

ADIM 3. $f(a)$ ve $f(b)$ bulunur.

ADIM 4. Bulunan bütün bu değerlerin en büyüğü ve en küçüğü alınır.

Örnek 12.2. $f(x) = x^2$ fonksiyonunun $[-2, 1]$ üzerindeki mutlak maksimum ve mutlak minimum değerlerini bulunuz.

çözüm:

1. $f(x) = x^2$'nin $[-2, 1]$'de türevlenebilirdir. Yani $f'(x)$ her $x \in (-2, 1)$ iç noktası için mevcuttur. Tek kritik nokta
$$0 = f'(x) = 2x \implies x = 0.$$
2. $f(0) = 0$.
3. $f(-2) = 4$ ve $f(1) = 1$.
4. $\{0, 1, 4\}$ kümesindeki en büyük ve en küçük sayılar 4 ve 0. Böylece $f(x) = x^2$'nin $[-2, 1]$ boyunca mutlak maksimum değeri 4 ve f'nin $[-2, 1]$ boyunca mutlak minimum değeri de 0 olur. Bunu
$$\text{maks}_{x\in[-2,1]} x^2 = 4 \quad \text{ve} \quad \min_{x\in[-2,1]} x^2 = 0$$

olarak yazabiliriz.

Örnek 12.3. $g(t) = 8t - t^4$'nin $[-2, 1]$ boyunca mutlak maksimum ve mutlak minimum değerlerini bulunuz.

çözüm:

1. Her $t \in (-2, 1)$ için $g'(t)$ mevcuttur .
$$0 = g'(t) = 8 - 4t^3 \implies t^3 = \frac{8}{4} = 2 \implies t = \sqrt[3]{2} > 1$$
olduğundan, g'nin $[-2, 1]$'de hiç kritik noktası yoktur.
2.
3. $g(-2) = -32$ ve $g(1) = 7$.
4. Böylece
$$\text{maks}_{t\in[-2,1]} g(t) = 7 \quad \text{and} \quad \min_{t\in[-2,1]} g(t) = -32.$$
Bkz. şekil 12.3.

4. Therefore

$$\max_{t\in[-2,1]} g(t) = 7 \qquad \text{and} \qquad \min_{t\in[-2,1]} g(t) = -32.$$

See figure 12.3.

Example 12.4. Find the absolute maximum and absolute minimum values of $h(x) = x^{\frac{2}{3}}$ on $[-2, 3]$.

solution:

1. We calculate that

$$h'(x) = \frac{d}{dx}\left(x^{\frac{2}{3}}\right) = \frac{2}{3}x^{-\frac{1}{3}} = \frac{2}{3\sqrt[3]{x}}.$$

Hence h' does not exist if $x = 0$. We can also see that $h'(x) \neq 0$ if $x \in [-2, 0)$ or $x \in (0, 3]$. The only critical point is $x = 0$

2. $h(0) = 0$.

3. $h(-2) = (-2)^{\frac{2}{3}} = (4)^{\frac{1}{3}} = \sqrt[3]{4}$ and $h(3) = (3)^{\frac{2}{3}} = (9)^{\frac{1}{3}} = \sqrt[3]{9}$.

4. Therefore

$$\max_{x\in[-2,3]} h(x) = \sqrt[3]{9} \approx 2.08 \qquad \text{and} \qquad \min_{x\in[-2,3]} h(t) = 0.$$

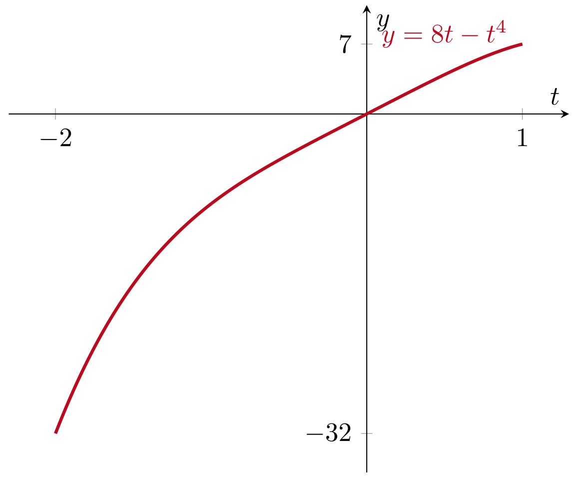


Figure 12.3: The graph of $g : [-2, 1] \to \mathbb{R}$, $g(t) = 8t - t^4$.
Şekil 12.3: $g : [-2, 1] \to \mathbb{R}$, $g(t) = 8t - t^4$ nin grafiği.

Örnek 12.4. $h(x) = x^{\frac{2}{3}}$'nin $[-2, 3]$ üzerindeki mutlak maksimum ve mutlak minimum değerlerini bulunuz.

çözüm:

1. Şöyle başlarsak

$$h'(x) = \frac{d}{dx}\left(x^{\frac{2}{3}}\right) = \frac{2}{3}x^{-\frac{1}{3}} = \frac{2}{3\sqrt[3]{x}}.$$

Dolayısıyla $x = 0$ ise h' mecut değildir. Ayrıca, $x \in [-2, 0)$ veya $x \in (0, 3]$ ise $h'(x) \neq 0$ olur. Tek kritik nokta $x = 0$'dır.

2. $h(0) = 0$.

3. $h(-2) = (-2)^{\frac{2}{3}} = (4)^{\frac{1}{3}} = \sqrt[3]{4}$ ve $h(3) = (3)^{\frac{2}{3}} = (9)^{\frac{1}{3}} = \sqrt[3]{9}$.

4. Bu nedenle

$$\text{maks}_{x\in[-2,3]} h(x) = \sqrt[3]{9} \approx 2.08 \qquad \text{and} \qquad \min_{x\in[-2,3]} h(t) = 0$$

bulunmuş olur.

Increasing and Decreasing Functions

Theorem 12.3. *Suppose that*

- *$f : [a, b] \to \mathbb{R}$ is continuous; and*
- *f is differentiable on (a, b).*

(i). If $f'(x) > 0$ for all $x \in (a, b)$, then f is increasing on $[a, b]$.

(ii). If $f'(x) < 0$ for all $x \in (a, b)$, then f is decreasing on $[a, b]$.

Example 12.5. Let $f(x) = x^3 - 12x - 5$.

(a). Find the critical points of f.

(b). Identify the intervals where f is increasing and the inter-

Artan ve Azalan Fonksiyonlar

Teorem 12.3. *Varsayalım ki*

- *$f : [a, b] \to \mathbb{R}$ sürekli ve*
- *f, (a, b)'de türevlenebilir .*

(i). Her $x \in (a, b)$ için, $f'(x) > 0$ ise o halde f fonksiyonu $[a, b]$'de artandır.

(ii). Her $x \in (a, b)$ için, $f'(x) < 0$ ise o halde f fonksiyonu $[a, b]$'de azalandır.

Örnek 12.5. Let $f(x) = x^3 - 12x - 5$.

(a). f'nin kritik noktalarını bulunuz .

vals where f is decreasing.

(b). f'nin arttığı aralıkları ve azaldığı aralıkları bulunuz.

solution: Clearly f is continuous and differentiable everywhere.

$$0 = f'(x) = 3x^2 - 12 = 3(x^2 - 4) = 3(x - 2)(x + 2) \implies x = -2 \text{ or } 2.$$

The critical points are $x = -2$ and $x = 2$. These critical points cut $(-\infty, \infty)$ into 3 open intervals: $(-\infty, -2)$, $(-2, 2)$ and $(2, \infty)$.

çözüm: Açıkçası f her yerde sürekli ve türevlenebilir.

$$0 = f'(x) = 3x^2 - 12 = 3(x^2 - 4) = 3(x - 2)(x + 2) \implies x = -2 \text{ or } 2.$$

Kritik noktalar $x = -2$ ve $x = 2$'dir. Bu noktalar $(-\infty, \infty)$ aralığını 3 açık aralığa ayırır: $(-\infty, -2)$, $(-2, 2)$ ve $(2, \infty)$.

Interval	$(-\infty, -2)$	$(-2, 2)$	$(2, \infty)$	Aralıklar
Calculate $f'(x_0)$ at one point	$f'(-3) = 15$	$f'(0) = -12$	$f'(3) = 15$	Hesaplanmış f'
f' is	> 0	< 0	> 0	f''nin işareti
f is	increasing artan	decreasing azalan	increasing artan	f'nin davranışı

Therefore f is increasing on $(-\infty, -2)$ and on $(2, \infty)$, and f is decreasing on $(-2, 2)$.

Dolayısıyla f, $(-\infty, -2)$ ve $(2, \infty)$ üzerinde artmakta ve f, $(-2, 2)$'de azalmaktadır.

The First Derivative Test For Local Extrema

Theorem 12.4. *Suppose that*

- $f : [a, b] \to \mathbb{R}$ *is continuous;*
- c *is a critical point of* f*; and*
- f *is differentiable on both* $(c - \delta, c)$ *and* $c, c + \delta)$ *for some* $\delta > 0$.

on the left of c	*on the right of c*	*at c*
$f' < 0$	$f' > 0$	*f has a local minimum*
$f' > 0$	$f' < 0$	*f has a local maximum*
$f' > 0$	$f' > 0$	*f does not have a local extremem*
$f' < 0$	$f' < 0$	*f does not have a local extremem*

Yerel Ekstrema İçin Birinci Türev Testi

Teorem 12.4. *Varsayalım ki*

- $f : [a, b] \to \mathbb{R}$ *sürekli;*
- c*,* f*'nin bir kritik noktası; ve*
- f*,* $(c-\delta, c)$ *ve* $(c, c+\delta)$*'nin her ikisinde türevli,* $\delta > 0$ *olsun.*

c'nin solunda	*c'nin sağında*	*c'de*
$f' < 0$	$f' > 0$	*f'nin bir yerel minimumu var*
$f' > 0$	$f' < 0$	*f'nin bir yerel maksimumu var*
$f' > 0$	$f' > 0$	*f'nin bir yerel ekstremumu yokm*
$f' < 0$	$f' < 0$	*f'nin bir yerel ekstremumu yok*

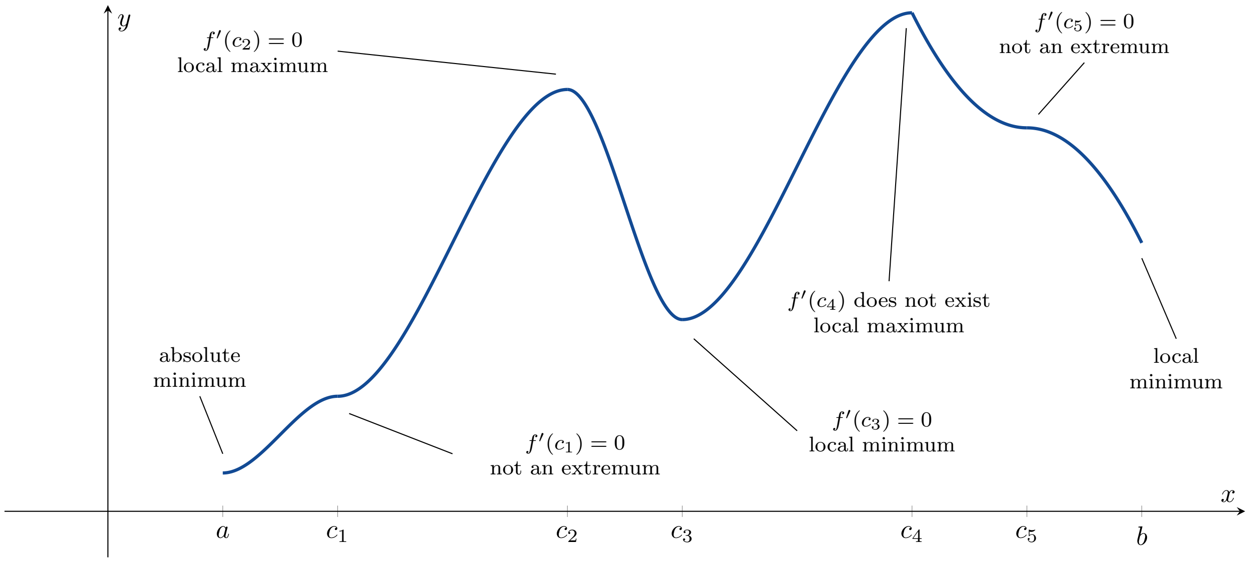


Figure 12.4: Local Extrema
Şekil 12.4:

Example 12.6. Let $f(x) = x^{\frac{1}{3}}(x-4) = x^{\frac{4}{3}} - 4x^{\frac{1}{3}}$.

(a). Find the critical points of f.

(b). Identify the intervals on which f is increasing/decreasing.

(c). Find the extreme values of f.

solution: f is continuous everywhere because $x^{\frac{1}{3}}$ and $(x-4)$ are continuous functions. We can calculate that

$$\begin{aligned} f'(x) &= \frac{d}{dx}\left(x^{\frac{4}{3}} - 4x^{\frac{1}{3}}\right) = \frac{4}{3}x^{\frac{1}{3}} - \frac{4}{3}x^{-\frac{2}{3}} \\ &= \frac{4}{3}x^{-\frac{2}{3}}(x-1) = \frac{4(x-1)}{3x^{\frac{2}{3}}}. \end{aligned}$$

$f'(x)$ does not exist if $x = 0$. $f'(x) = 0$ if and only if $x = 1$. The critical points of f are $x = 0$ and $x = 1$.

Using the critical points, we "cut" $(-\infty, \infty)$ into three subintervals: $(-\infty, 0)$, $(0, 1)$ and $(1, \infty)$.

$(-\infty, 0)$	$(0, 1)$	$(1, \infty)$
$f' < 0$	$f' < 0$	$f' > 0$
(e.g. $f'(-1) = -\frac{8}{3}$)		
f is decreasing	f is decreasing	f is increasing

We can see from this table that $x = 1$ is a local minimum and $x = 0$ is not an extremum. So

$$\min_{x\in\mathbb{R}} f(x) = f(1) = 1^{\frac{1}{3}}(1-4) = -3.$$

Note that f does not have an absolute maximum.

Note that $\lim_{x\to 0} f'(x) = -\infty$. Therefore the graph of $y = f(x)$ has a vertical tangent at $x = 0$. See figure 12.5 on page 65.

Örnek 12.6. $f(x) = x^{\frac{1}{3}}(x-4) = x^{\frac{4}{3}} - 4x^{\frac{1}{3}}$ olsun.

(a). f'nin kritik noktalarını bulunuz.

(b). f'nin artan/azalan olduğu aralıkları bulunuz.

(c). f'nin ekstremum değerlerini bulunuz.

çözüm: $x^{\frac{1}{3}}$ ve $(x-4)$ sürekli olduklarından, f de süreklidir. Hesaplayacak olursak

$$\begin{aligned} f'(x) &= \frac{d}{dx}\left(x^{\frac{4}{3}} - 4x^{\frac{1}{3}}\right) = \frac{4}{3}x^{\frac{1}{3}} - \frac{4}{3}x^{-\frac{2}{3}} \\ &= \frac{4}{3}x^{-\frac{2}{3}}(x-1) = \frac{4(x-1)}{3x^{\frac{2}{3}}}. \end{aligned}$$

$f'(x)$ mevcut değildir ancak ve ancak $x = 0$. $f'(x) = 0$ ancak ve ancak $x = 1$. Yani f'nin kritik noktaları $x = 0$ ve $x = 1$ olur.

Kritik noktalar kullanaraka, we "cut" $(-\infty, \infty)$'u şu üç alt aralığa ayırırız: $(-\infty, 0)$, $(0, 1)$ ve $(1, \infty)$.

$(-\infty, 0)$	$(0, 1)$	$(1, \infty)$
$f' < 0$	$f' < 0$	$f' > 0$
(e.g. $f'(-1) = -\frac{8}{3}$)		
f azalan	f azalan	f artan

Bu tablodan götülüyor ki $x = 1$ yerel minimum ve $x = 0$ ekstremum. So

$$\min_{x\in\mathbb{R}} f(x) = f(1) = 1^{\frac{1}{3}}(1-4) = -3.$$

Aynı zamanda f'nin hiç mutlak maksimumu yok.

$\lim_{x\to 0} f'(x) = -\infty$ olduğuna dikkat ediniz. Böylece $y = f(x)$ grafiğinin bir düşey teğeti $x = 0$ vardır. Bkz. şekil 12.5 sayfa 65.

Problems

Problem 12.1. Consider $g : [0, \frac{3}{2}] \to \mathbb{R}$, $g(x) = \sqrt{2x - x^2}$.

(a). Find all the critical points of g.

(b). Find the absolute maximum value and absolute minimum value of g on $[0, \frac{3}{2}]$ **and** state where they occur.

Sorular

Soru 12.1. $g : [0, \frac{3}{2}] \to \mathbb{R}$, $g(x) = \sqrt{2x - x^2}$ olsun.

(a). g'nin tüm kritik noktalarını bulunuz.

(b). g'nin $[0, \frac{3}{2}]$ üzerindeki mutlak maksimum ve minimumlarını bulunuz **ve** hangi noktalarda olduğunu belirtiniz.

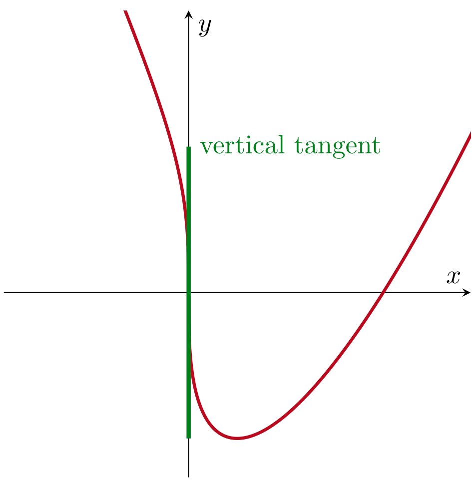
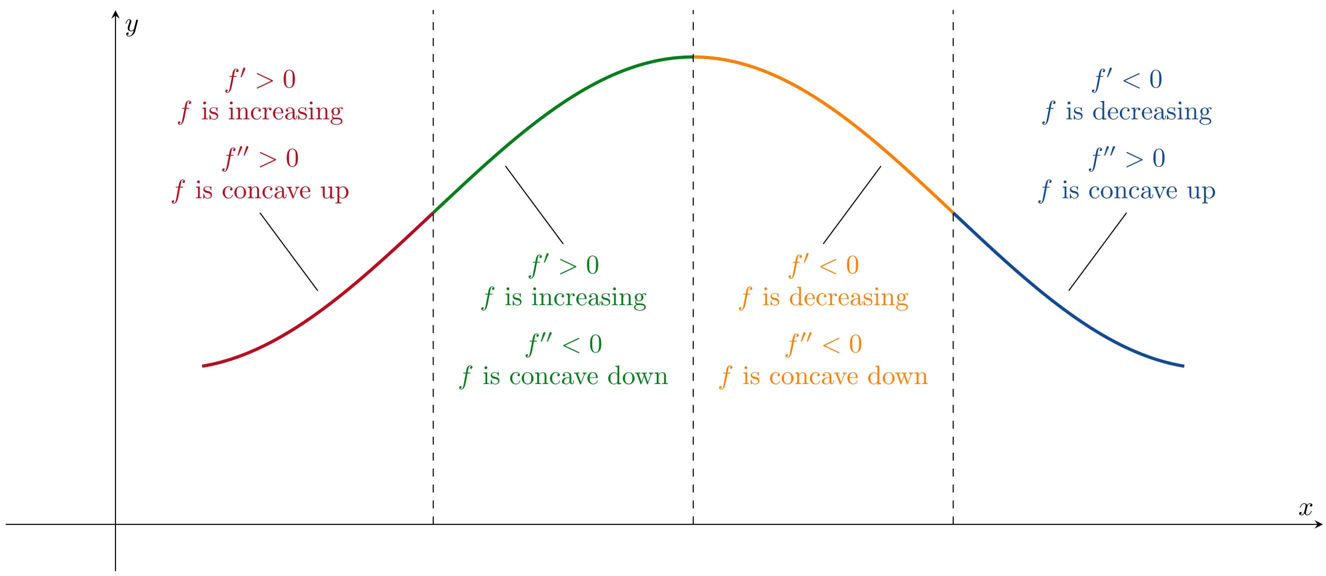


Figure 12.5: $y = x^{\frac{1}{3}}(x - 4) = x^{\frac{4}{3}} - 4x^{\frac{1}{3}}$ nin grafiği.
Şekil 12.5:

13

Concavity and Curve Sketching

Konkavlık ve Eğri Çizimi



Definition. $y = f(x)$ is

(i). ***concave up*** if f' is increasing; and

(ii). ***concave down*** if f' is decreasing.

Theorem 13.1 (The Second Derivative Test for Concavity)**.** *Suppose that $f : I \to \mathbb{R}$ is twice differentiable.*

(i). If $f'' > 0$ on I, then f is concave up on I.

(ii). If $f'' < 0$ on I, then f is concave down on I.

Example 13.1. Consider $y = x^3$. Then $y' = 3x^2$ and $y'' = 6x$.

$(-\infty, 0)$	$(0, \infty)$
$y'' < 0$	$y'' > 0$
$y = x^3$ is concave down	$y = x^3$ is concave up

Example 13.2. Consider $y = x^2$. Since $y' = 2x$ and $y'' = 2$, we have that $y'' > 0$ everywhere. Therefore $y = x^2$ is concave up everywhere.

Tanım. $y = f(x)$ grafiği

(i). eğer f' artansa ***yukarı konkav*** ; ve

(ii). eğer f' azalansa ***aşağı konkav*** denir.

Teorem 13.1 (Konkavlık İçin İkinci Türev Testi)**.** *$f : I \to \mathbb{R}$ iki kere türevli olsun.*

(i). Eğer I üzerinde $f'' > 0$ ise , I üzerinde f yukarı konkavdır .

(ii). Eğer I üzerinde $f'' < 0$ ise , I üzerinde f aşağı konkavdır .

Örnek 13.1. $y = x^3$ olsun. O zaman $y' = 3x^2$ ve $y'' = 6x$ olur.

$(-\infty, 0)$	$(0, \infty)$
$y'' < 0$	$y'' > 0$
$y = x^3$ aşağı konkav	$y = x^3$ yukarı konkav

Example 13.3. Determine the concavity of $y = 3 + \sin x$ on $[0, 2\pi]$.

solution: First we calculate that $y' = \cos x$ and $y'' = -\sin x$.

$(0, \pi)$	$(\pi, 2\pi)$
$y'' < 0$	$y'' > 0$
$y = 3 + \sin x$ is concave down	$y = 3 + \sin x$ is concave up

Graphs of $y = 3 + \sin x$ and $y'' = -\sin x$ are shown in figure 13.1.

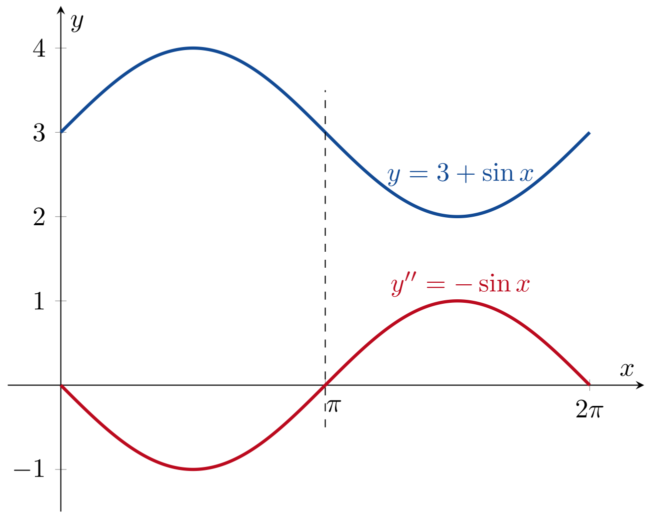


Figure 13.1: Graphs of $y = 3 + \sin x$ and $y'' = -\sin x$.
Şekil 13.1:

Definition. $(c, f(c))$ is a ***point of inflection*** of $y = f(x)$ if

- $y = f(x)$ has a tangent line at $x = c$; and
- the concavity of $y = f(x)$ changes at $x = c$.

Remark. If $(c, f(c))$ is a point of inflection, then either

- $f''(c) = 0$; or
- $f''(c)$ does not exist.

Example 13.4. Let $f(x) = x^{\frac{5}{3}}$. Then $f'(x) = \frac{5}{3}x^{\frac{2}{3}}$ and

$$f''(x) = \frac{d}{dx}\left(\frac{5}{3}x^{\frac{2}{3}}\right) = \frac{10}{9}x^{-\frac{1}{3}} = \frac{10}{9\sqrt[3]{x}}.$$

We can say that

- if $x < 0$, then $f''(x) < 0$;
- $f''(0)$ does not exist; and
- if $x > 0$, then $f''(x) > 0$.

Therefore $(0, 0)$ is an point of inflection of $y = x^{\frac{5}{3}}$. See figure 13.2.

Örnek 13.2. $y = x^2$ alalım. $y' = 2x$ ve $y'' = 2$ olduğu için, her noktada $y'' > 0$ olur. Bunun için $y = x^2$ her noktada yukarı konkavdır.

Örnek 13.3. $y = 3 + \sin x$ fonksiyonunun $[0, 2\pi]$ üzerinde konkavlığını belirleyiniz.

çözüm: İlk olarak $y' = \cos x$ ve $y'' = -\sin x$ olur.

$(0, \pi)$	$(\pi, 2\pi)$
$y'' < 0$	$y'' > 0$
$y = 3 + \sin x$ grafiği aşağı konkav	$y = 3 + \sin x$ gragiği yukarı konkav

$y = 3+\sin x$ ve $y'' = -\sin x$ grafikleri şekil 13.1 de görülmektedir.

Tanım. $(c, f(c))$ noktası $y = f(x)$'nin ***büküm noktasıdır*** eğer

- $y = f(x)$ grafiği $x = c$'de teğeti mevcutsa; ve
- $y = f(x)$'nin konkavlığı $x = c$'de değişiyorsa.

Not. $(c, f(c))$ bir büküm noktasıysa, ya

- $f''(c) = 0$ dır; veya
- $f''(c)$ mevcut değildir.

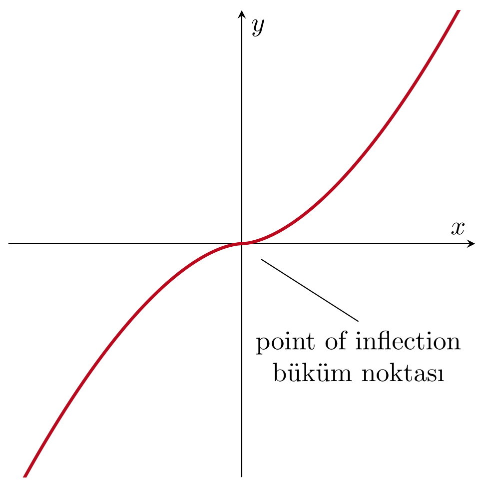


Figure 13.2: The graph of $y = x^{\frac{5}{3}}$.
Şekil 13.2: grafiği

Örnek 13.4. $f(x) = x^{\frac{5}{3}}$ olsun. O zaman $f'(x) = \frac{5}{3}x^{\frac{2}{3}}$ ve

$$f''(x) = \frac{d}{dx}\left(\frac{5}{3}x^{\frac{2}{3}}\right) = \frac{10}{9}x^{-\frac{1}{3}} = \frac{10}{9\sqrt[3]{x}}.$$

Şunu söylemek mümkün:

- $x < 0$ ise, $f''(x) < 0$;
- $f''(0)$ mevcut değil; ve
- $x > 0$ ise, bu durumda $f''(x) > 0$.

Bu sebeple $(0, 0)$ noktası $y = x^{\frac{5}{3}}$ grafiğinin bir büküm noktasıdır . Bkz. şekil 13.2.

Example 13.5. Let $y = x^4$. Then $y' = 4x^3$ and $y = 12x^2$. See figure 13.3. Note that $y'' = 0$ at $x = 0$, but the concavity of the graph does not change. Hence $(0, 0)$ is not a point of inflection of $y = x^4$.

Örnek 13.5. $y = x^4$ olsun. O halde $y' = 4x^3$ ve $y = 12x^2$ olur. Bkz. şekil 13.3. Dikkat edilirse $y'' = 0$ olduğu nokta $x = 0$, ama konkavlık değişmemekte. Bu debeple $(0, 0)$ noktası $y = x^4$'ün bir büküm noktası değildir.

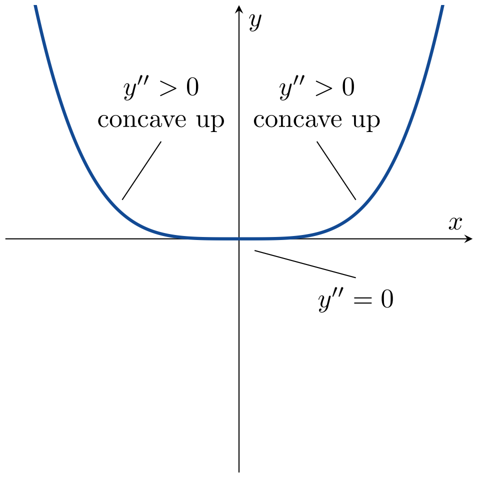


Figure 13.3: The graph of $y = x^4$.
Şekil 13.3:

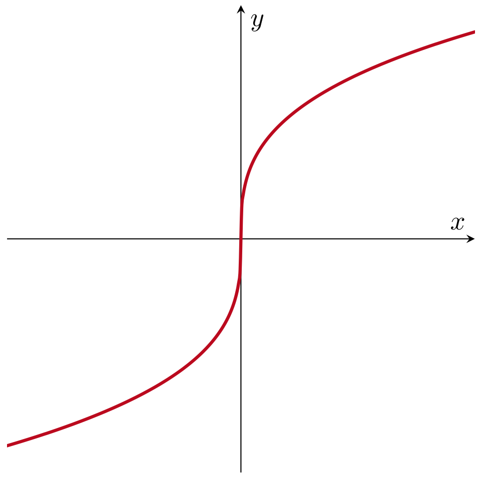


Figure 13.4: The graph of $y = x^{\frac{1}{3}}$.
Şekil 13.4:

Example 13.6. Let $y = x^{\frac{1}{3}}$. Then $y' = \frac{1}{3}x^{-\frac{2}{3}}$ and $y'' = -\frac{2}{9}x^{-\frac{5}{3}}$. Note that y'' does not exist at $x = 0$.

$(-\infty, 0)$	$(0, \infty)$
$y'' > 0$	$y'' < 0$
$y = x^{\frac{1}{3}}$ is concave up	$y = x^{\frac{1}{3}}$ is concave down

$(0, 0)$ is a point of inflection of $y = x^{\frac{1}{3}}$. See figure 13.4.

Örnek 13.6. $y = x^{\frac{1}{3}}$ alalım. Buradan $y' = \frac{1}{3}x^{-\frac{2}{3}}$ ve $y'' = -\frac{2}{9}x^{-\frac{5}{3}}$ olur. Dikkat edilirse $x = 0$'da y'' mevcut değil.

$(-\infty, 0)$	$(0, \infty)$
$y'' > 0$	$y'' < 0$
$y = x^{\frac{1}{3}}$ is yukarı konkav	$y = x^{\frac{1}{3}}$ is aşağı konkav

$(0, 0)$ noktası $y = x^{\frac{1}{3}}$'ün büküm noktası olur. Bkz şekil 13.4.

Theorem 13.2 (The Second Derivative Test for Local Extrema)**.** *Suppose that*

- *f'' is continuous on (a, b); and*
- *$c \in (a, b)$.*

(*i*). *If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at $x = c$.*

(*ii*). *If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at $x = c$.*

(*iii*). *If $f'(c) = 0$ and $f''(c) = 0$, then we don't know – we need to use a different theorem.*

Teorem 13.2 (Yerel Ekstrema için İkinci Türev Testi)**.** *Varsayalım ki*

- *f'' fonksiyonu (a, b)'de sürekli ; ve*
- *$c \in (a, b)$.*

(*i*). *$f'(c) = 0$ ve $f''(c) < 0$ ise, bu durumda f'nin $x = c$ noktasında bir yerel maksimumu vardır.*

(*ii*). *$f'(c) = 0$ ve $f''(c) > 0$ ise, f'nin $x = c$ de bir yerel minimumu vardır.*

(*iii*). *$f'(c) = 0$ ve $f''(c) = 0$ ise, bu test yetersiz kalır – başka bir teorem kullanmamız gerekir.*

Example 13.7. Let $f(x) = x^4 - 4x^3 + 10$.

(a). Find where the local extrema are.

(b). Find the intervals where f is increasing/decreasing.

(c). Find the intervals where f is concave up/concave down.

(d). Sketch the general shape of $y = f(x)$.

(e). Plot some points which satisfy $y = f(x)$.

(f). Graph $y = f(x)$.

solution: f is continuous because it is a polynomial. The domain of f is $(-\infty, \infty)$. Clearly $f'(x) = 4x^3 - 12x^2$. The domain of f' is also $(-\infty, \infty)$. To find the critical points, we need to solve $f'(x) = 0$.

$$0 = f'(x) = 4x^3 - 12x^2 = 4x^2(x-3) \implies x = 0 \text{ or } x = 3.$$

Interval	$(-\infty, 0)$	$(0, 3)$	$(3, \infty)$
f' is	$f' < 0$	$f' < 0$	$f' > 0$
f is	decreasing	decreasing	increasing

(a). By the First Derivative Test, $x = 3$ is a local minimum and $x = 0$ is not an extrema.

(b). f is decreasing on $(-\infty, 0]$ and on $[0, 3]$. f is increasing on $[3, \infty)$.

(c). Next we need to solve $f''(x) = 0$.

$$0 = f''(x) = 12x^2 - 24x \implies x = 0 \text{ or } x = 2.$$

Interval	$(-\infty, 0)$	$(0, 2)$	$(2, \infty)$
f'' is	$f'' > 0$	$f'' < 0$	$f'' > 0$
f is	concave up	concave down	concave up

f is concave up on $(-\infty, 0)$ and on $(2, \infty)$. f is concave down on $(0, 2)$.

(d). Putting the previous two tables together, we obtain

$(-\infty, 0)$	$(0, 2)$	$(2, 3)$	$(3, \infty)$
decreasing	decreasing	decreasing	increasing
concave up	concave down	concave up	concave up
			

Therefore the general shape of f is

Örnek 13.7. $f(x) = x^4 - 4x^3 + 10$ olsun.

(a). Yerel ekstremum noktaların olduğu noktaları bulunuz.

(b). f'nin arttığı/azaldığı aralıkları bulunuz.

(c). f'nin yukarı konkav/aşağı konkav olduğu aralıkları bulunuz.

(d). $y = f(x)$ grafiğini kabaca çiziniz.

(e). $y = f(x)$ üzerindeki bazı noktaları işaretleyiniz.

(f). $y = f(x)$'in bütün önemli noktaları göstererek grafiğini çiziniz.

çözüm: f polinom olduğundan süreklidir. f'nin tanım kümesi $(-\infty, \infty)$ dur. Aşikar ki $f'(x) = 4x^3 - 12x^2$. f''nün tanım kümesi $(-\infty, \infty)$ dur. Kritik noktaları bulmak için, $f'(x) = 0$ denklemini çözeriz.

$$0 = f'(x) = 4x^3 - 12x^2 = 4x^2(x-3) \implies x = 0 \text{ or } x = 3.$$

Aralık	$(-\infty, 0)$	$(0, 3)$	$(3, \infty)$
f''nün durumu	$f' < 0$	$f' < 0$	$f' > 0$
f is	azalan	azalan	artan

(a). Birinci Türev Testinden, $x = 3$ bir yerel minimum ve $x = 0$ bir ekstremum değildir.

(b). f, $(-\infty, 0]$ ve $[0, 3]$'de azalan . f, $[3, \infty)$'da artandır .

(c). Daha sonra $f''(x) = 0$ denklemi çözülür.

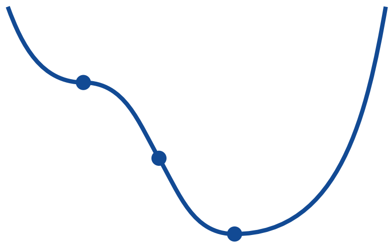
$$0 = f''(x) = 12x^2 - 24x \implies x = 0 \text{ veya } x = 2.$$

Aralık	$(-\infty, 0)$	$(0, 2)$	$(2, \infty)$
f'' is	$f'' > 0$	$f'' < 0$	$f'' > 0$
f	yukarı konkav	aşağı konkav	yukarı konkav

f, $(-\infty, 0)$ ve $(2, \infty)$ üzerinde yukarı konkav. f, $(0, 2)$ üzerinde aşağı konkav.

(d). Önceki iki tabloyu bir araya getirdiğimizde, şu elde edilir

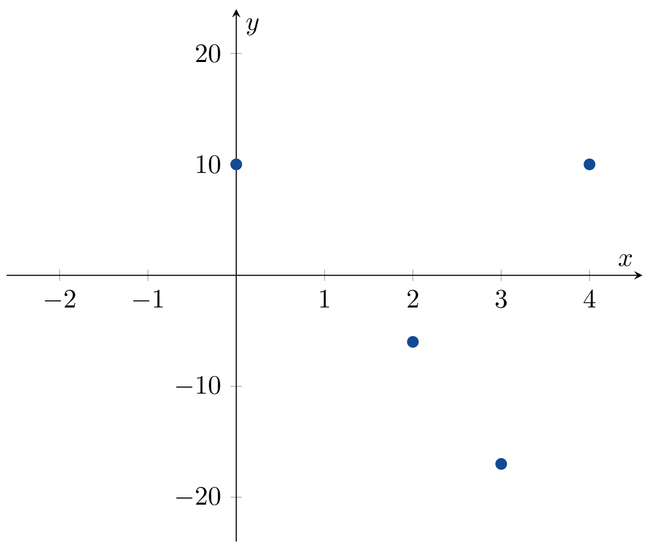
$(-\infty, 0)$	$(0, 2)$	$(2, 3)$	$(3, \infty)$
azalan	azalan	azalan	artan
yukarı konkav	aşağı konkav	yukarı konkav	yukarı konkav
			



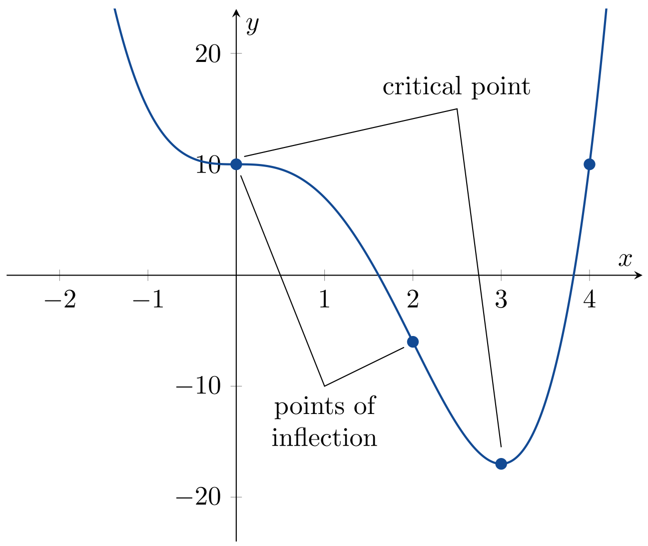
(e). We calculate some (x, y) points.

x	y
0	10
2	-6
3	-17
4	10

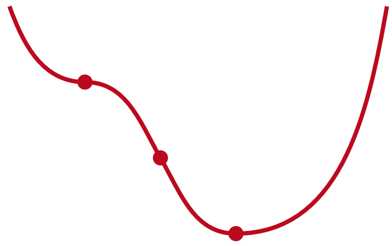
Then we plot these points.



(f). Finally, we have enough information to be able to graph $y = x^4 - 4x^3 + 10$.



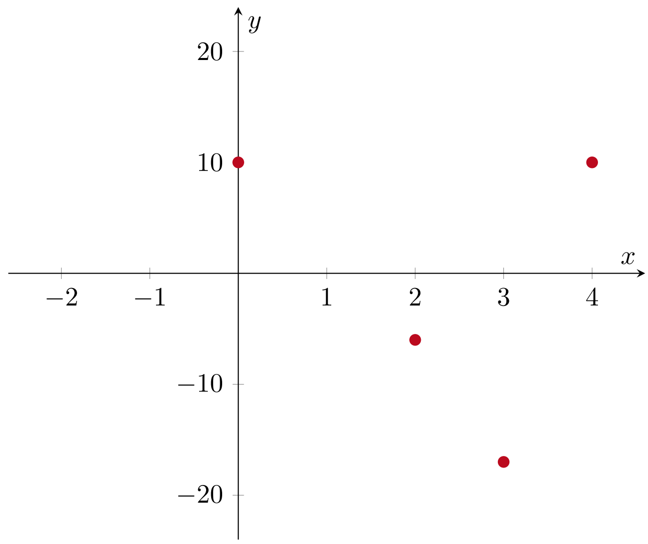
Dolayısıyla f'nin genel şekli şöyle olur.



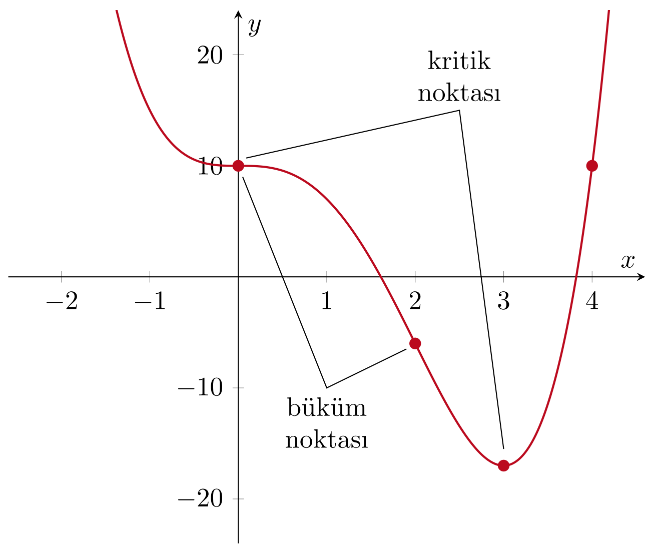
(e). (x, y) noktalarından bazılarını bulacak olursak,

x	y
0	10
2	-6
3	-17
4	10

Bulduğumuz bu noktaları düzlemde yerleştirirsek,



(f). Nihayet grafiği çizebilecek yeterli bilgiye artık sahibiz $y = x^4 - 4x^3 + 10$.



Problems

Problem 13.1. Let $f(x) = x^3(x+2)$. Note that $f'(x) = 2x^2(2x+3)$ and $f''(x) = 12x(x+1)$.

(a). Find all the critical points (if any) of $y = f(x)$.

(b). Find all the points of inflection (if any) of $y = f(x)$.

(c). Calculate $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$

(d). Find the intervals where f is increasing/decreasing.

(e). Find the intervals where f is concave up/down.

(f). Draw the graph of $y = f(x)$ (without using a computer/a calculator/a phone/the internet/etc.).

Sorular

Soru 13.1. $f(x) = x^3(x+2)$ olsun. dikkat edilirse $f'(x) = 2x^2(2x+3)$ ve $f''(x) = 12x(x+1)$ bulunabilir.

(a). $y = f(x)$'nin tüm kritik noktalarını (varsa) bulunuz.

(b). $y = f(x)$'in (varsa) tüm büküm noktalarını bulunuz.

(c). $\lim_{x\to\infty} f(x)$ ve $\lim_{x\to-\infty} f(x)$ limitlerini bulunuz

(d). f'nin arttığı/azalgığı aralıkları bulunuz.

(e). f'nin yukarı/aşağı konkav olduğu aralıkları bulunuz.down.

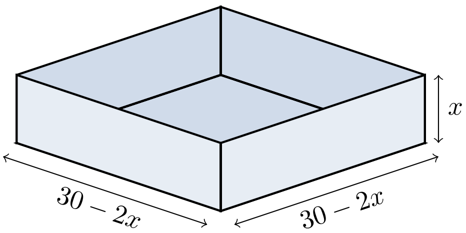
(f). $y = f(x)$'nin gragiğini çiziniz (bilgisayar/hesap makinesi/akıllı telefon/internet/vb. kullanmadan).

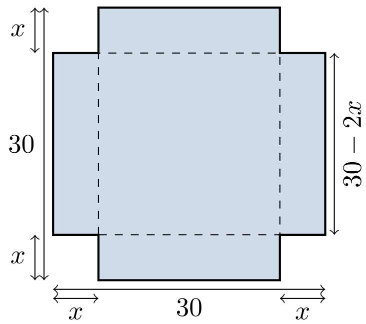
14 Applied Optimisation

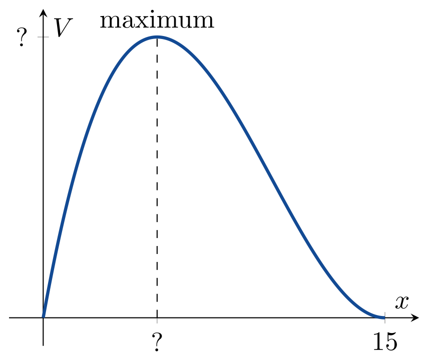
Uygulamalı Optimizasyon Problemleri

Example 14.1. An open-top box is to be made by cutting $x\,\text{cm} \times x\,\text{cm}$ squares from the corners of a $30\,\text{cm} \times 30\,\text{cm}$ piece of metal and bending the sides up. How large should the squares cut from the corners be to make the box hold as much as possible?

Örnek 14.1. Üstü açık bir kutu $30\,\text{cm} \times 30\,\text{cm}$ lik bir teneke levhanın köşelerinden kareler kesilip, kıvrılarak yapılacaktır. Kutunun mümkün olduğunca büyük hacimli olması için köşelerden kesilen $x\,\text{cm} \times x\,\text{cm}$ kareler ne büyüklükte olmalıdır?







solution: The volume of the box will be

$$V(x) = x(30-2x)^2.$$

Note that the domain of V is $[0, 15]$. We expect the graph of V to look like the graph above with a maximum somewhere in the middle.

We calculate that

$$\begin{aligned} 0 = \frac{dV}{dx} &= (x)'(30-2x)^2 + (x)\big((30-2x)^2\big)' \\ &= (1)(30-2x)^2 + (x)2(30-2x)(-2) \\ &= (30-2x)\big((30-2x)-4x\big) \\ &= (30-2x)(30-6x) = 12(15-x)(5-x). \end{aligned}$$

Therefore $x = 5$ or $x = 15$. Since $15 \notin (0, 15)$, the only critical point of V is $x = 5$. To make the largest possible box, we should choose $x = 5$. Such a box will have a volume of

$$V(5) = 5(30-10)^2 = 2000\,\text{cm}^3 = 2\,\text{litres}.$$

çözüm: Kutunun hacim formülü

$$V(x) = x(30-2x)^2.$$

Dikkat edilirse V'nin tanım kümesi $[0, 15]$ dir. V ye ait grafiğin yukarıdaki grafikte olduğu gibi ortalarda bir yerde maksimum olması beklenir.

Şunu elde ederiz:

$$\begin{aligned} 0 = \frac{dV}{dx} &= (x)'(30-2x)^2 + (x)\big((30-2x)^2\big)' \\ &= (1)(30-2x)^2 + (x)2(30-2x)(-2) \\ &= (30-2x)\big((30-2x)-4x\big) \\ &= (30-2x)(30-6x) = 12(15-x)(5-x). \end{aligned}$$

Bu sebeple $x = 5$ veya $x = 15$ olur. $15 \notin (0, 15)$ için, V'nin tek kritik noktası $x = 5$ olur. maksimum alanlı kutu yapmak için, $x = 5$ seçmeliyiz. Böyle bir kutunun hacmi şudur:

$$V(5) = 5(30-10)^2 = 2000\,\text{cm}^3 = 2\,\text{litre}.$$

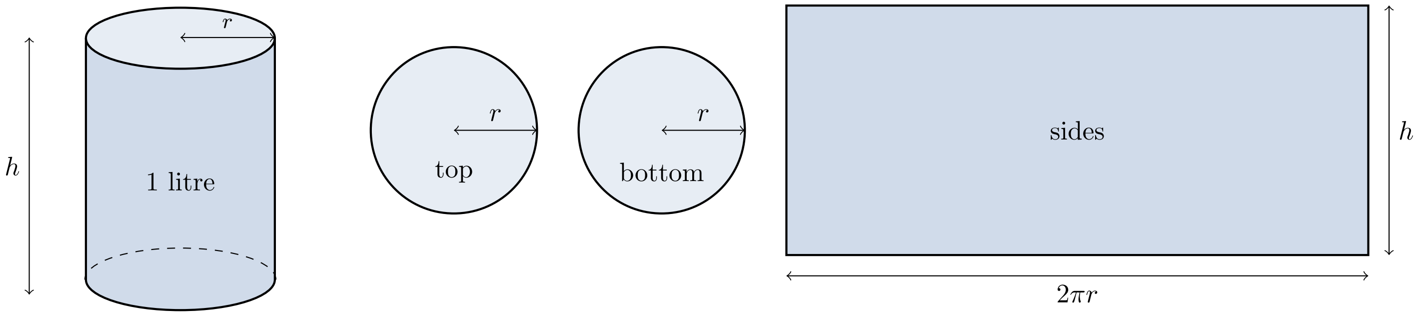
Example 14.2. You are designing a 1 litre drinks can. You will use the same metal and the same thickness of metal for

Örnek 14.2. Bir dik dairesel 1 litrelik kutu yamanız isteniyor. üst alt ve yanlar için aynı malzeme ve aynı kalınlık kullanmanız isteniyor. Hangi boyutlarda en az malzeme kullanılır?

the top, bottom and sides. What dimensions will use the least metal?

solution: We will use cm. Suppose that the radius of the can is r cm and the height of the can is h cm.

çözüm: Birim olarak cm kullanacağız. Diyelim ki yarıçap r cm ve yükseklik h cm.



Then the volume of the can is

$$\pi r^2 h = 1000 \text{ cm}^3$$

and the surface area of the can is

$$A = 2\pi r^2 + 2\pi r h.$$

We want to make A as small as we can.

Since

$$\pi r^2 h = 1000 \implies h = \frac{1000}{\pi r^2}$$

we have that

$$A(r) = 2\pi r^2 + 2\pi r h = 2\pi r^2 + 2\pi r\left(\frac{1000}{\pi r^2}\right) = 2\pi r^2 + \frac{2000}{r}.$$

See figure 14.2.

Then we calculate that

$$\begin{aligned} 0 &= \frac{dA}{dx} = 4\pi r - \frac{2000}{r^2} \\ 4\pi r &= \frac{2000}{r^2} \\ 4\pi r^3 &= 2000 \\ r &= \sqrt[3]{\frac{2000}{4\pi}} = \sqrt[3]{\frac{500}{\pi}} \approx 5.42 \text{ cm} \end{aligned}$$

and

$$h = \frac{1000}{\pi r^2} = 2\sqrt[3]{\frac{500}{\pi}} \approx 10.84 \text{ cm}.$$

O zaman kutunun hacmi

$$\pi r^2 h = 1000 \text{ cm}^3$$

yüzey alanı şöyle olur

$$A = 2\pi r^2 + 2\pi r h.$$

Şimdi A'yı minimum yapmak istiyoruz.

Burada

$$\pi r^2 h = 1000 \implies h = \frac{1000}{\pi r^2}$$

olduğundan

$$A(r) = 2\pi r^2 + 2\pi r h = 2\pi r^2 + 2\pi r\left(\frac{1000}{\pi r^2}\right) = 2\pi r^2 + \frac{2000}{r}.$$

Bkz. şekil 14.2.

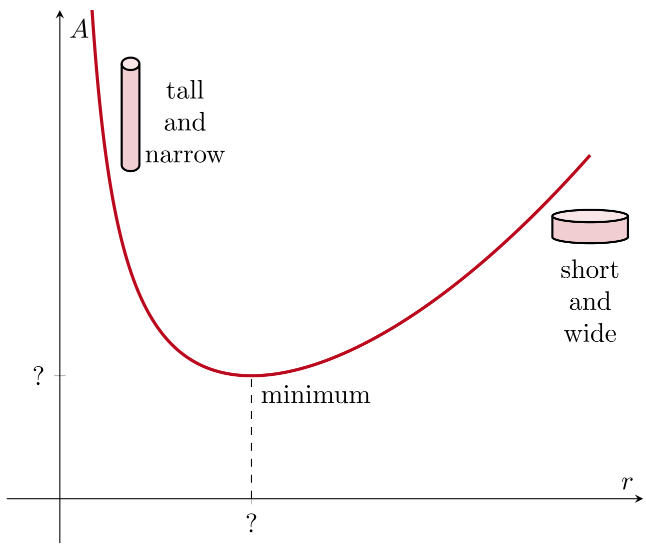


Figure 14.2: The surface area of a can of volume 1 litre and radius r cm.

Şekil 14.2: 1 litre hacimli ve r cm. yarıçaplı kutunun yan yüzey alanı

Example 14.3. A rectangle is to be inscribed in a semicircle of radius 2 as shown in figure 14.1. What is the largest possible area of the rectangle?

solution: Consider a rectangle with a vertex at the point (x, y). The area of this rectangle is clearly $A = 2xy$. Since the point (x, y) lies on the circle $x^2 + y^2 = 2^2$, we must have $y = \sqrt{4 - x^2}$. Hence the area of the rectangle is

$$A(x) = 2x\sqrt{4 - x^2}.$$

We want to find $\max_{x\in[0,2]} A(x)$.

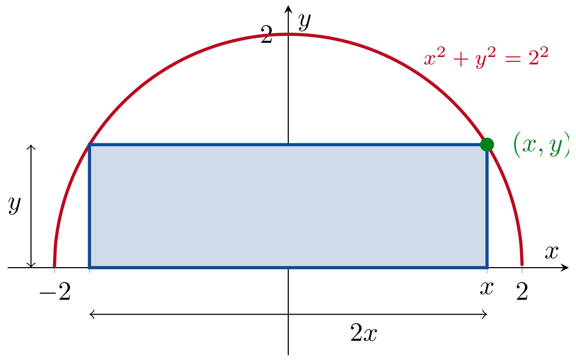


Figure 14.1: A rectangle inscribed inside a semicircle of radius 2.
Şekil 14.1:

By differentiating A, we see that

$$\begin{aligned} 0 = \frac{dA}{dx} &= \frac{d}{dx}\left(2x\sqrt{4-x^2}\right) \\ &= 2\sqrt{4-x^2} + 2x\left(\frac{-2x}{2\sqrt{4-x^2}}\right) \\ &= 2\sqrt{4-x^2} - \frac{2x^2}{\sqrt{4-x^2}}. \end{aligned}$$

Multiplying by $\sqrt{4-x^2}$ gives

$$0 = 2(4-x^2) - 2x^2 = 8 - 4x^2 = 4(2-x^2)$$

which implies that $x = \pm\sqrt{2}$. But $-\sqrt{2} \notin [0,2]$. So we must have $x = \sqrt{2}$. Therefore

$$\max_{x\in[0,2]} A(x) = A\left(\sqrt{2}\right) = 2\sqrt{2}\sqrt{4-2} = 2\sqrt{2}\sqrt{2} = 4.$$

Hesaplayacak olursak,

$$\begin{aligned} 0 &= \frac{dA}{dx} = 4\pi r - \frac{2000}{r^2} \\ 4\pi r &= \frac{2000}{r^2} \\ 4\pi r^3 &= 2000 \\ r &= \sqrt[3]{\frac{2000}{4\pi}} = \sqrt[3]{\frac{500}{\pi}} \approx 5.42 \text{ cm} \end{aligned}$$

ve

$$h = \frac{1000}{\pi r^2} = 2\sqrt[3]{\frac{500}{\pi}} \approx 10.84 \text{ cm}.$$

Örnek 14.3. Yarıçapı 2 olan yarı-çemberin içine şekil 14.1 deki gibi bir dikdörtgen yerleştirilecektir . Böyle bir dikdörtgenin alanı en fazla ne olmalıdır?

çözüm: Bir köşesi (x, y) noktasında olan dikdörtgen düşünelim. Bu dikdörtgenin alanı $A = 2xy$. Şimdi (x, y) noktası $x^2 + y^2 = 2^2$ çemberinin üzerinde olduğu için, $y = \sqrt{4-x^2}$ olur. Dolayısıyla dikdörtgenin alanı

$$A(x) = 2x\sqrt{4-x^2}.$$

Bulmak istediğimiz: $\text{maks}_{x\in[0,2]} A(x)$.
A türetitirse, şu elde edilir:

$$\begin{aligned} 0 = \frac{dA}{dx} &= \frac{d}{dx}\left(2x\sqrt{4-x^2}\right) \\ &= 2\sqrt{4-x^2} + 2x\left(\frac{-2x}{2\sqrt{4-x^2}}\right) \\ &= 2\sqrt{4-x^2} - \frac{2x^2}{\sqrt{4-x^2}}. \end{aligned}$$

$\sqrt{4-x^2}$ ile çarparsak

$$0 = 2(4-x^2) - 2x^2 = 8 - 4x^2 = 4(2-x^2)$$

buradan $x = \pm\sqrt{2}$ bulunur. Fakat $-\sqrt{2} \notin [0,2]$. Böylece $x = \sqrt{2}$ olur. Yani

$$\text{maks}_{x\in[0,2]} A(x) = A\left(\sqrt{2}\right) = 2\sqrt{2}\sqrt{4-2} = 2\sqrt{2}\sqrt{2} = 4.$$

Problems

Problem 14.1 (Selling Books)**.** You have 300 books to sell. If you price them at xTL each, then you will receive

$$R(x) = \begin{cases} 300x & x \le 40 \\ 500x - 5x^2 & 40 < x < 100 \\ 0 & x \ge 100 \end{cases}$$

liras.

(a). Draw the graph of $R(x)$.

(b). To receive the most money, at what price should you price your books?

Sorular

Soru 14.1 (Selling Books)**.** Elinizde satılmak üzere 300 kitap var. Eğer bunları her biri xTL olacak şekilde fiyatlandırırsanız, elinize

$$R(x) = \begin{cases} 300x & x \le 40 \\ 500x - 5x^2 & 40 < x < 100 \\ 0 & x \ge 100 \end{cases}$$

lira fonksiyonu geçsin.

(a). $R(x)$ grafiğini çiziniz.

(b). En fazla parayı kazanmak için, kitapları hangi fiyata fiyatlandırmalısınız?

15 Antiderivatives

Definition. F is an ***antiderivative*** of f on an interval I if $F'(x) = f(x)$ for all $x \in I$.

Example 15.1.
$2x$ is the derivative of x^2.
x^2 is an antiderivative of $2x$.

Example 15.2. If $g(x) = \cos x$, then an antiderivative of g is

$$G(x) = \sin x$$

because

$$G'(x) = \frac{d}{dx}(\sin x) = \cos x = g(x).$$

Example 15.3. If $h(x) = 2x + \cos x$, then $H(x) = x^2 + \sin x$ is an antiderivative of $h(x)$.

Remark. $F(x) = x^2$ is not the only antiderivative of $f(x) = 2x$.
$x^2 + 1$ is an antiderivative of $2x$ because $\frac{d}{dx}\left(x^2+1\right) = 2x$.
$x^2 + 5$ is an antiderivative of $2x$ because $\frac{d}{dx}\left(x^2+5\right) = 2x$.
$x^2 - 1234$ is an antiderivative of $2x$ because $\frac{d}{dx}\left(x^2-1234\right) = 2x$.

Theorem 15.1. *If F is an antiderivative of f on I, then the general antiderivative of f is*

$$F(x) + C$$

where C is a constant.

Example 15.4. Find an antiderivative of $f(x) = 3x^2$ that satisfies $F(1) = -1$.

solution: x^3 is an antiderivative of f because $\frac{d}{dx}\left(x^3\right) = 3x^2$. So the general antiderivative of f is

$$F(x) = x^3 + C.$$

Then we calculate that

$$-1 = F(1) = 1^3 + C = 1 + C \implies C = -2.$$

Therefore $F(x) = x^3 - 2$.

Ters Türevler

Tanım. Bir I aralığındaki her $x \in I$ için $F'(x) = f(x)$ olacak şekildeki F fonksiyonuna f fonksiyonunun bir ***ters türevi*** denir.

Örnek 15.1.
x^2 nin türevi $2x$ tir.
x^2 de $2x$ in bir ters türevidir.

Örnek 15.2. $g(x) = \cos x$ ise, g nin bir ters türevi

$$G(x) = \sin x$$

olur, çünkü

$$G'(x) = \frac{d}{dx}(\sin x) = \cos x = g(x).$$

Örnek 15.3. $h(x) = 2x + \cos x$ ise, $H(x) = x^2 + \sin x$ fonksiyonu $h(x)$ in bir ters türevidir.

Not. $F(x) = x^2$ fonksiyonu $f(x) = 2x$ in tek ters türevi değildir.
$x^2 + 1$ de $2x$ için bir ters türevdir çünkü $\frac{d}{dx}\left(x^2+1\right) = 2x$.
$x^2 + 5$ de $2x$ için bir ters türevdir çünkü $\frac{d}{dx}\left(x^2+5\right) = 2x$.
$x^2 - 1234$ de $2x$ için bir ters türevdir çünkü $\frac{d}{dx}\left(x^2-1234\right) = 2x$.

Teorem 15.1. *Eğer F fonksiyonu f nin I üzerindeki ters türevi ise, f nin genel ters türevi*

$$F(x) + C$$

burada C bir sabit oluyor.

Örnek 15.4. $F(1) = -1$ sağlayan $f(x) = 3x^2$ nin bir ters türevini bulunuz .

çözüm: x^3 fonksiyonu f nin bir ters türevidir çünkü $\frac{d}{dx}\left(x^3\right) = 3x^2$. Bu nedenle f nin genel ters türevi

$$F(x) = x^3 + C.$$

Şunları buluruz:

$$-1 = F(1) = 1^3 + C = 1 + C \implies C = -2.$$

Bu nedenle $F(x) = x^3 - 2$.

function, $f(x)$ fonksiyon, $f(x)$	derivative, $f'(x)$ türev, $f'(x)$
x^n	nx^{n-1}
$\sin kx$	$k\cos kx$
$\cos kx$	$-k\sin kx$
e^{kx}	ke^{kx}
$\ln\lvert x\rvert$	$\frac{1}{x}$

function, $f(x)$ fonksiyon, $f(x)$	general antiderivative, $F(x)$ genel ters türev, $F(x)$
x^n $(n \neq -1)$	$\frac{x^{n+1}}{n+1} + C$
$\sin kx$	$-\frac{1}{k}\cos kx + C$
$\cos kx$	$\frac{1}{k}\sin kx + C$
e^{kx}	$\frac{1}{k}e^{kx} + C$
$\frac{1}{x}$	$\ln\lvert x\rvert + C$

Table 15.1: Elementary derivatives and antiderivatives
Tablo 15.1:

The Sum Rule and the Constant Multiple Rule

Suppose that

- F is an antiderivative of f;
- G is an antiderivative of g;
- $k \in \mathbb{R}$.

The Sum Rule: The general antiderivative of $f+g$ is

$$F(x) + G(x) + C.$$

The Constant Multiple Rule: The general antiderivative of kf is

$$kF(x) + C.$$

Example 15.5. Find the general antiderivative of $f(x) = \frac{3}{\sqrt{x}} + \sin 2x$.

solution: We have $f = 3g + h$ where $g(x) = x^{-\frac{1}{2}}$ and $h(x) = \sin 2x$. An antiderivative of g is

$$G(x) = \frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} = \frac{x^{\frac{1}{2}}}{\frac{1}{2}} = 2\sqrt{x}.$$

An antiderivative of h is

$$H(x) = -\frac{1}{2}\cos 2x.$$

Therefore the general antiderivative of f is

$$F(x) = 6\sqrt{x} - \frac{1}{2}\cos 2x + C.$$

Definition. The general antiderivative of f is also called the ***indefinite integral*** of f with respect to x, and is denoted by

$$\int f(x)\, dx.$$

Toplam ve Sabitle çarpım Kuralı

Varsayalım ki

- F fonksiyonu f nin bir ters türevi;
- G fonksiyonu da g nin bir ters türevi;
- $k \in \mathbb{R}$.

Toplam Kuralı: $f+g$'nin ilkeli (ters türevi)

$$F(x) + G(x) + C.$$

Sabitle Çarpım Kuralı: kf'nin ilkeli

$$kF(x) + C.$$

Örnek 15.5. $f(x) = \frac{3}{\sqrt{x}} + \sin 2x$ nin ilkelini bulunuz.

çözüm: $g(x) = x^{-\frac{1}{2}}$ olmak üzere elimizde $f = 3g+h$ ve $h(x) = \sin 2x$ var. g'nin bir ilkeli

$$G(x) = \frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} = \frac{x^{\frac{1}{2}}}{\frac{1}{2}} = 2\sqrt{x}.$$

Ayrıca h'nin bir ilkeli

$$H(x) = -\frac{1}{2}\cos 2x.$$

Dıloayısıyla f fonksiyonunun bir ilkeli

$$F(x) = 6\sqrt{x} - \frac{1}{2}\cos 2x + C.$$

Tanım. f nin genel ters türev veya ilkeline aynı zamanda f nin x'e göre ***belirsiz integrali*** denir ve şöyle gösterilir:

$$\int f(x)\, dx.$$

$$\int f(x)\; dx$$

the integral sign
integral işareti

the integrand
integralin integrandı

x is the variable of integration
x ise integral değişkeni olarak tanımlanır

Example 15.6.

$$\int 2x\; dx = x^2 + C$$
$$\int \cos x = \sin x + C$$
$$\int (2x + \cos x)\; dx = x^2 + \sin x + C$$

Example 15.7. Calculate $\int (x^2 - 2x + 5)\; dx$.

solution 1. Since $\frac{d}{dx}\left(\frac{x^3}{3} - x^2 + 5x\right) = x^2 - 2x + 5$ we have that

$$\int (x^2 - 2x + 5)\; dx = \frac{x^3}{3} - x^2 + 5x + C.$$

solution 2.

$$\begin{aligned}\int (x^2 - 2x + 5)\; dx &= \int x^2\; dx - \int 2x\; dx + \int 5\; dx \\ &= \left(\frac{x^3}{3} + C_1\right) - \left(x^2 + C_2\right) + (5x + C_3) \\ &= \left(\frac{x^3}{3} - x^2 + 5x\right) + (C_1 - C_2 + C_3).\end{aligned}$$

Because we only need one constant, we can define $C := C_1 - C_2 + C_3$. Therefore

$$\int (x^2 - 2x + 5)\; dx = \frac{x^3}{3} - x^2 + 5x + C.$$

Örnek 15.6.

$$\int 2x\; dx = x^2 + C$$
$$\int \cos x = \sin x + C$$
$$\int (2x + \cos x)\; dx = x^2 + \sin x + C$$

Örnek 15.7. $\int (x^2 - 2x + 5)\; dx$ integralini bulunuz.

çözüm 1. $\frac{d}{dx}\left(\frac{x^3}{3} - x^2 + 5x\right) = x^2 - 2x + 5$ olduğundan

$$\int (x^2 - 2x + 5)\; dx = \frac{x^3}{3} - x^2 + 5x + C$$

buluruz.

çözüm 2.

$$\begin{aligned}\int (x^2 - 2x + 5)\; dx &= \int x^2\; dx - \int 2x\; dx + \int 5\; dx \\ &= \left(\frac{x^3}{3} + C_1\right) - \left(x^2 + C_2\right) + (5x + C_3) \\ &= \left(\frac{x^3}{3} - x^2 + 5x\right) + (C_1 - C_2 + C_3).\end{aligned}$$

Yalnızca bir sabite ihtiyacımız olduğundan, $C := C_1 - C_2 + C_3$ olarak tanımlarız. Yani

$$\int (x^2 - 2x + 5)\; dx = \frac{x^3}{3} - x^2 + 5x + C.$$

Example 15.8. You drop a box off the top of a tall building. The acceleration due to gravity is 9.8ms^{-2}. You can ignore air resistance. How far does the box fall in 5 seconds?

solution: The acceleration is

$$a(t) = 9.8\text{ms}^{-2}$$

downwards. Since

$$\text{acceleration} = \frac{d}{dt}(\text{velocity}),$$

the velocity is an antiderivative of the acceleration. Therefore the velocity is

$$v(t) = 9.8t + C \text{ ms}^{-1}.$$

You let go of the box at time $t = 0$. So $v(0) = 0$. Thus $C = 0$. Hence

$$v(t) = 9.8t \text{ ms}^{-1}.$$

Now velocity $= \frac{d}{dt}(\text{position})$. So the distance fallen is an antiderivative of velocity. Hence

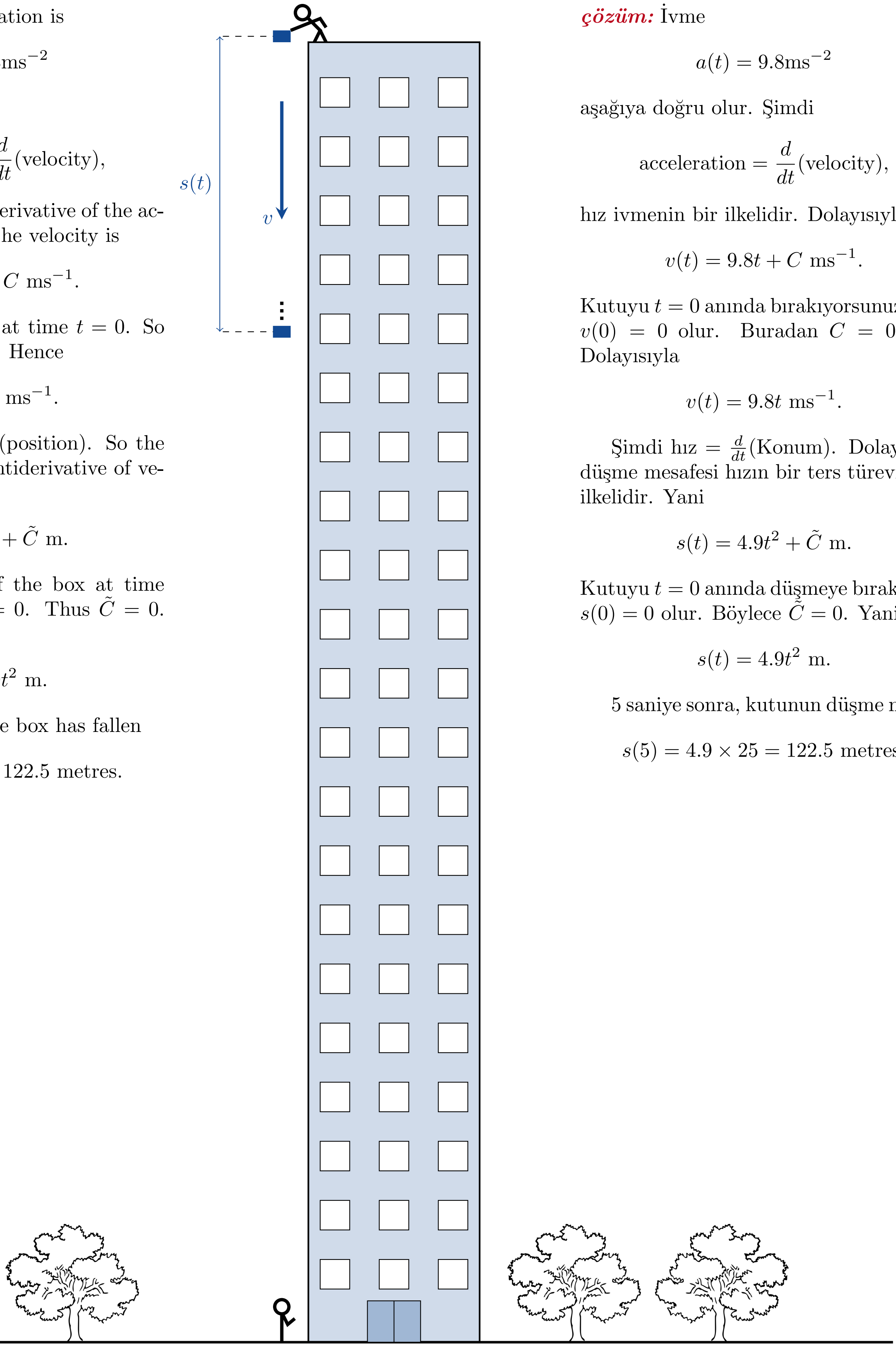
$$s(t) = 4.9t^2 + \tilde{C} \text{ m}.$$

Because you let go of the box at time $t = 0$, we have $s(0) = 0$. Thus $\tilde{C} = 0$. Therefore

$$s(t) = 4.9t^2 \text{ m}.$$

After 5 seconds, the box has fallen

$$s(5) = 4.9 \times 25 = 122.5 \text{ metres}.$$



Örnek 15.8. Bir binanın üstünden bir kutu bırakılıyor. Yerçekimi ivmesi 9.8ms^{-2} dir. Havadaki sürtünme ihmal edilebilir. Kutu 5 saniyede ne kadar yol alır?

çözüm: İvme

$$a(t) = 9.8\text{ms}^{-2}$$

aşağıya doğru olur. Şimdi

$$\text{acceleration} = \frac{d}{dt}(\text{velocity}),$$

hız ivmenin bir ilkelidir. Dolayısıyla hız

$$v(t) = 9.8t + C \text{ ms}^{-1}.$$

Kutuyu $t = 0$ anında bırakıyorsunuz. Böylece $v(0) = 0$ olur. Buradan $C = 0$ olur. Dolayısıyla

$$v(t) = 9.8t \text{ ms}^{-1}.$$

Şimdi hız $= \frac{d}{dt}(\text{Konum})$. Dolayısıyla düşme mesafesi hızın bir ters türevi veya ilkelidir. Yani

$$s(t) = 4.9t^2 + \tilde{C} \text{ m}.$$

Kutuyu $t = 0$ anında düşmeye bıraktığınızda, $s(0) = 0$ olur. Böylece $\tilde{C} = 0$. Yani

$$s(t) = 4.9t^2 \text{ m}.$$

5 saniye sonra, kutunun düşme mesafesi

$$s(5) = 4.9 \times 25 = 122.5 \text{ metres}.$$

Problems

Problem 15.1. Find an antiderivative for each function, then check your answer by differentiating it.

(a) $f(x) = 200x$.

(b) $g(x) = x^3 - \dfrac{1}{x^3}$.

(c) $h(x) = \sin(\pi x) - 3\sin(3x)$.

(d) $l(x) = x^7 - 6x + 8$.

Problem 15.2 (Right or Wrong?)**.** Consider

$$\int \left((2x+1)^2 + \cos x\right)\, dx = \frac{(2x+1)^3}{3} + \sin x + C.$$

Is this correct or incorrect? Why?

Problem 15.3. Find the following indefinite integrals.

(a) $\displaystyle\int 2x\, dx$.

(b) $\displaystyle\int \left(1 - x^2 - 3x^5\right)\, dx$.

(c) $\displaystyle\int \frac{4+\sqrt{t}}{t^3}\, dt$.

(d) $\displaystyle\int (2\cos 2\theta - 3\sin 3\theta)\, d\theta$.

Sorular

Soru 15.1. Aşağıdaki fonksiyonların birer ters türevini veya ilkelini bulup, sonra cevabınızı türev alarak bulup kontrol edin.

(a) $f(x) = 200x$.

(b) $g(x) = x^3 - \dfrac{1}{x^3}$.

(c) $h(x) = \sin(\pi x) - 3\sin(3x)$.

(d) $l(x) = x^7 - 6x + 8$.

Soru 15.2 (Doğru mu yoksa Yanlış mı?)**.**

$$\int \left((2x+1)^2 + \cos x\right)\, dx = \frac{(2x+1)^3}{3} + \sin x + C$$

yazalım. Bu doğru mu yoksa yanlış mı? Neden?

Soru 15.3. Aşağıdaki belirsiz integralleri bulunuz.

(a) $\displaystyle\int 2x\, dx$.

(b) $\displaystyle\int \left(1 - x^2 - 3x^5\right)\, dx$.

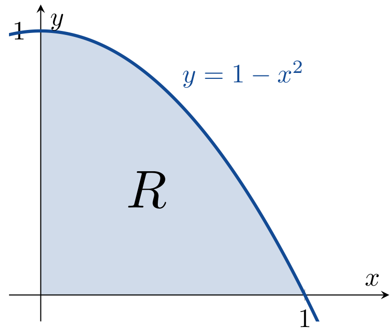
(c) $\displaystyle\int \frac{4+\sqrt{t}}{t^3}\, dt$.

(d) $\displaystyle\int (2\cos 2\theta - 3\sin 3\theta)\, d\theta$.

16 Integration

İntegral

Question: What is the area of R?

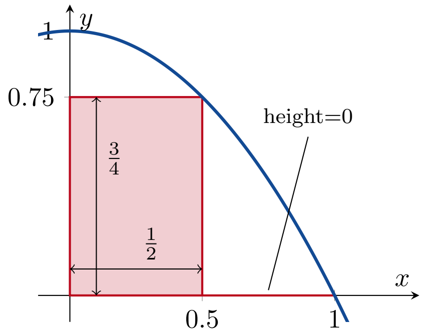


Soru: R bölgesinin alanı kaçtır?

We can use two rectangles to approximate the area of R. Then we have

$$\begin{aligned} \text{area of } R &\approx \text{area of 2 rectangles} \\ &= \left(\frac{3}{4} \times \frac{1}{2}\right) + \left(0 \times \frac{1}{2}\right) \\ &= \frac{3}{8} = 0.375. \end{aligned}$$

Can we do better than this? Yes! We could use more rectangles.



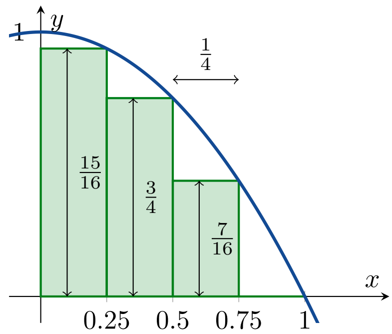
R nin alanını yaklaşık olarak hesaplamada iki dikdörtgen kullanırsak, Bu durumda

$$\begin{aligned} R\text{'nin alanı} &\approx \text{2 dikdörtgenin toplam alanı} \\ &= \left(\frac{3}{4} \times \frac{1}{2}\right) + \left(0 \times \frac{1}{2}\right) \\ &= \frac{3}{8} = 0.375. \end{aligned}$$

Bundan daha iyisini yapabilir miyiz? Evet! Daha fazla dikdörtgen kullanabiliriz.

We can say that

$$\begin{aligned} \text{area of } R &\approx \text{area of 4 rectangles} \\ &= \left(\frac{15}{16} \times \frac{1}{4}\right) + \left(\frac{3}{4} \times \frac{1}{4}\right) \\ &\quad + \left(\frac{7}{16} \times \frac{1}{4}\right) + \left(0 \times \frac{1}{4}\right) \\ &= \frac{17}{32} = 0.53125. \end{aligned}$$

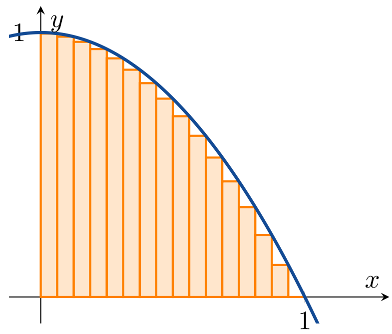


We can say that

$$\begin{aligned} \text{area of } R &\approx \text{area of 4 rectangles} \\ &= \left(\frac{15}{16} \times \frac{1}{4}\right) + \left(\frac{3}{4} \times \frac{1}{4}\right) \\ &\quad + \left(\frac{7}{16} \times \frac{1}{4}\right) + \left(0 \times \frac{1}{4}\right) \\ &= \frac{17}{32} = 0.53125. \end{aligned}$$

Every time we increase the number of rectangles, the total area of the rectangles gets closer and closer to the area of R.

$$\begin{aligned} \text{area of } R &\approx \text{area of 16 rectangles} \\ &= 0.63476. \end{aligned}$$



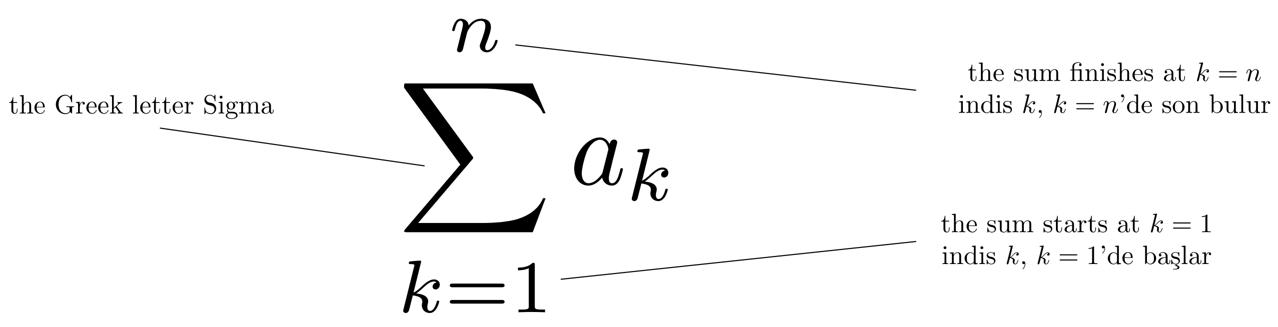
Dikdörtgenlerin sayısını her arttırdığımızda, dikdörtgenlerin toplam alanı, R alanına daha da yakınlaşıyor.

$$\begin{aligned} R\text{'nin alanı} &\approx \text{16 dikdörtgenin toplam alanı} \\ &= 0.63476. \end{aligned}$$

Sigma Notation

Sigma Notasyonu

$$\sum_{k=1}^{n} a_k = a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8 + \ldots + a_{n-1} + a_n$$



Example 16.1.

$$1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 + 7^2 + 8^2 + 9^2 + 10^2 + 11^2 = \sum_{k=1}^{11} k^2$$

$$f(1) + f(2) + f(3) + \ldots + f(99) + f(100) = \sum_{k=1}^{100} f(k)$$

$$\sum_{k=1}^{5} k = 1 + 2 + 3 + 4 + 5 = 15$$

Örnek 16.2.

$$\sum_{k=1}^{3} (-1)^k k = (-1)(1) + (-1)^2(2) + (-1)^3(3) = -1 + 2 - 3 = -2$$

$$\sum_{k=1}^{2} \frac{k}{k+1} = \frac{1}{1+1} + \frac{2}{2+1} = \frac{1}{2} + \frac{2}{3} = \frac{7}{6}$$

$$\sum_{k=4}^{5} \frac{k^2}{k-1} = \frac{4^2}{4-1} + \frac{5^2}{5-1} = \frac{16}{3} + \frac{25}{4} = \frac{139}{12}$$

Example 16.3. I want to find a formula for $1+2+3+\ldots+n$. Note that

Örnek 16.3. $1+2+3+\ldots+n$ için bir formül bulmak istiyoruz. Dikkat edilirse

$$\begin{aligned}
&2\big(1 + 2 + 3 + 4 + 5 + \ldots + (n-1) + n\big) \\
&= \quad 1 + 2 + 3 + 4 + 5 + \ldots + (n-1) + n \\
&\quad + n + (n-1) + (n-2) + (n-3) + (n-4) + \ldots + 2 + 1 \\
&= (n+1) + (n+1) + (n+1) + (n+1) + (n+1) + \ldots + (1+n) + (1+n) \\
&= n(n+1).
\end{aligned}$$

Therefore

$$\boxed{\sum_{k=1}^{n} k = \frac{n(n+1)}{2}.}$$

Similarly (but more difficult) we can find that

$$\boxed{\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}}$$

and

$$\boxed{\sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2.}$$

Dolayısıyla

$$\boxed{\sum_{k=1}^{n} k = \frac{n(n+1)}{2}.}$$

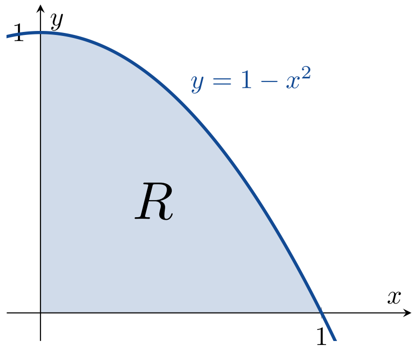
Benzer olarak (ama daha zor) şunu buluruz

$$\boxed{\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}}$$

ve

$$\boxed{\sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2.}$$

Limits of Finite Sums



Here's the plan:

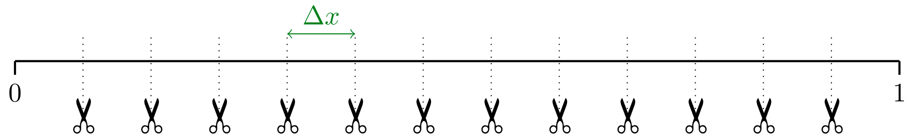
STEP 1. We will cut $[0, 1]$ in to n pieces of width

$$\Delta x = \frac{1-0}{n} = \frac{1}{n}.$$

STEP 2. We will use n rectangles to approximate the area of R. See figure 16.1.

STEP 3. Then we will take the limit as $n \to \infty$.

Sonlu Toplamların Limitleri



İşte izleyeceğimiz yol:

ADIM 1. $[0, 1]$'i n parçaya bölersek

$$\Delta x = \frac{1-0}{n} = \frac{1}{n}.$$

ADIM 2. n tane dikdörtgenle R'nin alanını yaklaşık olarak buluruz. Bkz. şekil 16.1.

ADIM 3. Daha sonra $n \to \infty$ iken limit alırız.

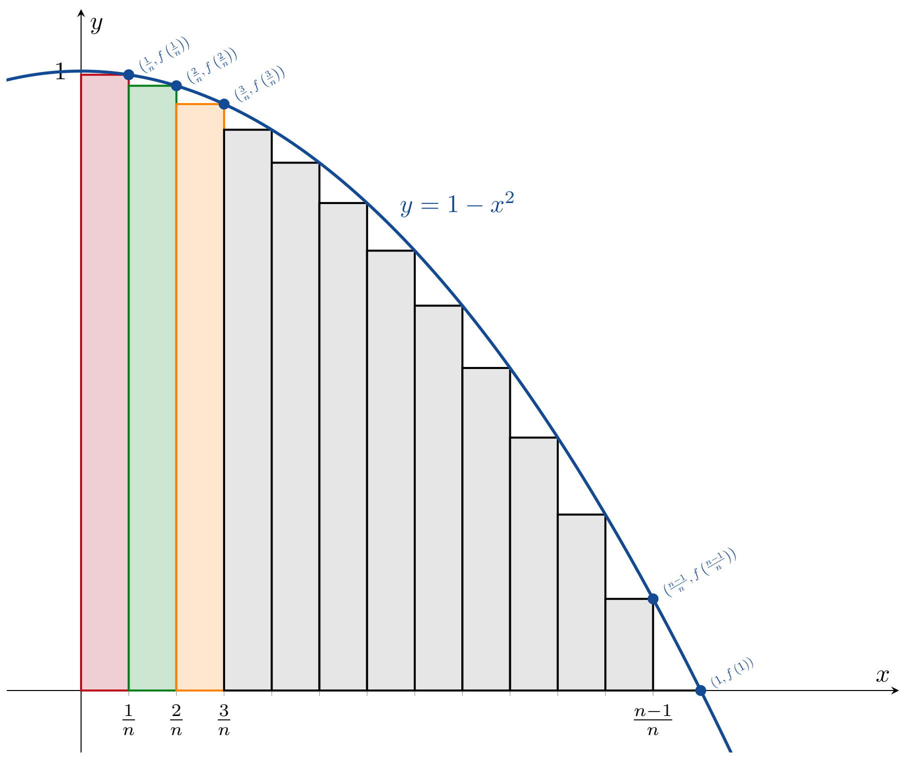


Figure 16.1: We can use n rectangles to approximate the area of R.
Şekil 16.1: n tane dikdörtgeni R'nin alanını yaklaşık hesaplamakta kullanabiliriz.

Let $f(x) = 1 - x^2$. Then

- the first rectangle has area $\frac{1}{n} f\left(\frac{1}{n}\right)$;
- the second rectangle has area $\frac{1}{n} f\left(\frac{2}{n}\right)$;
- the third rectangle has area $\frac{1}{n} f\left(\frac{3}{n}\right)$;

and so on.

Let $f(x) = 1 - x^2$. Then

- ilk dikdörtgen alanı $\frac{1}{n} f\left(\frac{1}{n}\right)$;
- ikinci dikdörtgen alanı $\frac{1}{n} f\left(\frac{2}{n}\right)$;
- üçüncü dikdörtgen alanı $\frac{1}{n} f\left(\frac{3}{n}\right)$;

ve saire.

The area of all n rectangles is

$$\begin{aligned}
\text{area} &= \sum_{k=1}^{n} (\text{area of the } k\text{th rectangle}) \\
&= \sum_{k=1}^{n} \frac{1}{n} f\left(\frac{k}{n}\right) \\
&= \sum_{k=1}^{n} \frac{1}{n} \left(1 - \left(\frac{k}{n}\right)^2\right) \\
&= \sum_{k=1}^{n} \left(\frac{1}{n} - \frac{k^2}{n^3}\right) \\
&= \sum_{k=1}^{n} \frac{1}{n} - \sum_{k=1}^{n} \frac{k^2}{n^3} \\
&= n\left(\frac{1}{n}\right) - \frac{1}{n^3} \sum_{k=1}^{n} k^2 \\
&= 1 - \frac{1}{n^3} \left(\frac{n(n+1)(2n+1)}{6}\right) \\
&= 1 - \frac{2n^2 + 3n + 1}{6n^2}.
\end{aligned}$$

Taking the limit gives

$$\begin{aligned}
\lim_{n\to\infty} \left(\sum_{k=1}^{n} \frac{1}{n} f\left(\frac{k}{n}\right)\right) &= \lim_{n\to\infty} \left(1 - \frac{2n^2+3n+1}{6n^2}\right) \\
&= 1 - \frac{2}{6} = \frac{2}{3}.
\end{aligned}$$

Therefore the area of R is $\frac{2}{3}$.

Riemann Sums

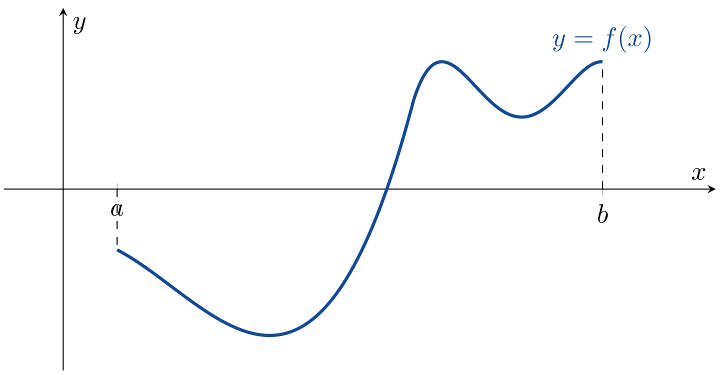


Figure 16.2: A function $f : [a, b] \to \mathbb{R}$.
Şekil 16.2: Bir fonksiyon $f : [a, b] \to \mathbb{R}$.

Now let $f[a, b] \to \mathbb{R}$ be a function. We will cut $[a, b]$ into n subintervals (the pieces don't have to all be the same size). In each subinterval we will choose one point $c_k \in [x_{k-1}, x_k]$, as shown in figure 16.3. The width of each subinterval is $\Delta x_k = x_k - x_{k-1}$.

On each subinterval $[x_{k-1}, x_k]$, we draw a rectangle of width Δx_k and height $f(c_k)$. See figure 16.4

n dikdörtgenin toplam alanı

$$\begin{aligned}
\text{area} &= \sum_{k=1}^{n} (k \text{ inci dikdörtgen}) \\
&= \sum_{k=1}^{n} \frac{1}{n} f\left(\frac{k}{n}\right) \\
&= \sum_{k=1}^{n} \frac{1}{n} \left(1 - \left(\frac{k}{n}\right)^2\right) \\
&= \sum_{k=1}^{n} \left(\frac{1}{n} - \frac{k^2}{n^3}\right) \\
&= \sum_{k=1}^{n} \frac{1}{n} - \sum_{k=1}^{n} \frac{k^2}{n^3} \\
&= n\left(\frac{1}{n}\right) - \frac{1}{n^3} \sum_{k=1}^{n} k^2 \\
&= 1 - \frac{1}{n^3} \left(\frac{n(n+1)(2n+1)}{6}\right) \\
&= 1 - \frac{2n^2 + 3n + 1}{6n^2}.
\end{aligned}$$

Limit alınırsa

$$\begin{aligned}
\lim_{n\to\infty} \left(\sum_{k=1}^{n} \frac{1}{n} f\left(\frac{k}{n}\right)\right) &= \lim_{n\to\infty} \left(1 - \frac{2n^2+3n+1}{6n^2}\right) \\
&= 1 - \frac{2}{6} = \frac{2}{3}.
\end{aligned}$$

Buradan R'nin alanı $\frac{2}{3}$ olur.

Riemann Sums

Şimdi $f : [a, b] \to \mathbb{R}$ bir fonksiyon olsun. $[a, b]$'yi n aralığa böleriz (parçaların hepsinin aynı genişlikte olması gerekmez). Her alt-aralıkta, Şekil 16.3'de gösterildiği gibi $[x_{k-1}, x_k]$ cinsinden bir nokta c_k seçeriz. Her alt aralığın genişliği $\Delta x_k = x_k - x_{k-1}$'dır.

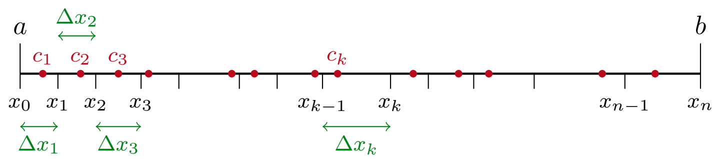


Figure 16.3: We split the interval $[a, b]$ into n subintervals. Note that $a = x_0 < x_1 < x_2 < x_3 < \ldots < x_{n-1} < x_n = b$.
Şekil 16.3: $[a, b]$ aralığını n alt-aralığa bölünür. Dikkat edilirse, $a = x_0 < x_1 < x_2 < x_3 < \ldots < x_{n-1} < x_n = b$ dir.

Her bir $[x_{k-1}, x_k]$ alt-aralığında, genişliği Δx_k ve yüksekliği $f(c_k)$ olan dikdörtgenler çizilir. Bkz. şekil 16.4

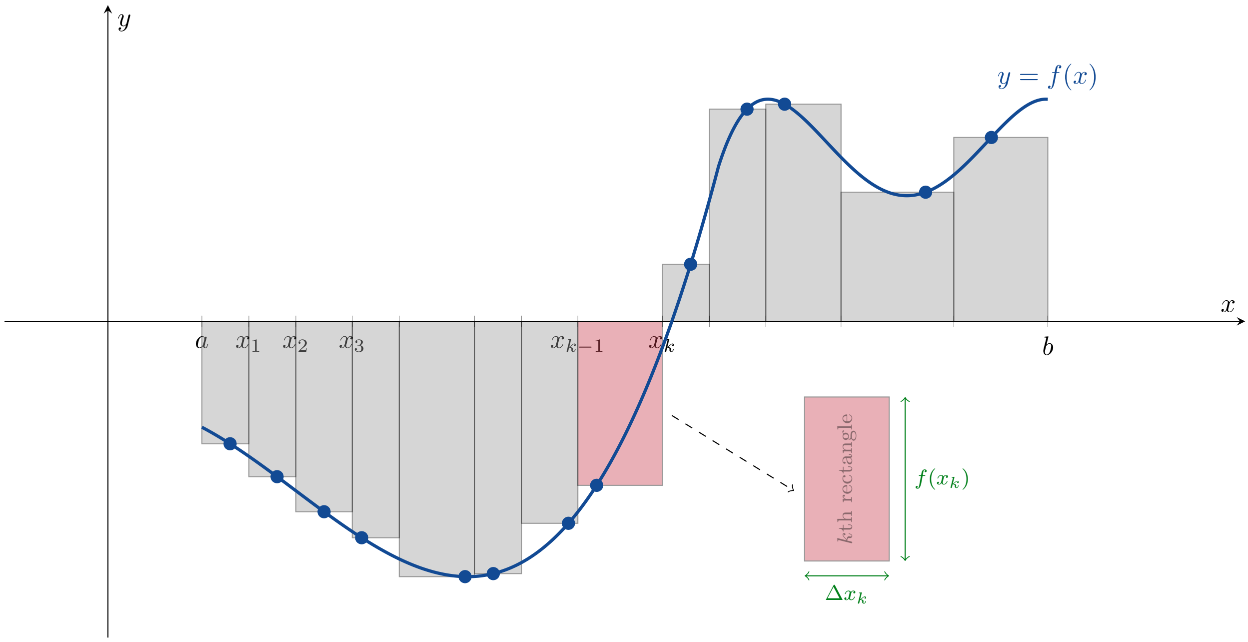


Figure 16.4: n rectangles.
Şekil 16.4:

Note that if $f(c_k) < 0$, then the rectangle on $[x_{k-1}, x_k]$ will have 'negative area' – this is ok.

The total of the n rectangles is

$$\sum_{k=1}^{n} f(x_k)\Delta x_k.$$

This is called a ***Riemann Sum for*** f ***on*** $[a, b]$. Then we want to take the limit as $n \to \infty$ (or more precisely, we want to take the limit as $\max\{\Delta x_1, \Delta x_2, \ldots, \Delta x_n\} \to 0$). Sometimes this limit exists, sometimes this limit does not exist.

$f(c_k) < 0$ olduğuna dikkat edersek, tabanı $[x_{k-1}, x_k]$ olan dikdörtgen 'negatif aalanlı' – olur.

n dikdörtgenin toplam alanı

$$\sum_{k=1}^{n} f(x_k)\Delta x_k.$$

Bu toplama bir f ***nin*** $[a, b]$ ***üzerindeki bir Riemann Toplamı*** denir. Sonra $n \to \infty$ iken limit alınır (veya daha doğrusu, $\text{maks}\{\Delta x_1, \Delta x_2, \ldots, \Delta x_n\} \to 0$ iken limit alınır). Bu limit bazen mevcuttur, bazen mevcut değil.

17

The Definite Integral

Definition. **If** the limit

$$\lim_{n\to\infty}\sum_{k=1}^{n} f(c_k)\Delta x_k$$

exists, then it is called the ***definite integral of*** f ***over*** $[a,b]$. We write

$$\int_a^b f(x)\;dx = \lim_{n\to\infty}\sum_{k=1}^{n} f(c_k)\Delta x_k$$

if the limit exists.

Belirli İntegral

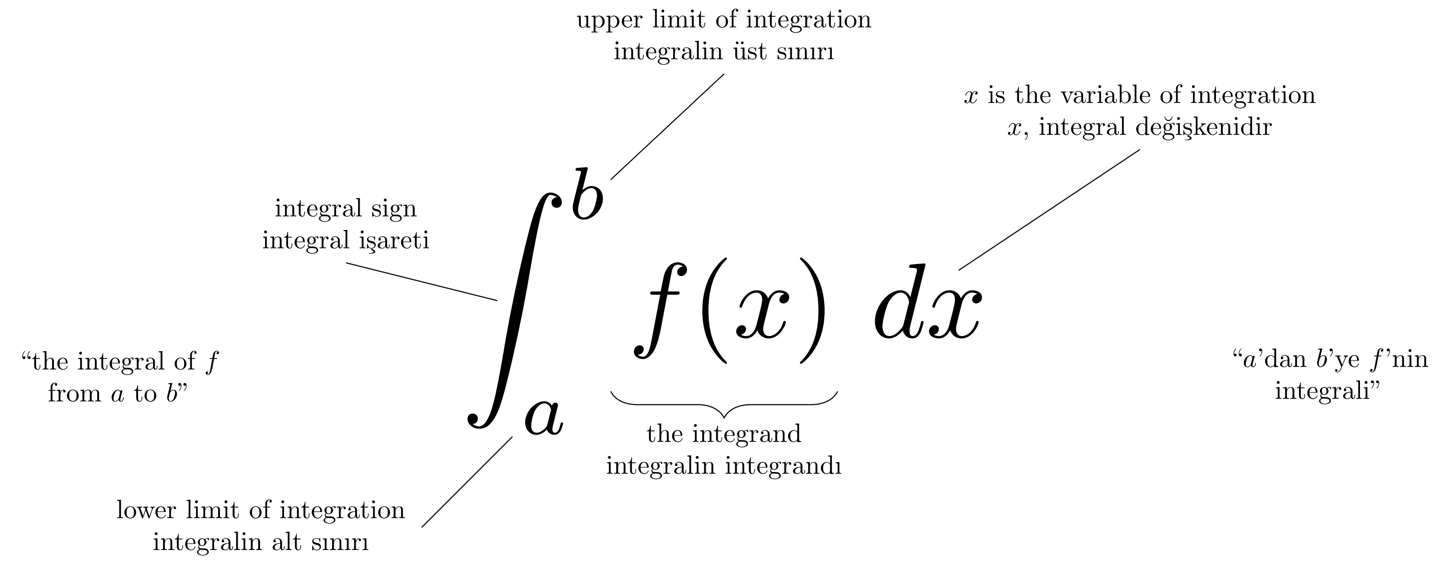
Tanım. **If** Eğer

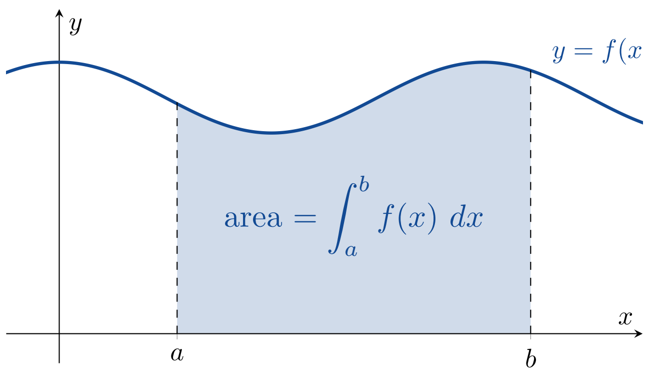
$$\lim_{n\to\infty}\sum_{k=1}^{n} f(c_k)\Delta x_k$$

limiti mevcutsa, bu limite f***'nin*** $[a,b]$ ***üzerindeki belirli integrali*** adı verilir. Şöyle gösteririz

$$\int_a^b f(x)\;dx = \lim_{n\to\infty}\sum_{k=1}^{n} f(c_k)\Delta x_k$$

tabi eğer limit mevcutsa.





Definition. If $\int_a^b f(x)\,dx$ exists, then we say that ***f is integrable on $[a,b]$***.

Example 17.1. $f(x) = 1 - x^2$ is integrable on $[0,1]$ and $\int_0^1 (1-x^2)\,dx = \frac{2}{3}$.

Remark.

$$\int_a^b f(x)\,dx = \int_a^b f(u)\,du = \int_a^b f(t)\,dt$$

It doesn't matter which letter we use for the ***dummy variable***.

Theorem 17.1. *If f is continuous on $[a,b]$, then f is integrable on $[a,b]$.*

If f has finitely many jump discontinuities but is otherwise continuous on $[a,b]$, then f is integrable on $[a,b]$.

Example 17.2. Define a function $g : [0,1] \to \mathbb{R}$ by

$$g(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q}. \end{cases}$$

See figure 17.1. This function is not integrable on $[0,1]$.

Tanım. Eğer $\int_a^b f(x)\,dx$ mevcutsa, ***f fonksiyonu $[a,b]$ üzerinde integrallenebilir*** denir.

Örnek 17.1. $f(x) = 1 - x^2$ fonksiyonu $[0,1]$ üzerinde integrallenebilir ve $\int_0^1 (1-x^2)\,dx = \frac{2}{3}$.

Not.

$$\int_a^b f(x)\,dx = \int_a^b f(u)\,du = \int_a^b f(t)\,dt$$

takma değişken için hangi sembol kullandığımızın bir önemi yok.

Teorem 17.1. *Eğer f fonksiyonu $[a,b]$'de sürekli ise, $[a,b]$'de f integrallenebilirdir.*

Eğer f sonlu sayıda sıçramalı süreksizliği varsa veya $[a,b]$'de sürekli ise, then $[a,b]$ üzerinde f integrallenebilirdir.

Örnek 17.2. Şu fonksiyonu tanımlarsak $g : [0,1] \to \mathbb{R}$ öyle ki

$$g(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q}. \end{cases}$$

Bkz. şekil 17.1. Bu fonksiyon $[0,1]$'de integrallenemez.

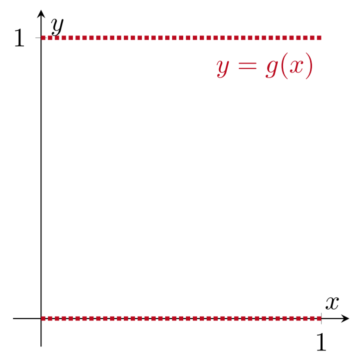


Figure 17.1: The graph of $g(x)$ defined in Example 17.2.
Şekil 17.1: Örnekteki $g(x)$ grafiği

Properties of Definite Integrals

Theorem 17.2. *Suppose that f and g are integrable. Let k be a number. Then*

(i). $\displaystyle\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx;$

(ii). $\displaystyle\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx;$

(iii). $\displaystyle\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx$

(iv). $\displaystyle\int_a^a f(x)\,dx = 0;$

(v). $\displaystyle\int_a^b \big(f(x)+g(x)\big)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx;$

(vi). $\displaystyle(b-a)\min f \le \int_a^b f(x)\,dx \le (b-a)\max f;$

(vii). *if* $f(x) \le g(x)$ *on* $[a,b]$, *then*

$$\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx;$$

$(viii)$. *if* $g(x) \ge 0$ *on* $[a,b]$, *then*

$$\int_a^b g(x)\,dx \ge 0;$$

(ix). *if* f *is an even function, then*

$$\int_{-a}^a f(x)\,dx = 2\int_0^a f(x)\,dx;$$

and

(x). *if* f *is an odd function, then*

$$\int_{-a}^a f(x)\,dx = 0.$$

Belirli İntegralin Özellikleri

Teorem 17.2. *f ve g integrallenebilir olsunlar. k bir sabit sayı olsun. Bu durumda*

(i). $\displaystyle\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx;$

(ii). $\displaystyle\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx;$

(iii). $\displaystyle\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx$

(iv). $\displaystyle\int_a^a f(x)\,dx = 0;$

(v). $\displaystyle\int_a^b \big(f(x)+g(x)\big)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx;$

(vi). $\displaystyle(b-a)\min f \le \int_a^b f(x)\,dx \le (b-a)\,\text{maks}\, f;$

(vii). $f(x) \le g(x)$ *on* $[a,b]$ *ise,*

$$\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx;$$

$(viii)$. $[a,b]$ *üzerinde* $g(x) \ge 0$ *ise,*

$$\int_a^b g(x)\,dx \ge 0;$$

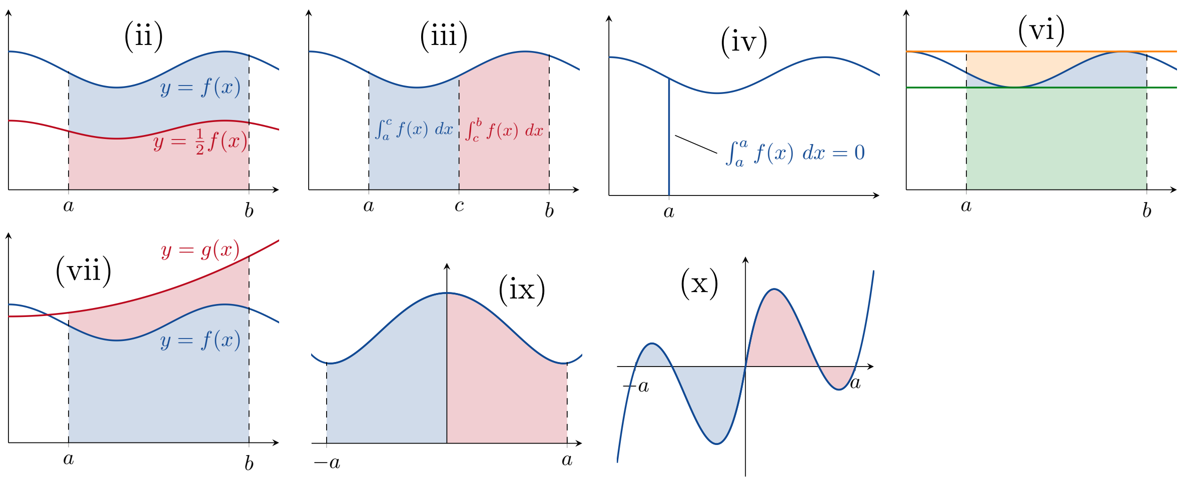
(ix). *f çift fonksiyon ise,*

$$\int_{-a}^a f(x)\,dx = 2\int_0^a f(x)\,dx;$$

ve

(x). *if f tek fonksiyon ise,*

$$\int_{-a}^a f(x)\,dx = 0.$$



Example 17.3. Suppose that $\int_{-1}^{1} f(x)\,dx = 5$, $\int_{1}^{4} f(x)\,dx = -2$ and $\int_{-1}^{1} h(x)\,dx = 7$. Then

$$\int_{4}^{1} f(x)\,dx = -\int_{1}^{4} f(x)\,dx = 2,$$

$$\begin{aligned}\int_{-1}^{1} \big(2f(x) + 3h(x)\big)\,dx &= 2\int_{-1}^{1} f(x)\,dx + 3\int_{-1}^{1} h(x)\,dx \\ &= 2(5) + 3(7) = 31\end{aligned}$$

and

$$\begin{aligned}\int_{-1}^{4} f(x)\,dx &= \int_{-1}^{1} f(x)\,dx + \int_{1}^{4} f(x)\,dx \\ &= 5 + (-2) = 3.\end{aligned}$$

Example 17.4. Show that $\int_{0}^{1} \sqrt{1+\cos x}\,dx \leq \sqrt{2}$.

solution: The maximum value of $\sqrt{1+\cos x}$ on $[0,1]$ is $\sqrt{1+1} = \sqrt{2}$. Therefore

$$\int_{0}^{1} \sqrt{1+\cos x}\,dx \leq (1-0)\max\sqrt{1+\cos x} = 1 \times \sqrt{2}.$$

Example 17.5. Calculate $\int_{-2}^{2} (x^3 + x)\,dx$.

solution: Because $(x^3 + x)$ is an odd function, we have that

$$\int_{-2}^{2} (x^3 + x)\,dx = 0.$$

Example 17.6. Calculate $\int_{-1}^{1} (1 - x^2)\,dx$.

solution: Because $(1 - x^2)$ is an even function, we have that

$$\int_{-1}^{1} (1 - x^2)\,dx = 2\int_{0}^{1} (1 - x^2)\,dx = 2 \times \frac{2}{3} = \frac{4}{3}.$$

Örnek 17.3. Varsayalım ki $\int_{-1}^{1} f(x)\,dx = 5$, $\int_{1}^{4} f(x)\,dx = -2$ ve $\int_{-1}^{1} h(x)\,dx = 7$. O zaman

$$\int_{4}^{1} f(x)\,dx = -\int_{1}^{4} f(x)\,dx = 2,$$

$$\begin{aligned}\int_{-1}^{1} \big(2f(x) + 3h(x)\big)\,dx &= 2\int_{-1}^{1} f(x)\,dx + 3\int_{-1}^{1} h(x)\,dx \\ &= 2(5) + 3(7) = 31\end{aligned}$$

ve

$$\begin{aligned}\int_{-1}^{4} f(x)\,dx &= \int_{-1}^{1} f(x)\,dx + \int_{1}^{4} f(x)\,dx \\ &= 5 + (-2) = 3.\end{aligned}$$

Örnek 17.4. Gösteriniz ki $\int_{0}^{1} \sqrt{1+\cos x}\,dx \leq \sqrt{2}$.

çözüm: $[0,1]$ üzerindeki $\sqrt{1+\cos x}$'nin maksimum değeri $\sqrt{1+1} = \sqrt{2}$. Buradan

$$\int_{0}^{1} \sqrt{1+\cos x}\,dx \leq (1-0)\,\text{maks}\sqrt{1+\cos x} = 1 \times \sqrt{2}.$$

Örnek 17.5. $\int_{-2}^{2} (x^3 + x)\,dx$ hesaplayınız.

çözüm: $(x^3 + x)$ tek fonksiyon olduğundan, şunu elde ederiz:

$$\int_{-2}^{2} (x^3 + x)\,dx = 0.$$

Örnek 17.6. $\int_{-1}^{1} (1 - x^2)\,dx$ hesaplayınız.

çözüm: $(1 - x^2)$ çift fonksiyon olduğu için,

$$\int_{-1}^{1} (1 - x^2)\,dx = 2\int_{0}^{1} (1 - x^2)\,dx = 2 \times \frac{2}{3} = \frac{4}{3}.$$

Example 17.7. Calculate $\int_0^b x\ dx$ for $b > 0$.

solution 1: We will use a Riemann Sum. First we cut $[0, b]$ in to n pieces using

$$0 < \frac{b}{n} < \frac{2b}{n} < \frac{3b}{n} < \ldots < \frac{(n-1)b}{n} < b$$

and $c_k = \frac{kb}{n}$. Note that $\Delta x_k = \frac{b}{n}$ for all k. See figure 17.2. Then

$$\begin{aligned}\sum_{k=1}^{n} f(c_k)\Delta x_k &= \sum_{k=1}^{n} \frac{kb}{n}\frac{b}{n} = \frac{b^2}{n^2}\sum_{k=1}^{n} k \\ &= \frac{b^2}{n^2}\left(\frac{n(n+1)}{2}\right) = \frac{b^2}{2}\left(1+\frac{1}{n}\right).\end{aligned}$$

Then

$$\begin{aligned}\int_0^b x\ dx &= \lim_{n\to\infty}\sum_{k=1}^{n} f(c_k)\Delta x_k \\ &= \lim_{n\to\infty}\frac{b^2}{2}\left(1+\frac{1}{n}\right) = \frac{b^2}{2}.\end{aligned}$$

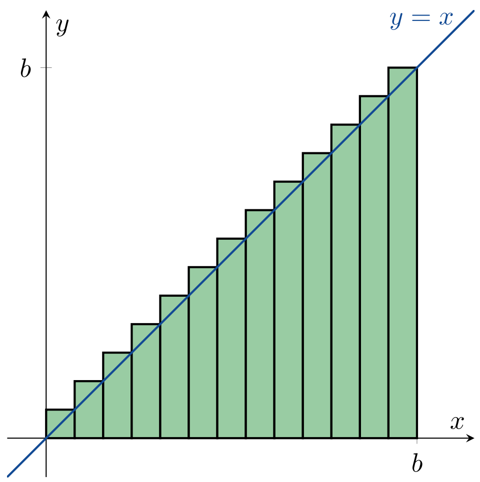


Figure 17.2: Approximating $\int_0^b x\ dx$ by n rectangles.
Şekil 17.2:

solution 2: Alternately, we can look at figure 17.3 and say that

$$\int_0^b x\ dx = \text{area of a triangle} = \frac{1}{2}\times b\times b = \frac{b^2}{2}.$$

Example 17.8.

$$\begin{aligned}\int_a^b x\ dx &= \int_a^0 x\ dx + \int_0^b x\ dx \\ &= -\int_0^a x\ dx + \int_0^b x\ dx \\ &= -\frac{a^2}{2} + \frac{b^2}{2} \\ &= \frac{b^2}{2} - \frac{a^2}{2}.\end{aligned}$$

Örnek 17.7. $b > 0$ ise $\int_0^b x\ dx$ integralini bulunuz.

çözüm 1: Riemann Toplamı kullanacağız. Önce $[0, b]$'yi n parçaya

$$0 < \frac{b}{n} < \frac{2b}{n} < \frac{3b}{n} < \ldots < \frac{(n-1)b}{n} < b$$

ve $c_k = \frac{kb}{n}$ kullanarak böleriz. Dikkat edilirse her k için $\Delta x_k = \frac{b}{n}$ olur. Bkz. şekil 17.2. Bu durumda

$$\begin{aligned}\sum_{k=1}^{n} f(c_k)\Delta x_k &= \sum_{k=1}^{n} \frac{kb}{n}\frac{b}{n} = \frac{b^2}{n^2}\sum_{k=1}^{n} k \\ &= \frac{b^2}{n^2}\left(\frac{n(n+1)}{2}\right) = \frac{b^2}{2}\left(1+\frac{1}{n}\right).\end{aligned}$$

O halde

$$\begin{aligned}\int_0^b x\ dx &= \lim_{n\to\infty}\sum_{k=1}^{n} f(c_k)\Delta x_k \\ &= \lim_{n\to\infty}\frac{b^2}{2}\left(1+\frac{1}{n}\right) = \frac{b^2}{2}.\end{aligned}$$

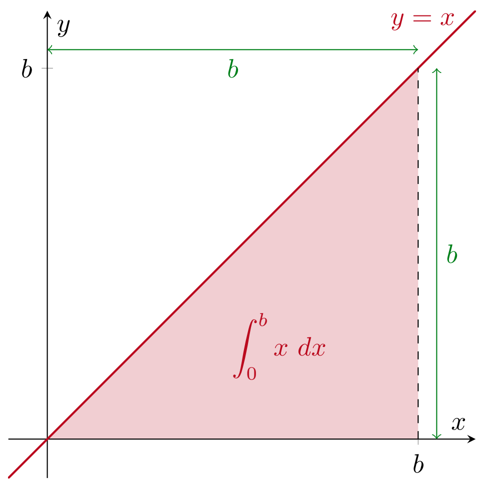


Figure 17.3: The integral of x from 0 to b.
Şekil 17.3: 0 dan b ye x in integrali

çözüm 2: Alternatif olarak, şekil 17.3 e bakarak

$$\int_0^b x\ dx = \text{area of a triangle} = \frac{1}{2}\times b\times b = \frac{b^2}{2}.$$

Örnek 17.8.

$$\begin{aligned}\int_a^b x\ dx &= \int_a^0 x\ dx + \int_0^b x\ dx \\ &= -\int_0^a x\ dx + \int_0^b x\ dx \\ &= -\frac{a^2}{2} + \frac{b^2}{2} \\ &= \frac{b^2}{2} - \frac{a^2}{2}.\end{aligned}$$

18

The Fundamental Theorem of Calculus

Kalkülüsün Temel Teoremi

We don't want to have to use Riemann sums every time we need to calculate a definite integral – we want a better way. The following theorem is the most important theorem in Calculus. If you can only memorise one theorem for the exams, it should be this one.

Bir belirli integrali hesaplamanız gerektiğinde her defasında Riemann toplamlarını kullanmamız gerekmiyor – daha iyi bir yol istiyoruz. Aşağıdaki teorem Kalkülüsün en önemli teoremidir. Sınavlar için bir teorem ezberleyeceğim diyorsanız, işte bu o teoremdir.

Theorem 18.1 (The Fundamental Theorem of Calculus)**.** *Suppose that $f : [a, b] \to \mathbb{R}$ is a continuous function.*

(*i*). *Then the function $F : [a, b] \to \mathbb{R}$ defined by*

$$F(x) = \int_a^x f(t)\ dt$$

is continuous on $[a, b]$; differentiable on (a, b); and its derivative is

$$\boxed{F'(x) = \frac{d}{dt}\int_a^x f(t)\ dt = f(x).}$$

(*ii*). *If F is any antiderivative of f on $[a, b]$, then*

$$\boxed{\int_a^b f(x)\ dx = F(b) - F(a).}$$

Teorem 18.1 (Kalkülüsün Temel Teoremi)**.** *$f : [a, b] \to \mathbb{R}$'nin sürekli bir fonksiyon olduğunu varsayalım.*

(*i*). *Bu durumda $F : [a, b] \to \mathbb{R}$,*

$$F(x) = \int_a^x f(t)\ dt$$

de $[a, b]$ üzerinde süreklidir; (a, b) üzerinde türevlenebilir; ve türevi $f(x)$'tir

$$\boxed{F'(x) = \frac{d}{dt}\int_a^x f(t)\ dt = f(x).}$$

(*ii*). *Eğer F de f'nin $[a, b]$ üzerindeki herhangi bir ters türevi ise, bu durumda*

$$\boxed{\int_a^b f(x)\ dx = F(b) - F(a).}$$

Remark. Part (i) of the theorem tells how to differentiate $\int_a^x f(t)\ dt$.

Not. Teoremin (i) kısmı $\int_a^x f(t)\ dt$'in türevini nasıl alacağımızı söyler.

Example 18.1. Find $\frac{dy}{dx}$ if $y = \int_a^x (t^3 + 1)\ dt$.

solution:

$$\frac{dy}{dx} = \frac{d}{dx}\int_a^x (t^3 + 1)\ dt = x^3 + 1.$$

Örnek 18.1. $y = \int_a^x (t^3 + 1)\ dt$ ise, $\frac{dy}{dx}$'i bulunuz.

çözüm:

$$\frac{dy}{dx} = \frac{d}{dx}\int_a^x (t^3 + 1)\ dt = x^3 + 1.$$

Example 18.2. Find $\frac{dy}{dx}$ if $y = \int_x^5 3t \sin t\ dt$.

solution:

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}\int_x^5 3t\sin t\ dt \\ &= \frac{d}{dx}\left(-\int_5^x 3t\sin t\ dt\right) \\ &= -3x\sin x.\end{aligned}$$

Örnek 18.2. $y = \int_x^5 3t \sin t\ dt$ ise, $\frac{dy}{dx}$'i bulunuz.

çözüm:

Example 18.3. Find $\frac{dy}{dx}$ if $y = \int_1^{x^2} \cos t\ dt$.

solution: This time we will need to use the Chain rule. Let $u = x^2$. Then

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du}\frac{du}{dx} \\ &= \left(\frac{d}{du}\int_1^u \cos t\ dt\right)\left(\frac{d}{dx}x^2\right) \\ &= (\cos u)\,(2x) = 2x\cos x^2.\end{aligned}$$

Remark. Part (ii) of the theorem tells us how to calculate the definite integral of f over $[a,b]$:

STEP 1. Find an antiderivative F of f.

STEP 2. Calculate $F(b) - F(a)$.

Notation. We will write

$$\Big[F(x)\Big]_a^b = F(b) - F(a).$$

Example 18.4.

$$\begin{aligned}\int_0^\pi \cos x\ dx &= \Big[\sin x\Big]_0^\pi \\ &\quad \left(\text{because } \tfrac{d}{dx}\sin x = \cos x\right) \\ &= \sin\pi - \sin 0 \\ &= 0 - 0 \\ &= 0\end{aligned}$$

Example 18.5.

$$\begin{aligned}\int_{-\frac{\pi}{4}}^0 \sec x\tan x &= \Big[\sec x\Big]_{-\frac{\pi}{4}}^0 \\ &\quad \left(\text{because } \tfrac{d}{dx}\sec x = \sec x\tan x\right) \\ &= \sec 0 - \sec -\frac{\pi}{4} \\ &= 1 - \sqrt{2}.\end{aligned}$$

Example 18.6.

$$\begin{aligned}\int_1^4 \left(\frac{3}{2}\sqrt{x} - \frac{4}{x^2}\right)\ dx &= \left[x^{\frac{3}{2}} + \frac{4}{x}\right]_1^4 \\ &\quad \left(\text{because } \tfrac{d}{dx}\left(x^{\frac{3}{2}} + \frac{4}{x}\right) = \frac{3}{2}\sqrt{x} - \frac{4}{x^2}\right) \\ &= \left(4^{\frac{3}{2}} + \frac{4}{4}\right) - \left(1^{\frac{3}{2}} + \frac{4}{1}\right) \\ &= (8+1) - (1+4) \\ &= 4.\end{aligned}$$

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}\int_x^5 3t\sin t\ dt \\ &= \frac{d}{dx}\left(-\int_5^x 3t\sin t\ dt\right) \\ &= -3x\sin x.\end{aligned}$$

Örnek 18.3. $y = \int_1^{x^2} \cos t\ dt$ ise, $\frac{dy}{dx}$'i bulunuz.

çözüm: Bu sefer Zincir kuralı kullanmamız gerekecek. $u = x^2$ diyelim. O zaman

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du}\frac{du}{dx} \\ &= \left(\frac{d}{du}\int_1^u \cos t\ dt\right)\left(\frac{d}{dx}x^2\right) \\ &= (\cos u)\,(2x) = 2x\cos x^2.\end{aligned}$$

Not. Teoremin (ii) kısmı f'nin $[a,b]$ üzerindeki belirli integrali nasıl hesaplayacağımızı söyler :

ADIM 1. f'nin bir ters türevi olan F'yi bulunuz.

ADIM 2. $F(b) - F(a)$ sayısını hesaplayınız.

Notasyon. We will write

$$\Big[F(x)\Big]_a^b = F(b) - F(a).$$

Örnek 18.4.

$$\begin{aligned}\int_0^\pi \cos x\ dx &= \Big[\sin x\Big]_0^\pi \\ &\quad \left(\text{çünkü } \tfrac{d}{dx}\sin x = \cos x\right) \\ &= \sin\pi - \sin 0 \\ &= 0 - 0 \\ &= 0\end{aligned}$$

Örnek 18.5.

$$\begin{aligned}\int_{-\frac{\pi}{4}}^0 \sec x\tan x &= \Big[\sec x\Big]_{-\frac{\pi}{4}}^0 \\ &\quad \left(\text{çünkü } \tfrac{d}{dx}\sec x = \sec x\tan x\right) \\ &= \sec 0 - \sec -\frac{\pi}{4} \\ &= 1 - \sqrt{2}.\end{aligned}$$

Örnek 18.6.

$$\begin{aligned}\int_1^4 \left(\frac{3}{2}\sqrt{x} - \frac{4}{x^2}\right)\ dx &= \left[x^{\frac{3}{2}} + \frac{4}{x}\right]_1^4 \\ &\quad \left(\text{çünkü } \tfrac{d}{dx}\left(x^{\frac{3}{2}} + \frac{4}{x}\right) = \frac{3}{2}\sqrt{x} - \frac{4}{x^2}\right) \\ &= \left(4^{\frac{3}{2}} + \frac{4}{4}\right) - \left(1^{\frac{3}{2}} + \frac{4}{1}\right) \\ &= (8+1) - (1+4) \\ &= 4.\end{aligned}$$

Total Area

Example 18.7. Let $f(x) = x^2 - 4$ and $g(x) = 4 - x^2$. See figure 18.1. We have that

$$\int_{-2}^{2} f(x)\ dx = \int_{-2}^{2} (x^2 - 4)\ dx = \left[\frac{x^3}{3} - 4x\right]_{-2}^{2}$$
$$= \left(\frac{8}{3} - 8\right) - \left(\frac{-8}{3} + 8\right) = -\frac{32}{3}$$

and

$$\int_{-2}^{2} g(x)\ dx = \int_{-2}^{2} (4 - x^2)\ dx = \left[4x - \frac{x^3}{3}\right]_{-2}^{2}$$
$$= \left(8 - \frac{8}{3}\right) - \left(8 + \frac{-8}{3}\right) = \frac{32}{3}.$$

The total area between the graph of $y = f(x)$ and the x-axis, over $[-2, 2]$, is $\left|-\frac{32}{3}\right| = \frac{32}{3}$. The total area between the graph of $y = g(x)$ and the x-axis, over $[-2, 2]$, is $\left|\frac{32}{3}\right| = \frac{32}{3}$.

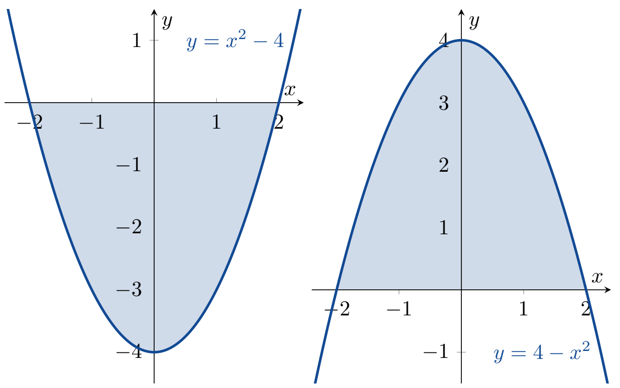


Figure 18.1: Graphs showing $\int_{-2}^{2}(x^2-4)\ dx$ and $\int_{-2}^{2}(4-x^2)\ dx$.
Şekil 18.1:

Example 18.8. Let $f(x) = \sin x$. Calculate

(a). the definite integral of f over $[0, 2\pi]$; and

(b). the total area between the graph of $y = f(x)$ and the x-axis over $[0, 2\pi]$.

solution:

(a).

$$\int_0^{2\pi} \sin x\ dx = \Big[-\cos x\Big]_0^{2\pi} = -\cos 2\pi + \cos 0$$
$$= -1 + 1 = 0.$$

(b).

$$\text{total area} = \int_0^{\pi} \sin x\ dx + \left|\int_{\pi}^{2\pi} \sin x\ dx\right|$$
$$= \Big[-\cos x\Big]_0^{\pi} + \left|\Big[-\cos x\Big]_{\pi}^{2\pi}\right|$$
$$= -\cos\pi + \cos 0 + |-\cos 2\pi + \cos\pi|$$
$$= -(-1) + 1 + |-1 + (-1)| = 4.$$

Toplam Alan

Örnek 18.7. $f(x) = x^2 - 4$ ve $g(x) = 4 - x^2$ olsun. Bkz. şekil 18.1. Burada

$$\int_{-2}^{2} f(x)\ dx = \int_{-2}^{2} (x^2 - 4)\ dx = \left[\frac{x^3}{3} - 4x\right]_{-2}^{2}$$
$$= \left(\frac{8}{3} - 8\right) - \left(\frac{-8}{3} + 8\right) = -\frac{32}{3}$$

ve

$$\int_{-2}^{2} g(x)\ dx = \int_{-2}^{2} (4 - x^2)\ dx = \left[4x - \frac{x^3}{3}\right]_{-2}^{2}$$
$$= \left(8 - \frac{8}{3}\right) - \left(8 + \frac{-8}{3}\right) = \frac{32}{3}.$$

$y = f(x)$ grafiği ve x-ekseni arasında kalan, $[-2, 2]$ üzerindeki toplam alan, is $\left|-\frac{32}{3}\right| = \frac{32}{3}$. $y = g(x)$ ve x-ekseni arasında kalan, $[-2, 2]$ üzerindeki toplam alan, ise $\left|\frac{32}{3}\right| = \frac{32}{3}$ olur.

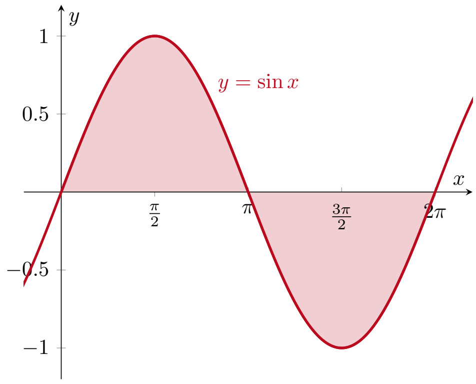


Figure 18.3: The total area between the graph $y = \sin x$ and the x-axis over $[0, 2\pi]$.
Şekil 18.3: $[0, 2\pi]$ üzerinde $y = \sin x$ grafiğiyle x-ekseni arasında kalan toplam alan

Örnek 18.8. $f(x) = \sin x$ olsun.

(a). f'nin $[0, 2\pi]$ üzerindeki belirli integralini; ve

(b). $y = f(x)$ grafiği ile x-ekseni arasında $[0, 2\pi]$ üzerinde kalan alanı bulunuz.

çözüm:

(a).

$$\int_0^{2\pi} \sin x\ dx = \Big[-\cos x\Big]_0^{2\pi} = -\cos 2\pi + \cos 0$$
$$= -1 + 1 = 0.$$

(b).

$$\text{toplam alan} = \int_0^{\pi} \sin x\ dx + \left|\int_{\pi}^{2\pi} \sin x\ dx\right|$$
$$= \Big[-\cos x\Big]_0^{\pi} + \left|\Big[-\cos x\Big]_{\pi}^{2\pi}\right|$$
$$= -\cos\pi + \cos 0 + |-\cos 2\pi + \cos\pi|$$
$$= -(-1) + 1 + |-1 + (-1)| = 4.$$

Summary

To find the ***total area*** between the graph of $y = f(x)$ and the x-axis over $[a, b]$:

STEP 1. Divide $[a, b]$ at the zeroes of f.

STEP 2. Integrate f over each subinterval.

STEP 3. Add the absolute values of the integrals.

Example 18.9. Find the total area between the graph of $y = x^3 - x^2 - 2x$ and the x-axis for $-1 \le x \le 2$.

solution:

1. Let $f(x) = x^3 - x^2 - 2x$. Since $0 = f(x) = x^3 - x^2 - 2x = x(x+1)(x-2)$ implies that $x = 0$ or $x - 1$ or $x = 2$, we divide $[-1, 2]$ into $[-1, 0]$ and $[0, 2]$.
2. We calculate that

$$\int_{-1}^{0} (x^3 - x^2 - 2x)\, dx = \left[\frac{x^4}{4} - \frac{x^3}{3} - x^2\right]_{-1}^{0} = (0 - 0 - 0) - \left(\frac{1}{4} - \frac{1}{3} - 1\right) = \frac{5}{12}$$

and

$$\int_{0}^{2} (x^3 - x^2 - 2x)\, dx = \left[\frac{x^4}{4} - \frac{x^3}{3} - x^2\right]_{0}^{2} = \left(\frac{16}{4} - \frac{8}{3} - 4\right) - (0 - 0 - 0) = -\frac{8}{3}.$$

3. Therefore

$$\text{total area} = \left|\frac{5}{12}\right| + \left|-\frac{8}{3}\right| = \frac{37}{12}.$$

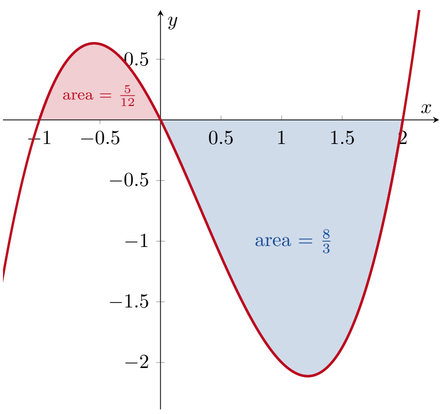


Figure 18.2: The total area between the graph $y = x^3 - x^2 - 2x$ and the x-axis over $[-1, 2]$.
Şekil 18.2: $[-1, 2]$ üzerinde olan, $y = x^3 - x^2 - 2x$ ve x-ekseni arasındaki toplam alan.

Summary

$[a, b]$ üzerindeki $y = f(x)$ grafiği ve x-ekseni arasında kalan ***toplam alanı*** bulmak için:

ADIM 1. f'nin köklerinin olduğu yerlerde $[a, b]$ bölünür .

ADIM 2. Her bir alt-aralık üzerinde f integre edilir.

ADIM 3. Her bir integralin mutlak değerleri toplanır.

Örnek 18.9. $-1 \le x \le 2$ ise $y = x^3 - x^2 - 2x$ grafiği ve x-ekseni arasında kalan alanı bulunuz.

çözüm:

1. $f(x) = x^3 - x^2 - 2x$ olsun. $0 = f(x) = x^3 - x^2 - 2x = x(x+1)(x-2)$ olduğundan $x = 0$ veya $x - 1$ veya $x = 2$ olduğundan, $[-1, 2]$'yi $[-1, 0]$ ve $[0, 2]$'ye ayırırız.
2. Kolayca hesaplanacağı üzere

$$\int_{-1}^{0} (x^3 - x^2 - 2x)\, dx = \left[\frac{x^4}{4} - \frac{x^3}{3} - x^2\right]_{-1}^{0} = (0 - 0 - 0) - \left(\frac{1}{4} - \frac{1}{3} - 1\right) = \frac{5}{12}$$

ve

$$\int_{0}^{2} (x^3 - x^2 - 2x)\, dx = \left[\frac{x^4}{4} - \frac{x^3}{3} - x^2\right]_{0}^{2} = \left(\frac{16}{4} - \frac{8}{3} - 4\right) - (0 - 0 - 0) = -\frac{8}{3}.$$

3. Dolayısıyla

$$\text{toplam alan} = \left|\frac{5}{12}\right| + \left|-\frac{8}{3}\right| = \frac{37}{12}.$$

olur.

The Average Value of a Continuous Function

The average of $\{1, 2, 2, 6, 9\}$ is $\frac{1+2+2+6+9}{5} = \frac{20}{5} = 4$. We can also calculate the average value of a continuous function.

Definition. If f is integrable on $[a, b]$, then the ***average value of f on $[a, b]$*** is

$$\operatorname{av}(f) = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

Example 18.10. Find the average value of $f(x) = \sqrt{4-x^2}$ on $[-2, 2]$.

solution: Since

$$\begin{aligned}\int_{-2}^{2} f(x)\,dx &= \frac{1}{2} \times \text{the area of a circle of radius } 2\\ &= \frac{1}{2}\pi 2^2 = 2\pi,\end{aligned}$$

we have that

$$\operatorname{av}(f) = \frac{1}{2-(-2)}\int_{-2}^{2} f(x)\,dx = \frac{2\pi}{4} = \frac{\pi}{2}.$$

Example 18.11. Find the average value of $g(x) = x^3 - x$ on $[0, 1]$.

solution:

$$\begin{aligned}\operatorname{av}(g) &= \frac{1}{1-0}\int_0^1 g(x)\,dx = \int_0^1 (x^3 - x)\,dx\\ &= \left[\frac{x^4}{4} - \frac{x^2}{2}\right]_0^1 = \frac{1}{4} - \frac{1}{2} = -\frac{1}{4}.\end{aligned}$$

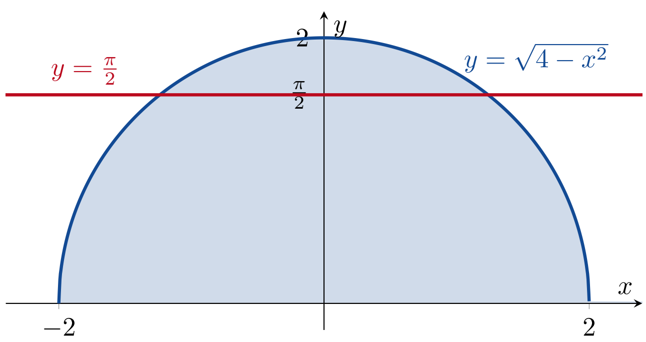


Figure 18.4: The average value of $f(x) = \sqrt{4-x^2}$ on $[-2, 2]$ is $\operatorname{av}(f) = \frac{\pi}{2}$.
Şekil 18.4:

Sürekli Bir Fonksiyonun Ortalama Değeri

$\{1, 2, 2, 6, 9\}$ kümesinin ortalaması $\frac{1+2+2+6+9}{5} = \frac{20}{5} = 4$ tür. Sürekli bir fonksiyonun ortalama değerini de hesaplayabiliriz..

Tanım. $[a, b]$ üzerinde f integrallenebilir ise, ***f'nin $[a, b]$ üzerinde ortalama değeri***

$$\operatorname{ort}(f) = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

Örnek 18.10. $f(x) = \sqrt{4-x^2}$ 'nin $[-2, 2]$ üzerindeki ortalama değerini bulunuz.

çözüm:

$$\begin{aligned}\int_{-2}^{2} f(x)\,dx &= \frac{1}{2} \times 2 \text{ yarıçaplı çemberin alanı}\\ &= \frac{1}{2}\pi 2^2 = 2\pi,\end{aligned}$$

olduğundan,

$$\operatorname{ort}(f) = \frac{1}{2-(-2)}\int_{-2}^{2} f(x)\,dx = \frac{2\pi}{4} = \frac{\pi}{2}.$$

Örnek 18.11. $g(x) = x^3 - x$ 'in $[0, 1]$ üzerindeki ortalama değerini bulunuz.

çözüm:

$$\begin{aligned}\operatorname{ort}(g) &= \frac{1}{1-0}\int_0^1 g(x)\,dx = \int_0^1 (x^3 - x)\,dx\\ &= \left[\frac{x^4}{4} - \frac{x^2}{2}\right]_0^1 = \frac{1}{4} - \frac{1}{2} = -\frac{1}{4}.\end{aligned}$$

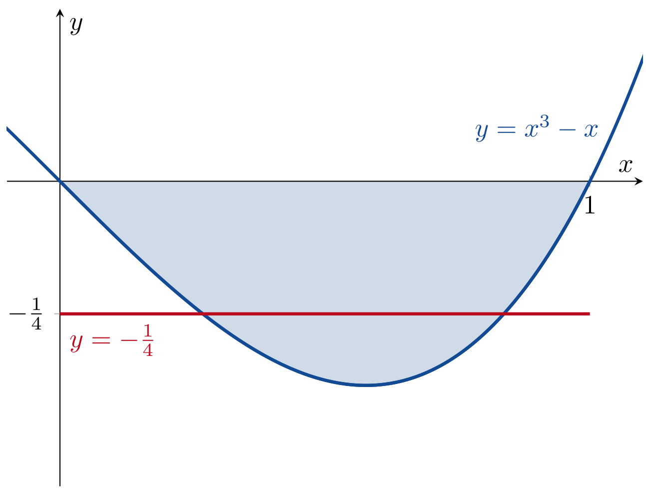


Figure 18.5: The average value of $g(x) = x^3 - x$ on $[0, 1]$ is $\operatorname{av}(g) = -\frac{1}{4}$.
Şekil 18.5: $[0, 1]$ üzerinde $g(x) = x^3 - x$'in ortalama değeri $\operatorname{ort}(g) = -\frac{1}{4}$.

Indefinite Integrals & Definite Integrals

Remember that

$\int f(x)\ dx$ is a function.

For example

$$\int x\ dx = \frac{x^2}{2} + C$$

and

$$\int \cos x\ dx = \sin x + C.$$

Remember that

$\int_a^b f(x)\ dx$ is a number.

For example

$$\int_0^1 x\ dx = \frac{1}{2}$$

and

$$\int_0^{\frac{\pi}{2}} \cos x\ dx = 1.$$

Problems

Problem 18.1 (Definite Integrals)**.** Find the following definite integrals.

(a). $\displaystyle\int_{-2}^{0} (2x+5)\ dx.$

(b). $\displaystyle\int_{-3}^{4} \left(5 - \frac{x}{2}\right)\ dx.$

(c). $\displaystyle\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1-\cos 2t}{2}\ dt.$

(d). $\displaystyle\int_{1}^{32} t^{-\frac{6}{5}}\ dt.$

Problem 18.2 (The Fundamental Theorem of Calculus)**.** Find $\dfrac{dy}{dx}$ if $y = x\displaystyle\int_2^{x^2} \sin(t^3)\ dt.$

Indefinite Integrals & Definite Integrals

Bilinmesi gereken

$\int f(x)\ dx$ bir fonksiyon.

Örneğin

$$\int x\ dx = \frac{x^2}{2} + C$$

ve

$$\int \cos x\ dx = \sin x + C.$$

Bilinmesi gereken

$\int_a^b f(x)\ dx$ bir sayı.

For example

$$\int_0^1 x\ dx = \frac{1}{2}$$

ve

$$\int_0^{\frac{\pi}{2}} \cos x\ dx = 1.$$

Sorular

Soru 18.1 (Belirli İntegraller)**.** Aşağıdaki belirli integralleri bulunuz.

(a). $\displaystyle\int_{-2}^{0} (2x+5)\ dx.$

(b). $\displaystyle\int_{-3}^{4} \left(5 - \frac{x}{2}\right)\ dx.$

(c). $\displaystyle\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1-\cos 2t}{2}\ dt.$

(d). $\displaystyle\int_{1}^{32} t^{-\frac{6}{5}}\ dt.$

Soru 18.2 (Kalkülüsün Temel Teorem of)**.** $y = x\displaystyle\int_2^{x^2} \sin(t^3)\ dt$ ise $\dfrac{dy}{dx}$'i bulunuz.

19

The Substitution Method

Yerine Koyma Yöntemi

The Substitution Method for Indefinite Integrals

By the Chain rule,

$$\frac{d}{dx}\left(\frac{u^{n+1}}{n+1}\right) = u^n \frac{du}{dx}.$$

So

$$\int u^n \frac{du}{dx}\ dx = \frac{u^{n+1}}{n+1} + C.$$

But we know that

$$\int u^n\ du = \frac{u^{n+1}}{n+1} + C$$

also. So it looks like

$$\boxed{du = \frac{du}{dx}\ dx.}$$

Example 19.1. Find $\int (x^3+x)^5(3x^2+1)\ dx$.

solution: Let $u = x^3 + x$. Then $du = \frac{du}{dx}\ dx = (3x^2+1)\ dx$. By substitution, we have that

$$\begin{aligned}\int (x^3+x)^5(3x^2+1)\ dx &= \int u^5\ du \\ &= \frac{u^6}{6} + C = \frac{1}{6}(x^3+x)^6 + C.\end{aligned}$$

Example 19.2. Find $\int \sqrt{2x+1}\ dx$.

solution: Let $u = 2x+1$. Then $du = \frac{du}{dx}\ dx = 2dx$. So $dx = \frac{1}{2}\ du$. Therefore

$$\begin{aligned}\int \sqrt{2x+1}\ dx &= \int u^{\frac{1}{2}}\left(\tfrac{1}{2}du\right) = \frac{1}{2}\int u^{\frac{1}{2}}\ du \\ &= \frac{1}{2}\left(\frac{u^{\frac{3}{2}}}{\frac{3}{2}}\right) + C = \frac{1}{3}(2x+1)^{\frac{3}{2}} + C.\end{aligned}$$

Belirsiz İntegralde Yerine Koyma Yöntemi

Zincir Kuralı gereğince,

$$\frac{d}{dx}\left(\frac{u^{n+1}}{n+1}\right) = u^n \frac{du}{dx}.$$

Bu yüzden

$$\int u^n \frac{du}{dx}\ dx = \frac{u^{n+1}}{n+1} + C.$$

Biliyoruz ki

$$\int u^n\ du = \frac{u^{n+1}}{n+1} + C$$

doğrudur. Yani şuna benziyor.

$$\boxed{du = \frac{du}{dx}\ dx.}$$

Örnek 19.1. $\int (x^3+x)^5(3x^2+1)\ dx$'i bulunuz.

çözüm: $u = x^3+x$. olsun. Öyleyse $du = \frac{du}{dx}\ dx = (3x^2+1)\ dx$. Değişken değiştirerek, şunu bulmak mümkün

$$\begin{aligned}\int (x^3+x)^5(3x^2+1)\ dx &= \int u^5\ du \\ &= \frac{u^6}{6} + C = \frac{1}{6}(x^3+x)^6 + C.\end{aligned}$$

Örnek 19.2. $\int \sqrt{2x+1}\ dx$'i bulunuz.

çözüm: Diyelim ki $u = 2x+1$. O zaman $du = \frac{du}{dx}\ dx = 2dx$ olur. Yani $dx = \frac{1}{2}\ du$. Böyle olunca

$$\begin{aligned}\int \sqrt{2x+1}\ dx &= \int u^{\frac{1}{2}}\left(\tfrac{1}{2}du\right) = \frac{1}{2}\int u^{\frac{1}{2}}\ du \\ &= \frac{1}{2}\left(\frac{u^{\frac{3}{2}}}{\frac{3}{2}}\right) + C = \frac{1}{3}(2x+1)^{\frac{3}{2}} + C.\end{aligned}$$

Theorem 19.1 (The Substitution Method). *If*

- $u = g(x)$ *is differentiable;*
- $g : \mathbb{R} \to I$*; and*
- $f : I \to \mathbb{R}$ *is continuous,*

then

$$\int f(g(x))g'(x)\ dx = \int f(u)\ du.$$

Example 19.3. Find $\int 5\sec^2(5t+1)\ dt$.

solution: Let $u = 5t+1$. Then $du = \frac{du}{dt}\ dt = 5dt$. So

$$\begin{aligned}\int 5\sec^2(5t+1)\ dt &= \int \sec^2 u\ du \\ &= \tan u + C \\ &\quad \left(\text{because } \frac{d}{du}\tan u = \sec^2 u\right) \\ &= \tan(5t+1) + C.\end{aligned}$$

Example 19.4. Find $\int \cos(7\theta+3)\ d\theta$.

solution: Let $u = 7\theta+3$. Then $du = \frac{du}{d\theta}\ d\theta = 7d\theta$. So $d\theta = \frac{1}{7}du$ and

$$\begin{aligned}\int \cos(7\theta+3)\ d\theta &= \frac{1}{7}\int \cos u\ du \\ &= \frac{1}{7}\sin u + C = \frac{1}{7}\sin(7\theta+3) + C.\end{aligned}$$

Example 19.5. Find $\int x^2\sin(x^3)\ dx$.

solution: Let $u = x^3$. Then $du = \frac{du}{dx}\ dx = 3x^2\ dx$. So $\frac{1}{3}du = x^2\ dx$ and

$$\begin{aligned}\int x^2\sin(x^3)\ dx &= \int \frac{1}{3}\sin u\ du = -\frac{1}{3}\cos u + C \\ &= -\frac{1}{3}\cos(x^3) + C.\end{aligned}$$

Example 19.6. Find $\int x\sqrt{2x+1}\ dx$.

solution: Let $u = 2x+1$. Then $du = \frac{du}{dx}\ dx = 2dx$. So $dx = \frac{1}{2}du$ and

$$\int x\sqrt{2x+1}\ dx = \int x\sqrt{u}\ \frac{1}{2}du.$$

But we still have an x here. We can't integrate until we change all the x terms to u terms. Note that

$$u = 2x+1 \implies u-1 = 2x \implies \frac{1}{2}(u-1) = x.$$

Teorem 19.1 (Yerine Koyma Yöntemi).

- $u = g(x)$ *türevlenebilir;*
- $g : \mathbb{R} \to I$*; ve*
- $f : I \to \mathbb{R}$ *sürekli,*

bunun üzerine

$$\int f(g(x))g'(x)\ dx = \int f(u)\ du.$$

Örnek 19.3. $\int 5\sec^2(5t+1)\ dt$'yi bulunuz.

çözüm: $u = 5t+1$ diyelim. Buradan $du = \frac{du}{dt}\ dt = 5dt$ olur. Yani

$$\begin{aligned}\int 5\sec^2(5t+1)\ dt &= \int \sec^2 u\ du \\ &= \tan u + C \\ &\quad \left(\frac{d}{du}\tan u = \sec^2 u \text{ olduğundan}\right) \\ &= \tan(5t+1) + C.\end{aligned}$$

Örnek 19.4. $\int \cos(7\theta+3)\ d\theta$'yı bulunuz.

çözüm: $u = 7\theta+3$ olsun. Buradan $du = \frac{du}{d\theta}\ d\theta = 7d\theta$. Böylece $d\theta = \frac{1}{7}du$ ve

$$\begin{aligned}\int \cos(7\theta+3)\ d\theta &= \frac{1}{7}\int \cos u\ du \\ &= \frac{1}{7}\sin u + C = \frac{1}{7}\sin(7\theta+3) + C.\end{aligned}$$

bulunur.

Örnek 19.5. $\int x^2\sin(x^3)\ dx$'i bulunuz.

çözüm: $u = x^3$ olsun. Yani $du = \frac{du}{dx}\ dx = 3x^2\ dx$. Böylece $\frac{1}{3}du = x^2\ dx$ ve

$$\begin{aligned}\int x^2\sin(x^3)\ dx &= \int \frac{1}{3}\sin u\ du = -\frac{1}{3}\cos u + C \\ &= -\frac{1}{3}\cos(x^3) + C.\end{aligned}$$

bulunur.

Örnek 19.6. $\int x\sqrt{2x+1}\ dx$'i bulunuz.

çözüm: $u = 2x+1$ diyelim. Bu durumda $du = \frac{du}{dx}\ dx = 2dx$ olur. Yani $dx = \frac{1}{2}du$ ve

$$\int x\sqrt{2x+1}\ dx = \int x\sqrt{u}\ \frac{1}{2}du$$

buluruz. Elimizde hala x var. Bütün x'li terimleri u'lu terimlere dönüştürmedikçe integre edemiyoruz. Şunu akılda tutarak,

$$u = 2x+1 \implies u-1 = 2x \implies \frac{1}{2}(u-1) = x.$$

Therefore

$$\begin{aligned}\int x\sqrt{2x+1}\,dx &= \int \frac{1}{2}(u-1)\sqrt{u}\,\frac{1}{2}du\\ &= \frac{1}{4}\int u^{\frac{3}{2}} - u^{\frac{1}{2}}\,du\\ &= \frac{1}{4}\left(\frac{2}{5}u^{\frac{5}{2}} - \frac{2}{3}u^{\frac{3}{2}}\right) + C\\ &= \frac{1}{10}u^{\frac{5}{2}} - \frac{1}{6}u^{\frac{3}{2}} + C\\ &= \frac{1}{10}(2x+1)^{\frac{5}{2}} - \frac{1}{6}(2x+1)^{\frac{3}{2}} + C.\end{aligned}$$

Example 19.7. Find $\int \frac{2z}{\sqrt[3]{z^2+1}}\,dz$.

solution: Let $u = z^2+1$. Then $du = \frac{du}{dx}\,dx = 2z\,dz$ and

$$\begin{aligned}\int \frac{2z}{\sqrt[3]{z^2+1}}\,dz &= \int \frac{du}{u^{\frac{1}{3}}}\\ &= \int u^{-\frac{1}{3}}\,du\\ &= \frac{u^{\frac{2}{3}}}{\frac{2}{3}} + C\\ &= \frac{3}{2}u^{\frac{2}{3}} + C\\ &= \frac{3}{2}(z^2+1)^{\frac{2}{3}} + C.\end{aligned}$$

Example 19.8. Find $\int \sin^2 x\,dx$.

solution: We use the identity

$$\sin^2 x = \frac{1-\cos 2x}{2}$$

to calculate that

$$\begin{aligned}\int \sin^2 x\,dx &= \int \frac{1-\cos 2x}{2}\,dx\\ &= \frac{1}{2}\int (1-\cos 2x)\,dx\\ &= \frac{1}{2}\left(x - \frac{1}{2}\sin 2x\right) + C\\ &= \frac{1}{2}x - \frac{1}{4}\sin 2x + C.\end{aligned}$$

Example 19.9. Similarly

$$\int \cos^2 x\,dx = \int \frac{1+\cos 2x}{2}\,dx = \frac{1}{2}x + \frac{1}{4}\sin 2x + C.$$

Bu yüzden

$$\begin{aligned}\int x\sqrt{2x+1}\,dx &= \int \frac{1}{2}(u-1)\sqrt{u}\,\frac{1}{2}du\\ &= \frac{1}{4}\int u^{\frac{3}{2}} - u^{\frac{1}{2}}\,du\\ &= \frac{1}{4}\left(\frac{2}{5}u^{\frac{5}{2}} - \frac{2}{3}u^{\frac{3}{2}}\right) + C\\ &= \frac{1}{10}u^{\frac{5}{2}} - \frac{1}{6}u^{\frac{3}{2}} + C\\ &= \frac{1}{10}(2x+1)^{\frac{5}{2}} - \frac{1}{6}(2x+1)^{\frac{3}{2}} + C.\end{aligned}$$

bulunmuş olur.

Örnek 19.7. $\int \frac{2z}{\sqrt[3]{z^2+1}}\,dz$ integralini bulunuz.

çözüm: $u = z^2+1$ diyelim. Buradan $du = \frac{du}{dx}\,dx = 2z\,dz$ ve oradan da

$$\begin{aligned}\int \frac{2z}{\sqrt[3]{z^2+1}}\,dz &= \int \frac{du}{u^{\frac{1}{3}}}\\ &= \int u^{-\frac{1}{3}}\,du\\ &= \frac{u^{\frac{2}{3}}}{\frac{2}{3}} + C\\ &= \frac{3}{2}u^{\frac{2}{3}} + C\\ &= \frac{3}{2}(z^2+1)^{\frac{2}{3}} + C.\end{aligned}$$

elde edilir

Örnek 19.8. $\int \sin^2 x\,dx$ integralini bulunuz.

çözüm: Burada kullanacağımız özdeşlik

$$\sin^2 x = \frac{1-\cos 2x}{2}$$

ve buradan da

$$\begin{aligned}\int \sin^2 x\,dx &= \int \frac{1-\cos 2x}{2}\,dx\\ &= \frac{1}{2}\int (1-\cos 2x)\,dx\\ &= \frac{1}{2}\left(x - \frac{1}{2}\sin 2x\right) + C\\ &= \frac{1}{2}x - \frac{1}{4}\sin 2x + C\end{aligned}$$

bulunur.

Örnek 19.9. Benzer şekilde

$$\int \cos^2 x\,dx = \int \frac{1+\cos 2x}{2}\,dx = \frac{1}{2}x + \frac{1}{4}\sin 2x + C$$

bulunur.

The Substitution Method for Definite Integrals

Theorem 19.2 (The Substitution Method)**.** *If*

- $u = g(x)$ *is differentiable on* $[a, b]$*;*
- g' *is continuous on* $[a, b]$*; and*
- f *is continous on the range of* g*,*

then

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du.$$

Example 19.10. Calculate $\displaystyle\int_{-1}^{1} 3x^2\sqrt{x^3+1}\,dx$.

solution 1. We can use the previous theorem to solve this example. Let $u = x^3 + 1$. Then $du = 3x^2\,dx$. Moreover $x = -1 \implies u = 0$ and $x = 1 \implies u = 2$. So

$$\int_{x=-1}^{x=1} 3x^2\sqrt{x^3+1}\,dx = \int_{u=0}^{u=2} \sqrt{u}\,du = \left[\frac{2}{3}u^{\frac{3}{2}}\right]_0^2$$
$$= \frac{2}{3}\left(2^{\frac{3}{2}} - 0^{\frac{3}{2}}\right) = \frac{2}{3}2\sqrt{2} = \frac{4\sqrt{2}}{3}.$$

solution 2. Alternately, we can first find the indefinite integral, then find the required definite integral.

Let $u = x^3 + 1$. Then $du = 3x^2\,dx$. So

$$\int 3x^2\sqrt{x^3+1}\,dx = \int \sqrt{u}\,du = \frac{2}{3}u^{\frac{3}{2}} + C = \frac{2}{3}(x^3+1)^{\frac{3}{2}} + C.$$

Therefore

$$\int_{-1}^{1} 3x^2\sqrt{x^3+1}\,dx = \left[\frac{2}{3}(x^3+1)^{\frac{3}{2}}\right]_{-1}^{1}$$
$$= \left(\frac{2}{3}(1+1)^{\frac{3}{2}}\right) - \left(\frac{2}{3}(-1+1)^{\frac{3}{2}}\right)$$
$$= \frac{2}{3} \times 2^{\frac{3}{2}} = \frac{4\sqrt{2}}{3}.$$

Example 19.11. Calculate $\displaystyle\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos\theta\,\mathrm{cosec}^2\,\theta\,d\theta$.

solution: Let $u = \cot\theta$. Then $du = \frac{du}{d\theta}\,d\theta = -\mathrm{cosec}^2\,\theta\,d\theta$. So $-du = \mathrm{cosec}^2\,\theta\,d\theta$. Moreover $\theta = \frac{\pi}{4} \implies u = \cot\frac{\pi}{4} = 1$ and $\theta = \frac{\pi}{2} \implies u = \cot\frac{\pi}{2} = 0$. Hence

$$\int_{\theta=\frac{\pi}{4}}^{\theta=\frac{\pi}{2}} \cos\theta\,\mathrm{cosec}^2\,\theta\,d\theta = \int_{u=1}^{u=0} u\,(-du) = -\int_1^0 u\,du$$
$$= -\left[\frac{u^2}{2}\right]_1^0 = -\left(\frac{0^2}{2} - \frac{1^2}{2}\right) = \frac{1}{2}.$$

Belirli İntegralde Değişken Değiştirme

Teorem 19.2 (Değişken Değiştirme Yöntemi)**.** *Eğer*

- $u = g(x)$ *fonksiyonu* $[a, b]$*'de türevliyse;*
- g' *fonksiyonu* $[a, b]$*'de sürekliyse; ve*
- f *fonksiyonu da* g*'nin görüntü kümesinde sürekliyse,*

bu durumda

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$$

olur.

Örnek 19.10. $\displaystyle\int_{-1}^{1} 3x^2\sqrt{x^3+1}\,dx$ integralini bulunuz.

çözüm 1. Bu soruyu yapmak için önceki teoremi kullanabiliriz. Diyelim ki, $u = x^3 + 1$ olsun. Bu durumda $du = 3x^2\,dx$ olur. Ayrıca $x = -1 \implies u = 0$ ve $x = 1 \implies u = 2$ olur. Buradan

$$\int_{x=-1}^{x=1} 3x^2\sqrt{x^3+1}\,dx = \int_{u=0}^{u=2} \sqrt{u}\,du = \left[\frac{2}{3}u^{\frac{3}{2}}\right]_0^2$$
$$= \frac{2}{3}\left(2^{\frac{3}{2}} - 0^{\frac{3}{2}}\right) = \frac{2}{3}2\sqrt{2} = \frac{4\sqrt{2}}{3}$$

bulunmuş olur.

çözüm 2. Değişimli olarak, önce belirsiz integrali bulur, daha sonra da belirli integrali bulabiliriz.

Şimdi $u = x^3 + 1$ olsun. Buradan $du = 3x^2\,dx$ olur. Bu sebeple

$$\int 3x^2\sqrt{x^3+1}\,dx = \int \sqrt{u}\,du = \frac{2}{3}u^{\frac{3}{2}} + C = \frac{2}{3}(x^3+1)^{\frac{3}{2}} + C.$$

Böylece

$$\int_{-1}^{1} 3x^2\sqrt{x^3+1}\,dx = \left[\frac{2}{3}(x^3+1)^{\frac{3}{2}}\right]_{-1}^{1}$$
$$= \left(\frac{2}{3}(1+1)^{\frac{3}{2}}\right) - \left(\frac{2}{3}(-1+1)^{\frac{3}{2}}\right)$$
$$= \frac{2}{3} \times 2^{\frac{3}{2}} = \frac{4\sqrt{2}}{3}.$$

Örnek 19.11. $\displaystyle\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos\theta\,\mathrm{cosec}^2\,\theta\,d\theta$'yı bulunuz.

çözüm: $u = \cot\theta$ olsun. Buradan $du = \frac{du}{d\theta}\,d\theta = -\mathrm{cosec}^2\,\theta\,d\theta$ olur. Böylece $-du = \mathrm{cosec}^2\,\theta\,d\theta$ bulunur. Ayrıca $\theta = \frac{\pi}{4} \implies u = \cot\frac{\pi}{4} = 1$ ve $\theta = \frac{\pi}{2} \implies u = \cot\frac{\pi}{2} = 0$ bulunur. Bunun sonucu olarak da

$$\int_{\theta=\frac{\pi}{4}}^{\theta=\frac{\pi}{2}} \cos\theta\,\mathrm{cosec}^2\,\theta\,d\theta = \int_{u=1}^{u=0} u\,(-du) = -\int_1^0 u\,du$$
$$= -\left[\frac{u^2}{2}\right]_1^0 = -\left(\frac{0^2}{2} - \frac{1^2}{2}\right) = \frac{1}{2}$$

bulunur

Problems

Problem 19.1. Use a substitution to evaluate the following indefinite integrals. You must show your working.

(a). $\displaystyle\int r^4 \left(7 - \frac{r^5}{10}\right)^3 \, dr.$

(b). $\displaystyle\int x(x-1)^{10} \, dx.$

Problem 19.2. Use a substitution to evaluate the following definite integrals. You must show your working.

(a). $\displaystyle\int_{-\frac{\pi}{2}}^{0} \frac{\sin x}{(3 + 2\cos x)^2} \, dx.$

(b). $\displaystyle\int_{0}^{\frac{\pi}{2}} \frac{\sin x}{(3 + 2\cos x)^2} \, dx.$

Sorular

Soru 19.1. Yerine koyma (değişken değiştirme) yöntemi kullanarak, aşağıdaki belirsiz integralleri bulunuz. İşlemlerini açıklamalı

(a). $\displaystyle\int r^4 \left(7 - \frac{r^5}{10}\right)^3 \, dr.$

(b). $\displaystyle\int x(x-1)^{10} \, dx.$

Soru 19.2. Yerine koyma (değişken değiştirme) yöntemi kullanarak, aşağıdaki belirli integralleri bulunuz. İşlemlerini açıklamalısı

(a). $\displaystyle\int_{-\frac{\pi}{2}}^{0} \frac{\sin x}{(3 + 2\cos x)^2} \, dx.$

(b). $\displaystyle\int_{0}^{\frac{\pi}{2}} \frac{\sin x}{(3 + 2\cos x)^2} \, dx.$

20

Area Between Curves

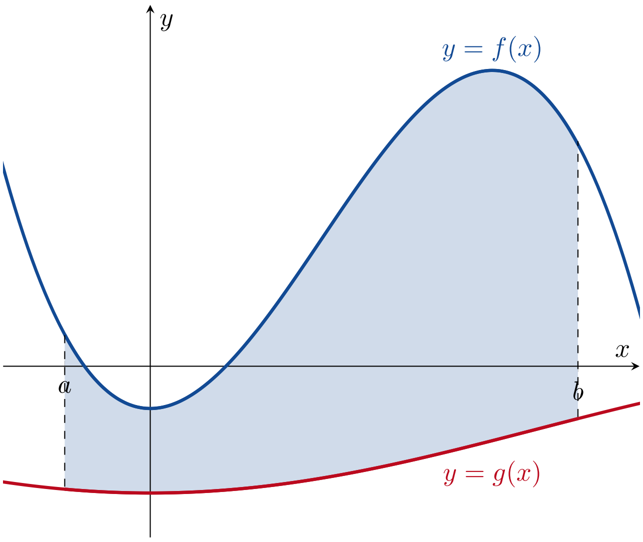


Figure 20.1: The region between the curves $y = f(x)$ and $y = g(x)$ for $a \leq x \leq b$.
Şekil 20.1:

Definition. If

- f is continous;
- g is continous; and
- $f(x) \geq g(x)$ on $[a, b]$,

then the ***area of the region between the curves*** $y = f(x)$ ***and*** $y = g(x)$ ***for*** $a \leq x \leq b$ is

$$\text{area} = \int_a^b \Big(f(x) - g(x) \Big)\, dx.$$

Example 20.1. Find the area between $y = 2 - x^2$ and $y = -x$.

solution: First we need to find the limits of integration:

Eğriler Arasındaki Alanlar

Tanım. Eğer

- f sürekli;
- g sürekli; ve
- $[a, b]$ üzerinde $f(x) \geq g(x)$ 'se,

o zaman $a \leq x \leq b$ ***oldukça*** $y = f(x)$ ***ve*** $y = g(x)$ ***eğrileri arasındaki alan***

$$\text{alan} = \int_a^b \Big(f(x) - g(x) \Big)\, dx.$$

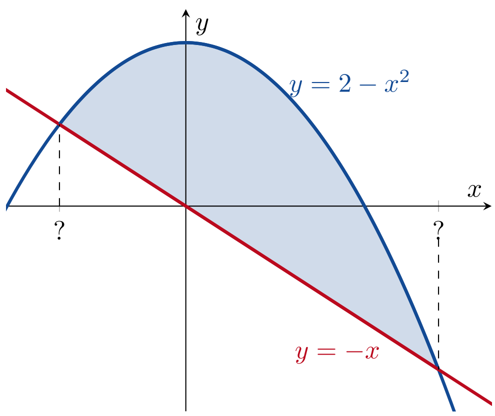


Figure 20.2: The region between the curves $y = 2 - x^2$ and $y = -x$.
Şekil 20.2: $y = 2 - x^2$ ve $y = -x$ arasındaki bölge

Örnek 20.1. $y = 2 - x^2$ ve $y = -x$ arasındaki alanı bulunuz.

çözüm: İlk olarak integrasyon sınırlarını buluruz:

$$\begin{aligned} 2 - x^2 &= -x \\ 0 &= x^2 - x - 2 \\ 0 &= (x+1)(x-2) \implies x = -1 \text{ veya } 2. \end{aligned}$$

$$\begin{aligned} 2 - x^2 &= -x \\ 0 &= x^2 - x - 2 \\ 0 &= (x+1)(x-2) \implies x = -1 \text{ or } 2. \end{aligned}$$

We need to integrate from $x = -1$ to $x = 2$. Therefore

$$\begin{aligned} \text{area} &= \int_{-1}^{2} \Big((2-x^2) - (-x) \Big)\, dx \\ &= \int_{-1}^{2} (2 + x - x^2)\, dx \\ &= \left[2x + \tfrac{1}{2}x^2 - \tfrac{1}{3}x^3 \right]_{-1}^{2} \\ &= \left(4 + \frac{4}{2} - \frac{8}{3}\right) - \left(-2 + \frac{1}{2} + \frac{1}{3}\right) \\ &= \frac{9}{2}. \end{aligned}$$

Example 20.2. Find the area bounded by $y = \sqrt{x}$, $y = x - 2$ and the x-axis, for $x \geq 0$ and $y \geq 0$.

solution: First we calculate that

$$\begin{aligned} \sqrt{x} &= x - 2 \\ x &= (x-2)^2 = x^2 - 4x + 4 \\ 0 &= x^2 - 5x + 4 = (x-1)(x-4) \implies x = 1 \text{ or } 4. \end{aligned}$$

Since $\sqrt{1} \neq 1 - 2$, we must have $x = 4$. See figure 20.3. Therefore

$$\begin{aligned} \text{area} &= \text{blue area} + \text{red area} \\ &= \int_0^2 \sqrt{x}\, dx + \int_2^4 \Big(\sqrt{x} - (x-2) \Big)\, dx \\ &= \int_0^2 x^{\frac{1}{2}}\, dx + \int_2^4 (x^{\frac{1}{2}} - x + 2)\, dx \\ &= \left[\frac{2}{3}x^{\frac{3}{2}} \right]_0^2 + \left[\frac{2}{3}x^{\frac{3}{2}} - \frac{1}{2}x^2 + 2x \right]_2^4 \\ &= \left(\frac{2}{3}(2)^{\frac{3}{2}} - 0 \right) + \left(\frac{2}{3}(4)^{\frac{3}{2}} - \frac{1}{2}(16) + 2(4) \right) \\ &\quad - \left(\frac{2}{3}(2)^{\frac{3}{2}} - \frac{1}{2}(4) + 2(2) \right) \\ &= \frac{4\sqrt{2}}{3} + \frac{16}{3} - 8 + 8 - \frac{4\sqrt{2}}{3} + 2 - 4 \\ &= \frac{10}{3}. \end{aligned}$$

$x = -1$ den $x = 2$'ye integre ederiz. Böylece

$$\begin{aligned} \text{area} &= \int_{-1}^{2} \Big((2-x^2) - (-x) \Big)\, dx \\ &= \int_{-1}^{2} (2 + x - x^2)\, dx \\ &= \left[2x + \tfrac{1}{2}x^2 - \tfrac{1}{3}x^3 \right]_{-1}^{2} \\ &= \left(4 + \frac{4}{2} - \frac{8}{3}\right) - \left(-2 + \frac{1}{2} + \frac{1}{3}\right) \\ &= \frac{9}{2}. \end{aligned}$$

Örnek 20.2. $x \geq 0$ ve $y \geq 0$ olmak üzere $y = \sqrt{x}$, $y = x - 2$ ve x-ekseni ile sınırlı alanı bulunuz.

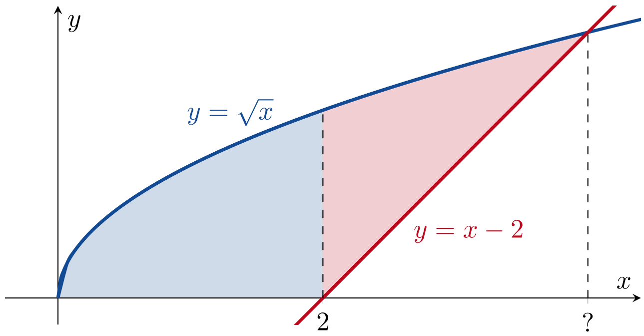


Figure 20.3: The region between the curves $y = \sqrt{x}$, $y = x - 2$ and the x-axis for $x \geq 0$ and $y \geq 0$.
Şekil 20.3: $x \geq 0$ and $y \geq 0$ olduğunda $y = \sqrt{x}$, $y = x - 2$ ve x-ekseni ile sınırlı bölge.

çözüm: İlk olarak

$$\begin{aligned} \sqrt{x} &= x - 2 \\ x &= (x-2)^2 = x^2 - 4x + 4 \\ 0 &= x^2 - 5x + 4 = (x-1)(x-4) \implies x = 1 \text{ veya } 4. \end{aligned}$$

$\sqrt{1} \neq 1 - 2$ olduğundan, $x = 4$ buluruz. Bkz. şekil 20.3. Buradan

$$\begin{aligned} \text{area} &= \text{mavi alan} + \text{kırmızı alan} \\ &= \int_0^2 \sqrt{x}\, dx + \int_2^4 \Big(\sqrt{x} - (x-2) \Big)\, dx \\ &= \int_0^2 x^{\frac{1}{2}}\, dx + \int_2^4 (x^{\frac{1}{2}} - x + 2)\, dx \\ &= \left[\frac{2}{3}x^{\frac{3}{2}} \right]_0^2 + \left[\frac{2}{3}x^{\frac{3}{2}} - \frac{1}{2}x^2 + 2x \right]_2^4 \\ &= \left(\frac{2}{3}(2)^{\frac{3}{2}} - 0 \right) + \left(\frac{2}{3}(4)^{\frac{3}{2}} - \frac{1}{2}(16) + 2(4) \right) \\ &\quad - \left(\frac{2}{3}(2)^{\frac{3}{2}} - \frac{1}{2}(4) + 2(2) \right) \\ &= \frac{4\sqrt{2}}{3} + \frac{16}{3} - 8 + 8 - \frac{4\sqrt{2}}{3} + 2 - 4 \\ &= \frac{10}{3} \end{aligned}$$

elde edilir.

Problems

Problem 20.1 (Total Area)**.** Calculate the total area between the curve $y = 2x^2$ and the curve $y = x^4 - 2x^2$ for $-2 \le x \le 2$.

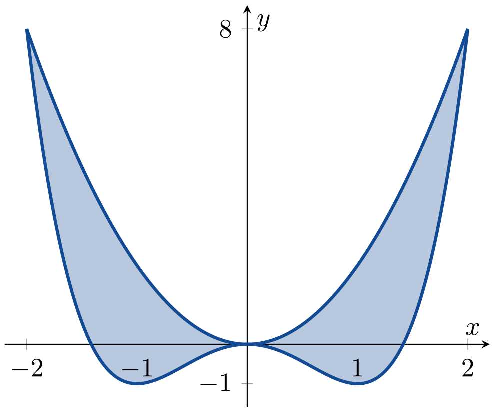


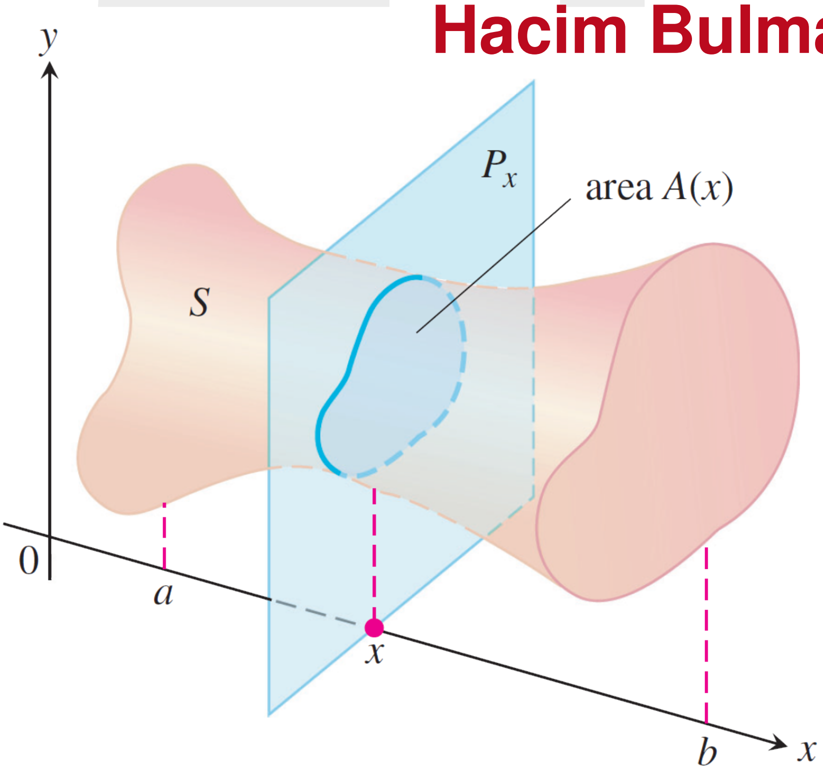
Figure 20.4: The total area between the curve $y = 2x^2$ and the curve $y = x^4 - 2x^2$ for $-2 \le x \le 2$.
Şekil 20.4:

Sorular

Soru 20.1 (Toplam Alan)**.** $y = 2x^2$ eğrisiyle $y = x^4 - 2x^2$ eğrisi arasındaki alanı $-2 \le x \le 2$ ise bulunuz.

Volumes Using Cross Sections

Dik-Kesitler Kullanarak Hacim Bulmak



Definition. The ***volume*** of a solid of integrable cross-sectional area $A(x)$ from $x = a$ to $x = b$ is

$$\boxed{\text{volume} = \int_a^b A(x)\ dx.}$$

Tanım. İntegrallenebilir $A(x)$ kesitinin $x = a$'dan $x = b$'ye olan alanının ***hacmi***

$$\boxed{\text{hacim} = \int_a^b A(x)\ dx.}$$

Example 21.1. A 3 metres tall pyramid has a square 3 metres × 3 metres base, as shown in figure 21.1. The cross-section x metres from the vertex is an x m × x m square. Find the volume of the pyramid.

solution:

STEP 1. Draw a picture: See figure 21.2.

STEP 2. Find a formula for $A(x)$: $A(x) = x^2$.

STEP 3. Find the limits of integration: $0 \le x \le 3$.

STEP 4. Integrate:

$$\text{volume} = \int_a^b A(x)\ dx = \int_0^3 x^2\ dx = \left[\frac{1}{3}x^3\right]_0^3 = 9\,\text{m}^3.$$

Örnek 21.1. 3 metre yüksekliğinde bir piramitin tabanı kenarı 3 metre olan bir karedir, şekil 21.1'de gösterildiği gibi. Kesit köşesinden x metre olan bir x m × x m karedir. Piramitin hacmini bulunuz.

çözüm:

ADIM 1. Şekil çizilir: Bkz şekil 21.2.

ADIM 2. $A(x)$: $A(x) = x^2$.'e ait bir formül bulunur

ADIM 3. İntegrasyon limitleri bulunur: $0 \le x \le 3$.

ADIM 4. İntegral hesaplanır:

$$\text{volume} = \int_a^b A(x)\ dx = \int_0^3 x^2\ dx = \left[\frac{1}{3}x^3\right]_0^3 = 9\,\text{m}^3.$$

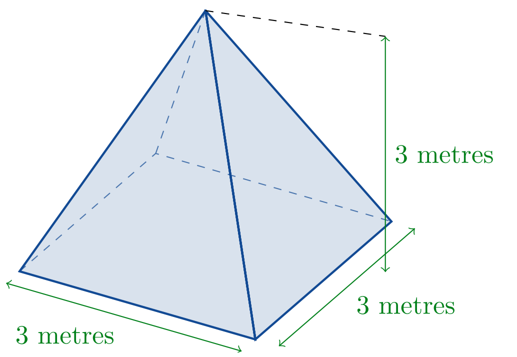


Figure 21.1: A 3 metres tall pyramid with 3m×3m base.
Şekil 21.1:

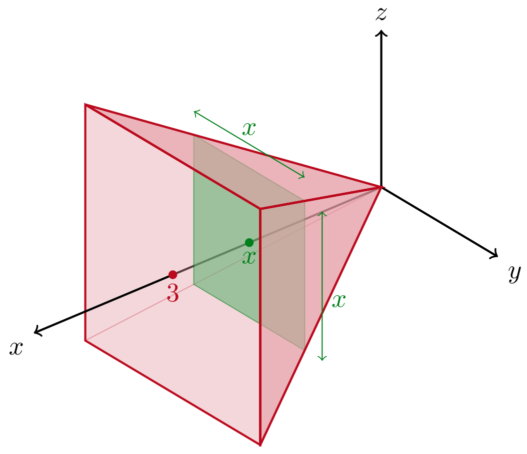


Figure 21.2: A 3 metres tall pyramid with 3m×3m base.
Şekil 21.2:

Example 21.2. A curved wedge is cut from a cylinder of radius 3 by two planes. The first plane is perpendicular to the the axis of the cylinder. The second plane crosses the first plane with an angle of $45° = \frac{\pi}{4}$ at the centre of the cylinder. See figure 21.3. Find the volume of the wedge.

Örnek 21.2. Şekildeki gibi eğri bir takoz 3 yarıçaplı silindir iki düzlemle kesilerek elde ediliyor. Birinci düzlem silindirin eksenine diktir. İkinci düzlem de birinci düzlemle silindirin merkezinde $45° = \frac{\pi}{4}$'lik açı yapıyor. Bkz şekil 21.3. Takozun hacmini bulunuz.

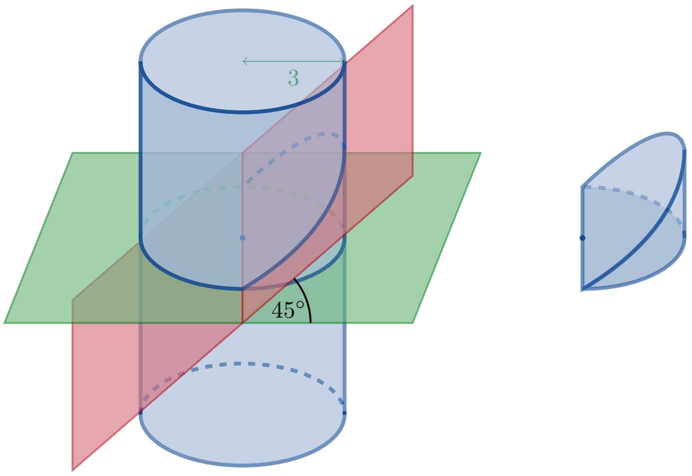


Figure 21.3: A wedge cut from a cylinder.
Şekil 21.3:

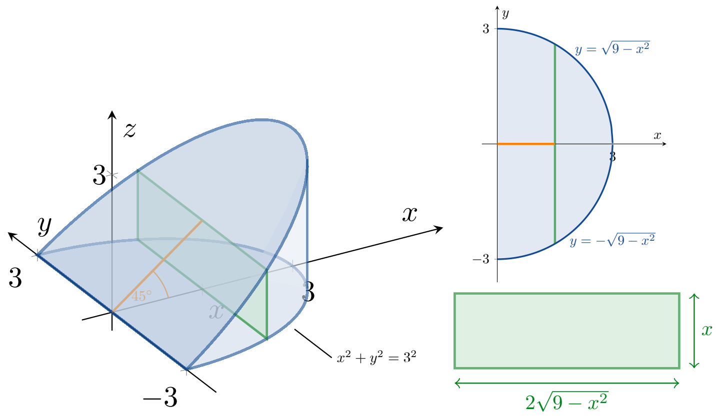


Figure 21.4: A wedge cut from a cylinder.
Şekil 21.4:

solution: The cross-sectional area is

$$A(x) = 2x\sqrt{9 - x^2}$$

for $0 \le x \le 3$. Therefore

$$\text{volume} = \int_0^3 2x\sqrt{9 - x^2}\, dx$$

We need to make a substitution. Let $u = 9 - x^2$. Then $du = -2x\, dx$ and

$$\begin{aligned}\text{volume} &= \int_{x=0}^{x=3} -u^{\frac{1}{2}}\, du = \left[-\frac{2}{3}u^{\frac{3}{2}}\right]_{x=0}^{x=3} \\ &= \left[-\frac{2}{3}(9 - x^2)^{\frac{3}{2}}\right]_{x=0}^{x=3} = 0 - \frac{2}{3}(9)^{\frac{3}{2}} \\ &= 18.\end{aligned}$$

çözüm: Kesit alanı

$$A(x) = 2x\sqrt{9 - x^2}$$

for $0 \le x \le 3$. Buradan

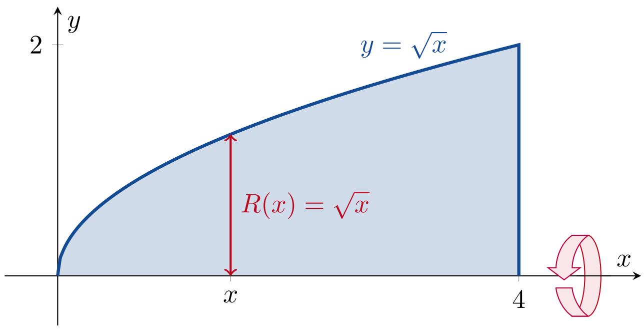
$$\text{hacim} = \int_0^3 2x\sqrt{9 - x^2}\, dx$$

Değişken değiştirilir. Burada $u = 9 - x^2$ denilir. Bu durumda

$du = -2x\ dx$ ve

$$\begin{aligned}\text{hacim} &= \int_{x=0}^{x=3} -u^{\frac{1}{2}}\ du = \left[-\frac{2}{3}u^{\frac{3}{2}}\right]_{x=0}^{x=3} \\ &= \left[-\frac{2}{3}(9-x^2)^{\frac{3}{2}}\right]_{x=0}^{x=3} = 0 - \frac{2}{3}(9)^{\frac{3}{2}} \\ &= 18.\end{aligned}$$

Solids of Revolution



Definition. The solid generated by rotating a plane region about a line in the plane is called a ***solid of revolution***.

$$\text{volume} = \int_a^b A(x)\ dx = \int_a^b \pi\Big(R(x)\Big)^2\ dx$$

Example 21.3. The region between the curve $y = \sqrt{x}$ and the x-axis, for $0 \leq x \leq 4$, is rotated about the x-axis to generate a sold. Find its volume.

solution:

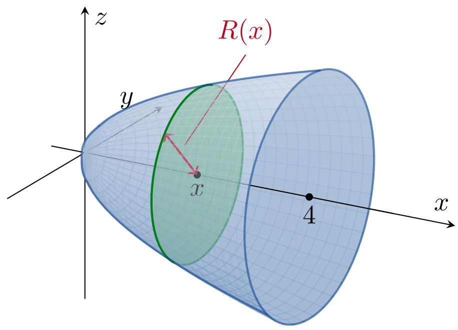
$$\begin{aligned}\text{volume} &= \int_a^b \pi\Big(R(x)\Big)^2\ dx = \int_0^4 \pi\Big(\sqrt{x}\Big)^2\ dx \\ &= \pi\int_0^4 x\ dx = \pi\left[\frac{1}{2}x^2\right]_0^4 = 8\pi.\end{aligned}$$

Example 21.4. Find the volume of a sphere of radius a.

solution: To generate a sphere, we rotate the area between $y = \sqrt{a^2 - x^2}$ and the x-axis about the x-axis. Its volume is

$$\begin{aligned}\text{volume} &= \int_{-a}^a \pi\Big(R(x)\Big)^2\ dx = \int_{-a}^a \pi\Big(\sqrt{a^2-x^2}\Big)^2\ dx \\ &= \pi\int_{-a}^a \left(a^2 - x^2\right)\ dx = \pi\left[a^2x - \frac{x^3}{3}\right]_{-a}^a \\ &= \frac{4}{3}\pi a^3.\end{aligned}$$

Dönel Cisimler



Tanım. Düzlemde bir bölgenin bir doğru etrafında döndürülmesiyl oluşan cisme bir ***dönel cisim*** denir.

$$\text{volume} = \int_a^b A(x)\ dx = \int_a^b \pi\Big(R(x)\Big)^2\ dx$$

Örnek 21.3. $y = \sqrt{x}$ ve x-ekseni arasındaki bölge, $0 \leq x \leq 4$ olmak üzere, x-ekseni etrafında döndürülüyor. Oluşan cismin hacmini bulunuz.

çözüm:

$$\begin{aligned}\text{hacim} &= \int_a^b \pi\Big(R(x)\Big)^2\ dx = \int_0^4 \pi\Big(\sqrt{x}\Big)^2\ dx \\ &= \pi\int_0^4 x\ dx = \pi\left[\frac{1}{2}x^2\right]_0^4 = 8\pi.\end{aligned}$$

Örnek 21.4. Yarıçapı a olan kürenin hacmini buluz.

çözüm: Bir küre oluşması için, $y = \sqrt{a^2 - x^2}$ ile x-ekseni arasındaki bölgeyi x-ekseni etrafında döndürürüz. Hacmi de

$$\begin{aligned}\text{hacim} &= \int_{-a}^a \pi\Big(R(x)\Big)^2\ dx = \int_{-a}^a \pi\Big(\sqrt{a^2-x^2}\Big)^2\ dx \\ &= \pi\int_{-a}^a \left(a^2 - x^2\right)\ dx = \pi\left[a^2x - \frac{x^3}{3}\right]_{-a}^a \\ &= \frac{4}{3}\pi a^3.\end{aligned}$$

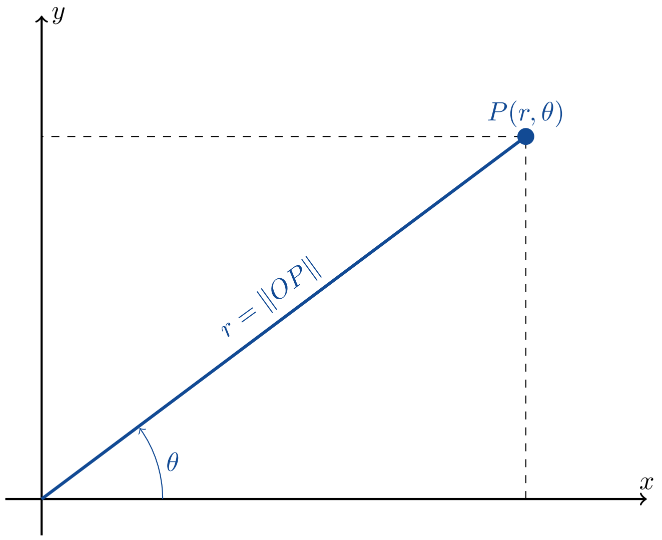
Part III

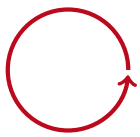
The Geometry of Space

22

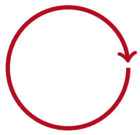
Polar Coordinates

Kutupsal Koordinatlar

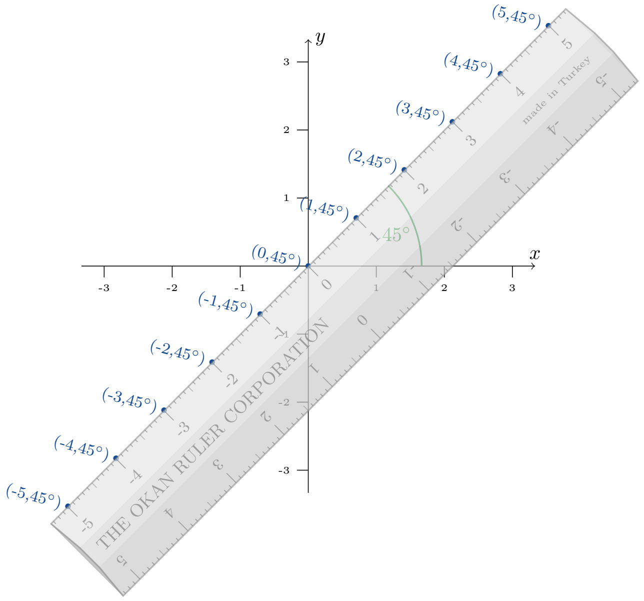


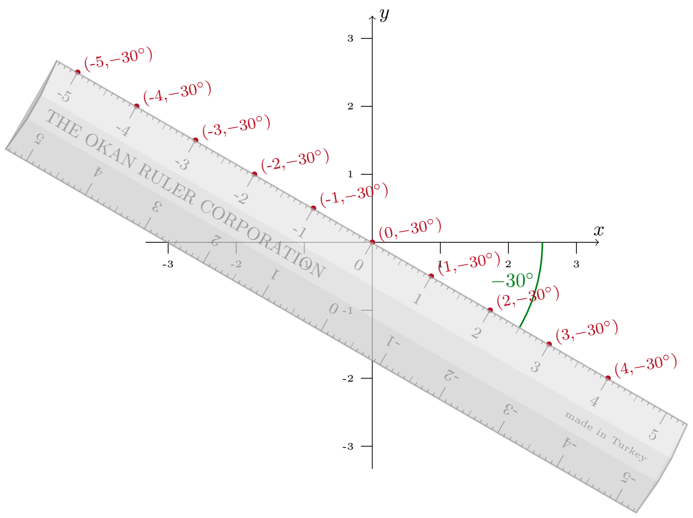


anticlockwise = positive angle
saat yönünün tersi = pozitif açı

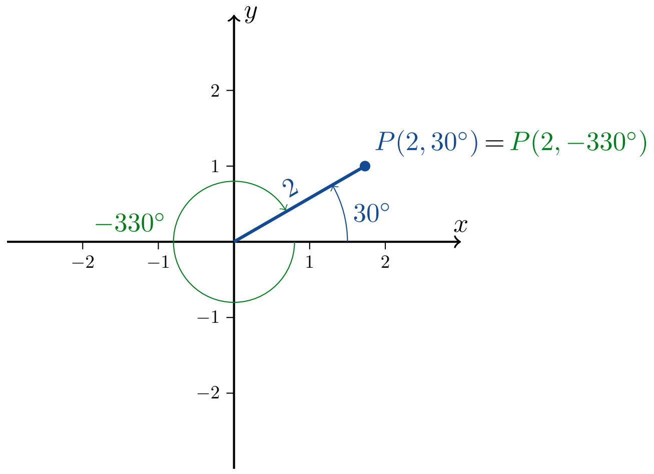


clockwise = negative angle
saat yönünde = negatif açı

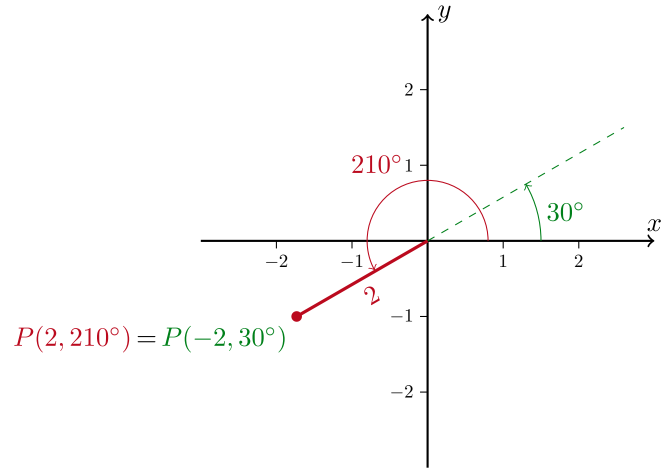




Example 22.1.



Örnek 22.2.



Example 22.3. Find all the polar coordinates of $P(2, 30°)$.

solution: We can have either $r = 2$ or $r = -2$. For $r = 2$, we can have

$$\theta = 30°, 30° \pm 360°, 30° \pm 720°, 30° \pm 1080°, \ldots$$

For $r = -2$, we can have

$$\theta = 210°, 210° \pm 360°, 210° \pm 720°, 210° \pm 1080°, \ldots$$

Therefore

$$P(2, 30°) = P\big(2, (30 + 360n)°\big) = P\big(-2, (210 + 360m)°\big)$$

for all $m, n \in \mathbb{Z}$.

Örnek 22.3. $P(2, 30°)$ noktasının tüm kutupsal koordinatlarını bulunuz.

çözüm: Ya $r = 2$ ya da $r = -2$ olmalıdır. $r = 2$ ise,

$$\theta = 30°, 30° \pm 360°, 30° \pm 720°, 30° \pm 1080°, \ldots$$

olmalıdır. $r = -2$ olduğunda ise,

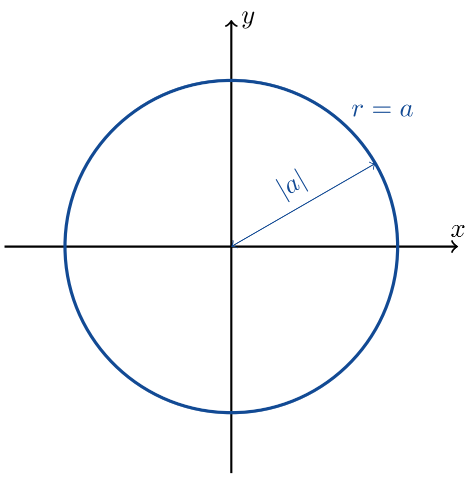
$$\theta = 210°, 210° \pm 360°, 210° \pm 720°, 210° \pm 1080°, \ldots$$

olmalıdır. Böylece her $m, n \in \mathbb{Z}$ için,

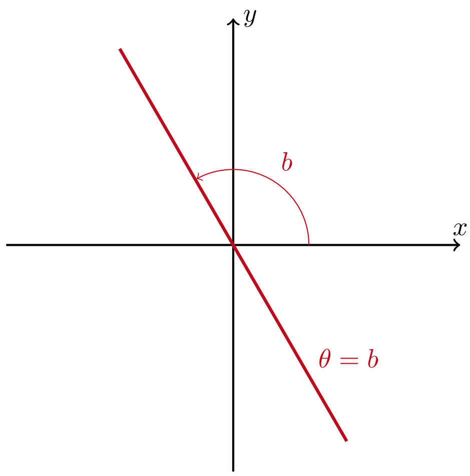
$$P(2, 30°) = P\big(2, (30 + 360n)°\big) = P\big(-2, (210 + 360m)°\big)$$

.

Example 22.4.



Örnek 22.5.



Example 22.6.

(a). $r = 1$ and $r = -1$ are both equations for a circle of radius 1 centred at the origin.

(b). $\theta = 30°$, $\theta = 210°$ and $\theta = -150°$ are all equations for the same line.

Örnek 22.6.

(a). $r = 1$ ve $r = -1$ her ikisi merkezi orijin yarıçapı 1 olan çemberin denklemleridir.

(b). $\theta = 30°$, $\theta = 210°$ ve $\theta = -150°$ herbiri aynı doğruya ait denklemlerdir.

Example 22.7. Draw the sets of points whose polar coordinates satisfy the following:

(a). $1 \le r \le 2$ and $0 \le \theta \le 90°$.

Örnek 22.7. Polar koordinatları aşağıdakileri sağlayan noktaları çiziniz:

(a). $1 \le r \le 2$ ve $0 \le \theta \le 90°$.

(b). $-3 \le r \le 2$ and $\theta = 45°$.

(c). $r \le 0$ and $\theta = 60°$.

(d). $120° \le \theta \le 150°$.

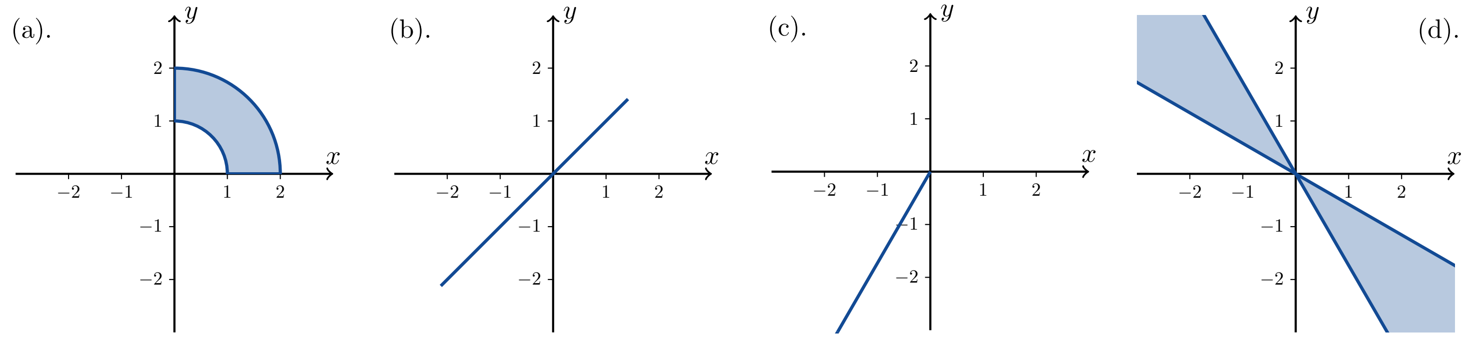
solution:

(b). $-3 \le r \le 2$ ve $\theta = 45°$.

(c). $r \le 0$ ve $\theta = 60°$.

(d). $120° \le \theta \le 150°$.

çözüm:

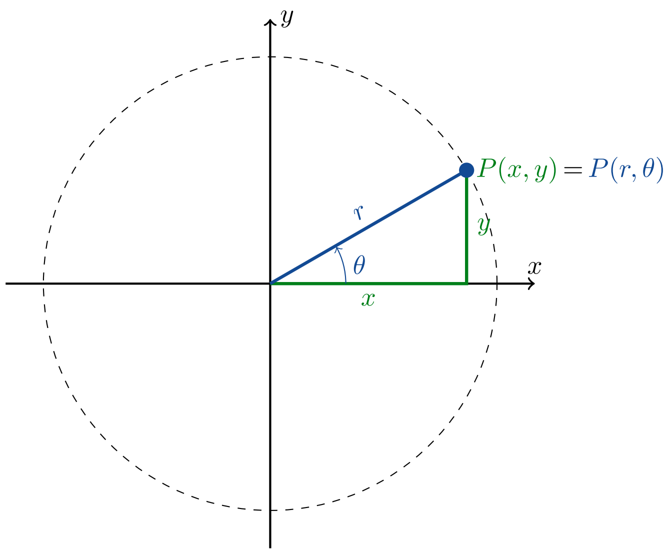


Relating Polar and Cartesian Coordinates

Kutupsal ve Kartezyen Koordinatlar Arasındaki İlişki

$$x = r\cos\theta \qquad x^2 + y^2 = r^2$$

$$y = r\sin\theta \qquad \tan\theta = \frac{y}{x}$$



Example 22.8. Convert the polar coordinates $(r, \theta) = (-3, 90°)$ into Cartesian coordinates.

solution:

$$(x, y) = (r\cos\theta, r\sin\theta) = (-3\cos 90°, -3\sin 90°) = (0, -3).$$

Example 22.9. Find polar coordinates for the Cartesian coordinates $(x, y) = (5, -12)$.

solution: Choosing $r > 0$, we calculate that

$$r = \sqrt{x^2 + y^2} = \sqrt{25 + 144} = \sqrt{169} = 13.$$

To find θ we use the equation $y = r\sin\theta$ to calculate that

$$\theta = \sin^{-1}\frac{y}{r} = \sin^{-1}\frac{-12}{13} = -67.38°.$$

Örnek 22.8. $(r, \theta) = (-3, 90°)$ kutupsal koordinatlarını kartezyen koordinatlarına dönüştürünüz.

çözüm:

$$(x, y) = (r\cos\theta, r\sin\theta) = (-3\cos 90°, -3\sin 90°) = (0, -3).$$

Örnek 22.9. $(x, y) = (5, -12)$ noktasının kutupsal koordinatlarını bulunuz.

çözüm: $r > 0$ alarak,

$$r = \sqrt{x^2 + y^2} = \sqrt{25 + 144} = \sqrt{169} = 13$$

buluruz. Şimdi θ'yı bulmak için $y = r\sin\theta$ denklemi kullanılır ve

$$\theta = \sin^{-1}\frac{y}{r} = \sin^{-1}\frac{-12}{13} = -67.38°$$

Therefore

$$(r, \theta) = (13, -67.38°).$$

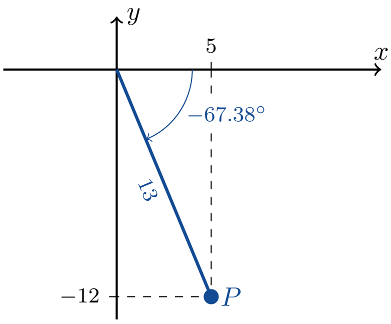


Figure 22.1: The point P had Cartesian coordinates $(x, y) = (5, -12)$ and polar coordinates $(r, \theta) = (13, -67.38°)$
Şekil 22.1:

elde edilir. Dolayısıyla

$$(r, \theta) = (13, -67.38°).$$

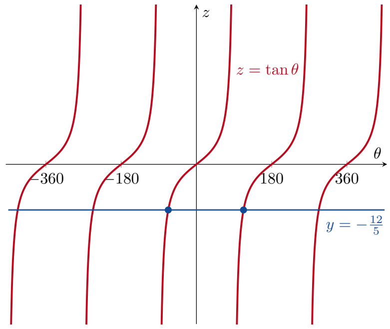


Figure 22.2: The graph of $z = \tan\theta$. Note that $\tan -67.38° = -\frac{12}{5} = \tan 112.62°$.
Şekil 22.2: $z = \tan\theta$ gragiği gösterilmektedir. Dikkat edilirse, $\tan -67.38° = -\frac{12}{5} = \tan 112.62°$.

Problems

Problem 22.1. Convert the following polar coordinates to Cartesian coordinates.

(a). $(3, 0)$ (d). $(2, 420°)$ (g). $(-2, -60°)$

(b). $(-3, 0)$ (e). $(2, 60°)$ (h). $(1, 180°)$

(c). $(2, 120°)$ (f). $(-3, 360°)$ (i). $(2\sqrt{2}, 45°)$.

Problem 22.2. Find polar coordinates for each of the following sets of Cartesian coordinates.

(a). $(1, 1)$ (c). $(\sqrt{3}, -1)$ (e). $(-2, -2)$

(b). $(-3, 0)$ (d). $(-3, 4)$ (f). $(-\sqrt{3}, 1)$.

Problem 22.3. Draw the sets of points whose polar coordinates satisfy the following:

(a). $r = 2$

(b). $0 \le r \le 2$

(c). $r \ge 2$

(d). $0 \le \theta \le 30°$ and $r \ge 0$

(e). $\theta = 120°$ and $r \le -2$

(f). $0 \le \theta \le 90°$ and $1 \le |r| \le 2$.

Sorular

Soru 22.1. Aşağıdaki kutupsal koordinatları Kartezyen koordinatlara dönüştürünüz.

(a). $(3, 0)$ (d). $(2, 420°)$ (g). $(-2, -60°)$

(b). $(-3, 0)$ (e). $(2, 60°)$ (h). $(1, 180°)$

(c). $(2, 120°)$ (f). $(-3, 360°)$ (i). $(2\sqrt{2}, 45°)$.

Soru 22.2. Aşağıdaki Kartezyen koordinatların herbiri için bir kutupsal koordinat bulunuz.

(a). $(1, 1)$ (c). $(\sqrt{3}, -1)$ (e). $(-2, -2)$

(b). $(-3, 0)$ (d). $(-3, 4)$ (f). $(-\sqrt{3}, 1)$.

Soru 22.3. Kutupsal koordinatları aşağıdakilari sağlayan noktaların kümesini çiziniz:

(a). $r = 2$

(b). $0 \le r \le 2$

(c). $r \ge 2$

(d). $0 \le \theta \le 30°$ ve $r \ge 0$

(e). $\theta = 120°$ ve $r \le -2$

(f). $0 \le \theta \le 90°$ ve $1 \le |r| \le 2$.

23

Graphing in Polar Coordinates

Kutupsal Koordinatlarla Grafik Çizimi

Example 23.1. Graph the curve $r = 1 - \cos\theta$.

solution: We will use the following steps to draw this graph

STEP 1. First we will create a table of θ and r values which satisfy the equation.

STEP 2. Then we will plot these points in $\mathbb{R}^2$.

STEP 3. Finally we will draw a smooth curve through these points.

Örnek 23.1. $r = 1 - \cos\theta$ eğrisinin grafiğini çiziniz.

çözüm: Bu grafiği çizmek için aşağıdaki adımları izleyeceğiz

ADIM 1. önce denklemi sağlayan θ ve r'ye ait değerler tablosu yapılır values which satisfy the equation.

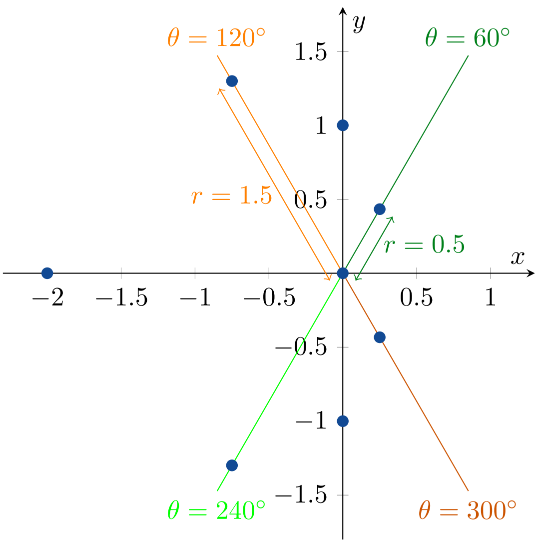
ADIM 2. Sonra bu noktaları $\mathbb{R}^2$'de işaretleriz.

ADIM 3. Son olarak işaretlediğimiz noktalardan geçen düzgün bir eğri çizilir.

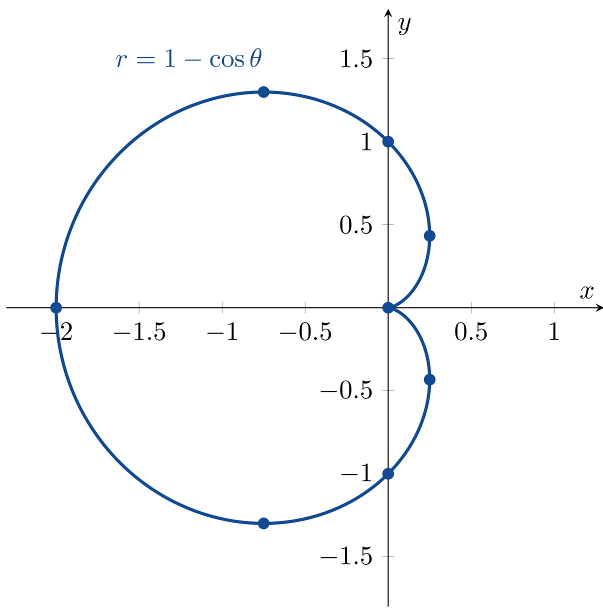
1.

θ	$r = 1 - \cos\theta$
0	0
$60° = \frac{\pi}{3}$	$\frac{1}{2}$
$90° = \frac{\pi}{2}$	1
$120° = \frac{2\pi}{3}$	$\frac{3}{2}$
$180° = \pi$	2
$240° = \frac{4\pi}{3}$	$\frac{3}{2}$
$270° = \frac{3\pi}{2}$	1
$300° = \frac{5\pi}{2}$	$\frac{1}{2}$
$360° = 2\pi$	0

2.



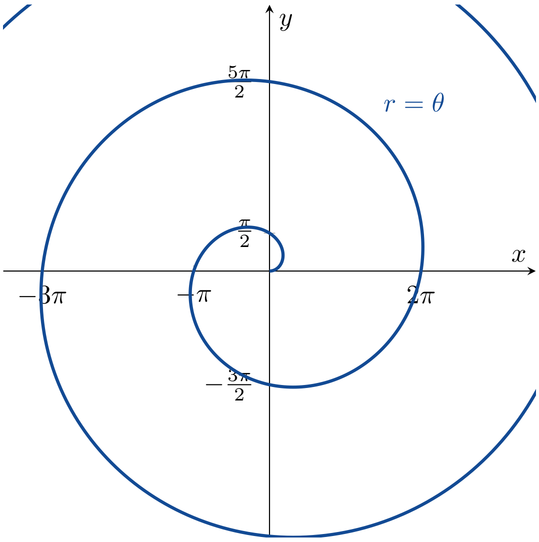
3.



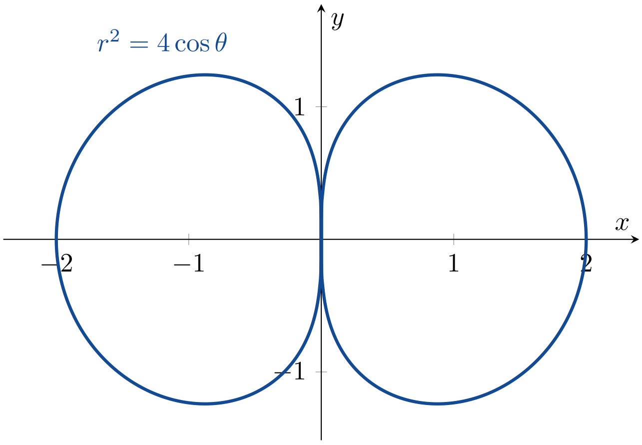
Note that $r = 1 - \cos\theta$ is a nice simple equation, yet its graph is the interesting curve that we plotted above. Below we will look at some more examples. We are studying these purely because they give nice shapes. You will not be asked to graph a polar equation in your exam.

$r = 1 - \cos\theta$'nın güzel basit bir denklem, grafiğinin de yukarıda çizdiğimiz ilginç eğri olduğuna dikkat ediniz. Aşağıda başka örnekleri de inceliyoruz. Bu güzel eğrileri tamaıyle çiziyoruz. Sınavda sorulmayacaktır.

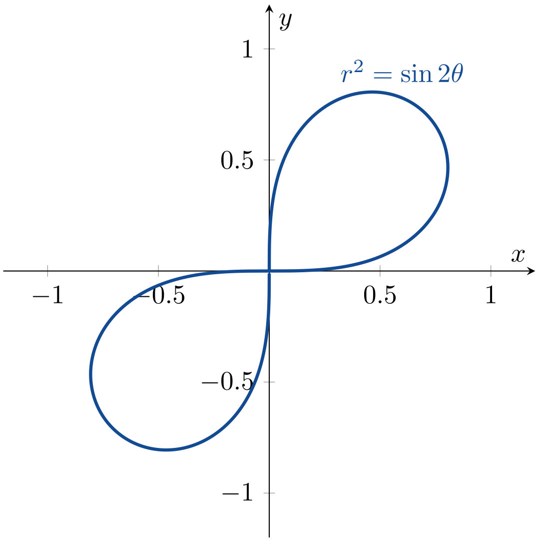
Example 23.2.



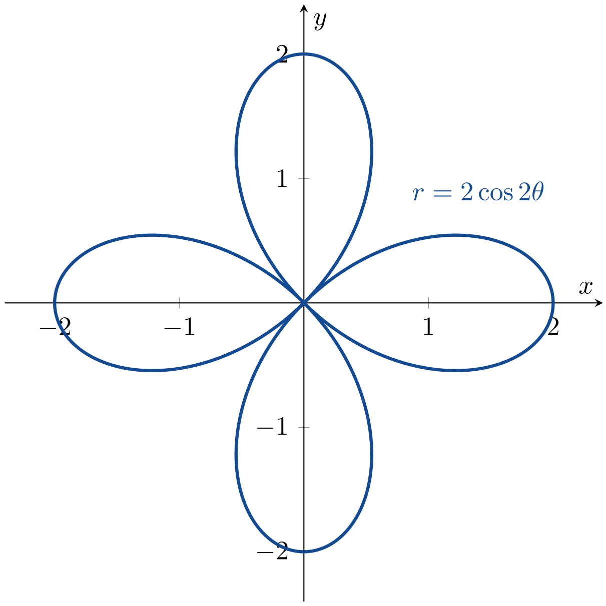
Example 23.3.



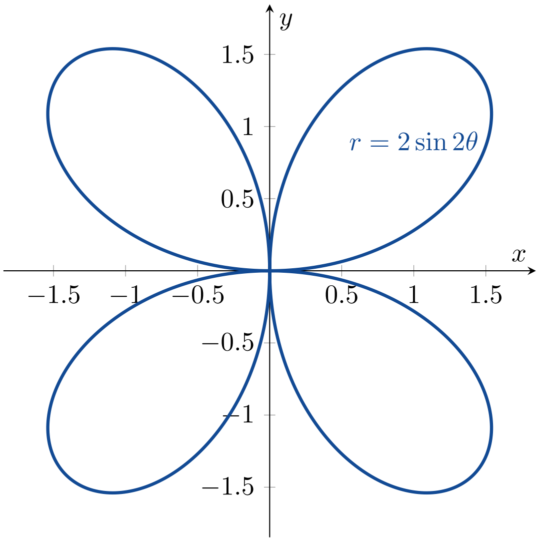
Example 23.4.



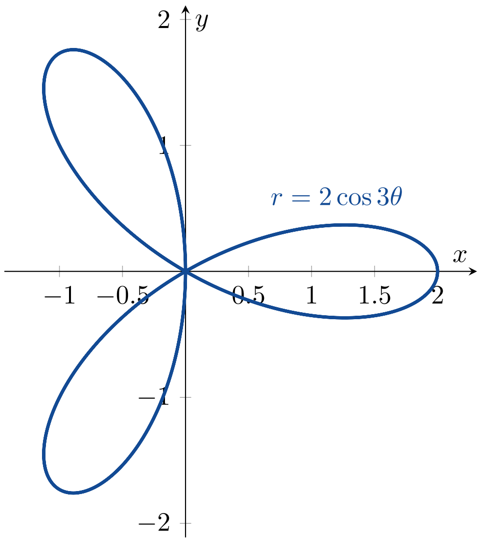
Example 23.5.



Example 23.6.

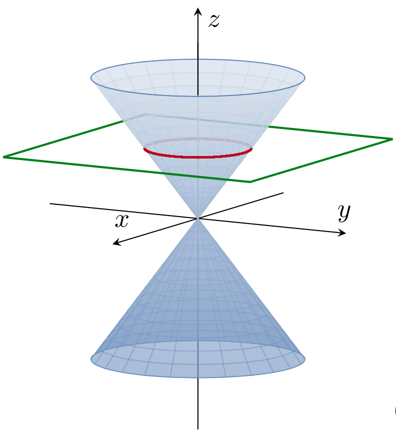


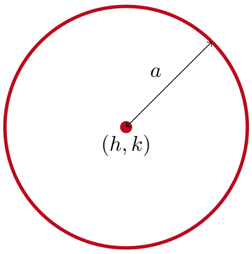
Example 23.7.



24 Conic Sections

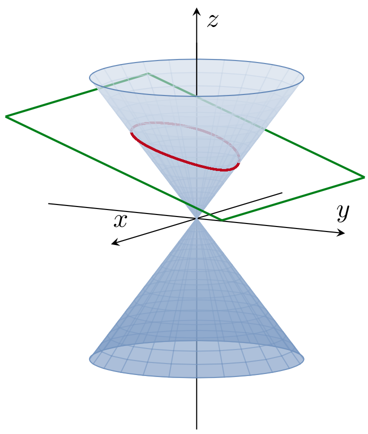
Konik Kesitler

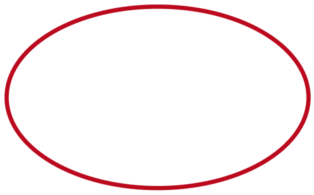




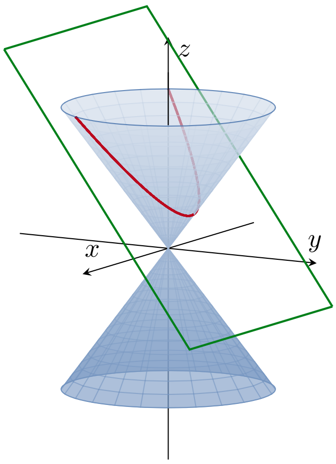
a circle
çember

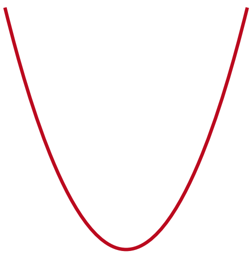
$$(x-h)^2+(y-k)^2=a^2$$



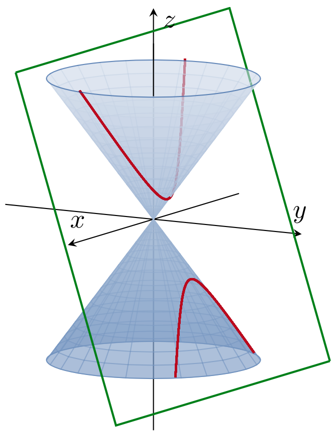


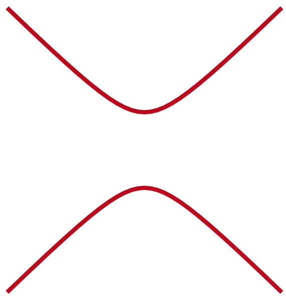
an ellipse
elips





a parabola
parabol





a hyperbola
hiperbol

Parabolas

Figure 24.1: Clifton suspension bridge, Bristol, UK. The cables of a suspension bridges hang in a shape which is almost (but not exactly) a parabola.
Şekil 24.1:

To describe a parabola, we need a point called a ***focus*** and a line called a ***directrix***. See figure 24.2.

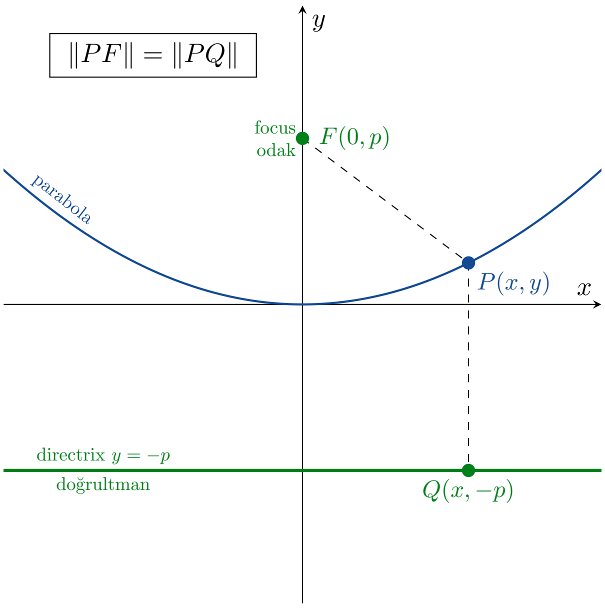


Figure 24.2: A parabola with focus at $F(0,p)$ and directrix $y=-p$.
Şekil 24.2:

Definition. A point $P(x,y)$ lies on the ***parabola*** if and only if

$$\|PF\| = \|PQ\|.$$

Now

$$\begin{aligned}\|PF\| &= \text{distance between } P(x,y) \text{ and } F(0,p)\\ &= \sqrt{(x-0)^2+(y-p)^2} = \sqrt{x^2+(y-p)^2}\end{aligned}$$

Paraboller

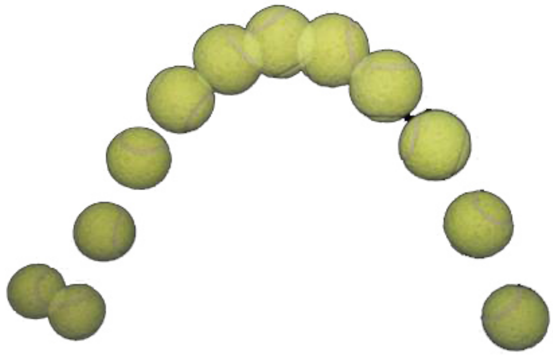


Figure 24.3: The motion of a tennis ball.
Şekil 24.3: Bir tenis topunun hareketi

Figure 24.4: Satellite dishes.
Şekil 24.4: Uydu antenleri.

Bir parabolü tanımlamak için, ***odak*** adı verilen bir noktaya ve ***doğrultman*** adı verilen bir doğruya ihtiyaç var. Bkz şekil 24.2.

Tanım. Bir $P(x,y)$ noktası bir ***parabol*** üzerindedir ancak ve ancak

$$\|PF\| = \|PQ\|.$$

Şimdi

$$\begin{aligned}\|PF\| &= P(x,y) \text{ ile } F(0,p) \text{ arasındaki uzaklık}\\ &= \sqrt{(x-0)^2+(y-p)^2} = \sqrt{x^2+(y-p)^2}\end{aligned}$$

ve

$$\begin{aligned}\|PQ\| &= P(x,y) \text{ ile } Q(x,-p) \text{ arasındaki uzaklık}\\ &= \sqrt{(x-x)^2+(y+p)^2} = \sqrt{(y+p)^2} = y+p.\end{aligned}$$

and

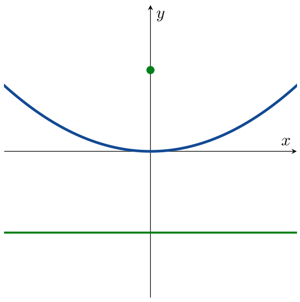
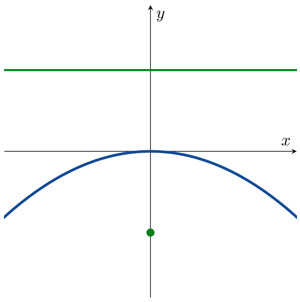
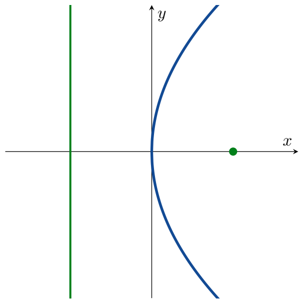
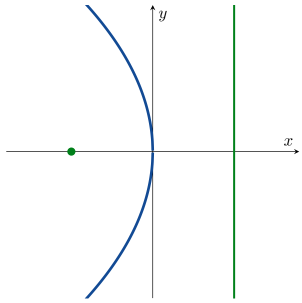
$$\begin{aligned}\|PQ\| &= \text{distance between } P(x,y) \text{ and } Q(x,-p)\\ &= \sqrt{(x-x)^2+(y+p)^2} = \sqrt{(y+p)^2} = y+p.\end{aligned}$$

Therefore

$$\begin{aligned}\|PF\| &= \|PQ\|\\ \sqrt{x^2+(y-p)^2} &= y+p\\ x^2+(y-p)^2 &= (y+p)^2\\ x^2+y^2-2py+p^2 &= y^2+2py+p^2\\ x^2-2py &= 2py\\ \boxed{x^2 = 4py}\end{aligned}$$

Bu nedenle

$$\begin{aligned}\|PF\| &= \|PQ\|\\ \sqrt{x^2+(y-p)^2} &= y+p\\ x^2+(y-p)^2 &= (y+p)^2\\ x^2+y^2-2py+p^2 &= y^2+2py+p^2\\ x^2-2py &= 2py\\ \boxed{x^2 = 4py}\end{aligned}$$

graph graf				
equation denklem	$x^2 = 4py$	$x^2 = -4py$	$y^2 = 4px$	$y^2 = -4px$
focus odak	$F(0,p)$	$F(0,-p)$	$F(p,0)$	$F(-p,0)$
directrix doğrultman	$y = -p$	$y = p$	$x = -p$	$x = p$

Example 24.1. Find the focus and directrix of the parabola $y^2 = 10x$.

solution: Our equation $y^2 = 10x$ looks like $y^2 = 4px$ with $p = \frac{10}{4} = 2.5$. Therefore the focus is at the point $F(2.5,0)$ and the directrix is the line $x = -2.5$.

Example 24.2. Find the equation for the parabola which has focus $F(0,-10)$ and directrix $y = 10$.

solution: Clearly $p = 10$ and $x^2 = -4py$. Therefore the answer is $x^2 = -40y$.

Örnek 24.1. $y^2 = 10x$ parabolünün odak noktasını ve doğrultmanın bulunuz.

çözüm: $y^2 = 10x$ denklemimiz olmak üzere $y^2 = 4px$ biçimindedir. Yani odak noktası $F(2.5,0)$ ve doğrultmanı da $x = -2.5$ olur.

Örnek 24.2. Odağı $F(0,-10)$ noktası ve doğrultmanı $y = 10$ doğrusu olan parabolün denklemini yazınız.

çözüm: Şurası açık ki $p = 10$ ve $x^2 = -4py$ dir. Bu nedenle yanıt $x^2 = -40y$ olur.

Ellipses

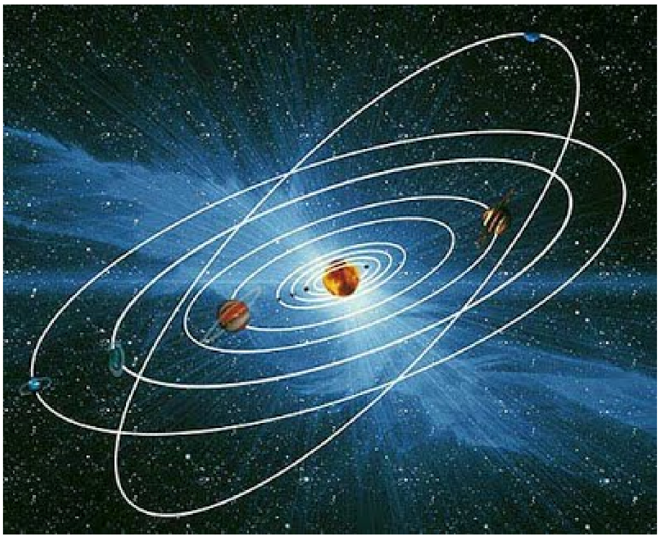


Figure 24.5: Our solar system.
Şekil 24.5: Güneş sistemimiz

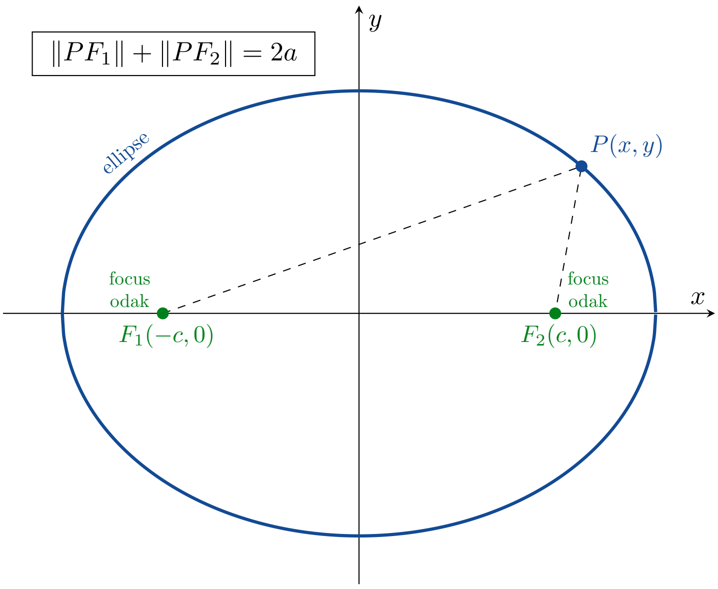


Figure 24.7: An ellipse with foci at $F_1(-c, 0)$ and $F_2(c, 0)$.
Şekil 24.7: Odakları $F_1(-c, 0)$ ve $F_2(c, 0)$ olan elips.

To describe an ellipse, we need two foci. See figure 24.7.

Definition. A point $P(x, y)$ is on the ***ellipse*** if and only if

$$\|PF_1\| + \|PF_2\| = 2a.$$

So

$$\sqrt{(x+c)^2 + y^2} + \sqrt{(x-c)^2 + y^2} = 2a.$$

This rearranges to

$$\frac{x^2}{a^2} + \frac{y^2}{a^2 - c^2} = 1.$$

Elipsler

Figure 24.6: Tycho Brahe Planetarium, Copenhagen, Denmark.
Şekil 24.6: Tycho Brahe Planetaryumu, Kopenhag, Danimarka.

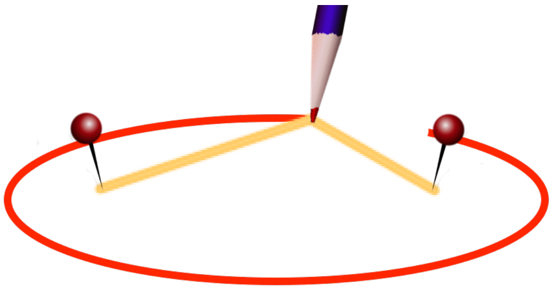


Figure 24.8: Drawing an ellipse with a pencil, two pins and a piece of string.
Şekil 24.8: İki toplu iğne, bir kalem ve biraz ip kullanarak elips çizmek

Elipsi tanımlamak için, we need two foci. Bkz. şekil 24.7.

Tanım. Bir $P(x, y)$ noktası ***ellips*** üzerindedir ancak ve ancak

$$\|PF_1\| + \|PF_2\| = 2a.$$

Buradan hareketle

$$\sqrt{(x+c)^2 + y^2} + \sqrt{(x-c)^2 + y^2} = 2a.$$

Bunu da düzenlersek

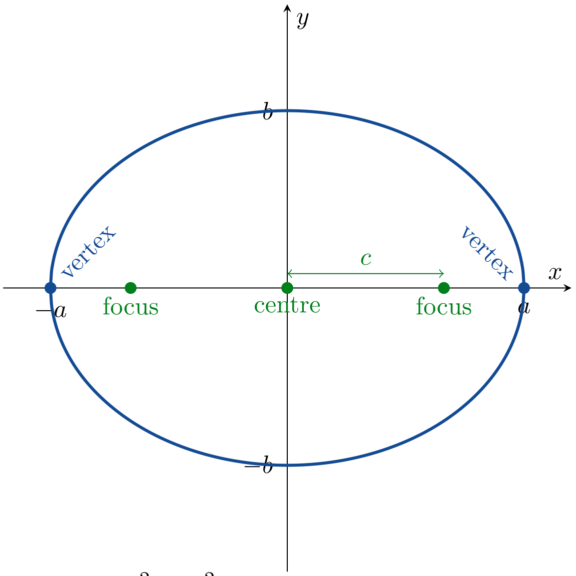
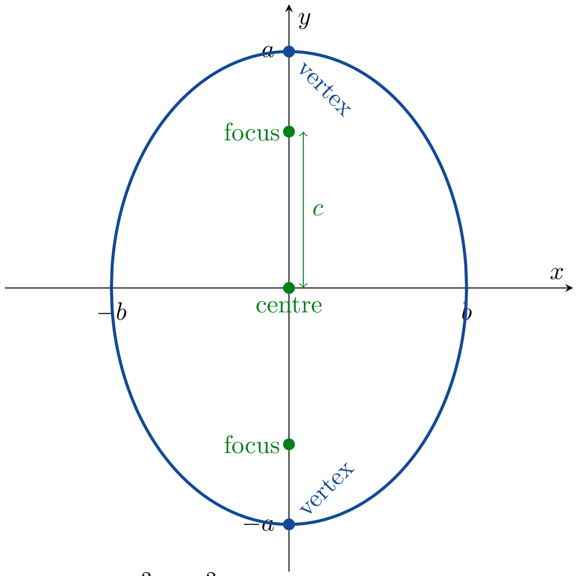
$$\frac{x^2}{a^2} + \frac{y^2}{a^2 - c^2} = 1$$

If we set $b = \sqrt{a^2 - c^2}$, then we have

$$\boxed{\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1} \qquad (0 < b < a).$$

buluruz. $b = \sqrt{a^2 - c^2}$ dersek, o zaman

$$\boxed{\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1} \qquad (0 < b < a).$$

graph			graf
equation	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad (0 < b < a)$	$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1 \quad (0 < b < a)$	denklem
centre-to-focus distance	$c = \sqrt{a^2 - b^2}$	$c = \sqrt{a^2 - b^2}$	merkez-odak uzaklığı
foci	$F_1(-c, 0)$ & $F_2(c, 0)$	$F_1(0, -c)$ & $F_2(0, c)$	odaklar
vertices	$(-a, 0)$ & $(a, 0)$	$(0, -a)$ & $(0, a)$	tepe noktaları

Example 24.3. The ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$ has

- $a = 4$ and $b = 3$;
- centre-to-focus distance $c = \sqrt{a^2 - b^2} = \sqrt{16 - 9} = \sqrt{7}$;
- centre at $(0, 0)$;
- foci at $(-\sqrt{7}, 0)$ and $(\sqrt{7}, 0)$; and
- vertices at $(-4, 0)$ and $(4, 0)$.

Örnek 24.3. $\frac{x^2}{16} + \frac{y^2}{9} = 1$ elipsinin

- $a = 4$ ve $b = 3$;
- merkez-odak uzaklığı $c = \sqrt{a^2 - b^2} = \sqrt{16 - 9} = \sqrt{7}$;
- merkezi $(0, 0)$;
- odakları $(-\sqrt{7}, 0)$ ve $(\sqrt{7}, 0)$; and
- tepe noktaları $(-4, 0)$ ve $(4, 0)$.

Example 24.4. The ellipse $\frac{x^2}{16} + \frac{y^2}{25} = 1$ has

- $a = 5$ and $b = 4$;
- centre-to-focus distance
  $c = \sqrt{a^2 - b^2} = \sqrt{25 - 16} = \sqrt{9} = 3$;
- centre at $(0, 0)$;
- foci at $(0, -3)$ and $(0, 3)$; and
- vertices at $(0, -5)$ and $(0, 5)$.

Örnek 24.4. $\frac{x^2}{16} + \frac{y^2}{25} = 1$ elipsi

- $a = 5$ ve $b = 4$;
- merkez-odak uzaklığı
  $c = \sqrt{a^2 - b^2} = \sqrt{25 - 16} = \sqrt{9} = 3$;
- merkezi $(0, 0)$;
- odakları $(0, -3)$ and $(0, 3)$; ve
- tepe noktaları da $(0, -5)$ ve $(0, 5)$.

Hyperbolas

Hiperboller

Figure 24.9: Cooling towers.
Şekil 24.9:

Figure 24.10: Twin Arch 138, Ichinomiya City, Japan.
Şekil 24.10:

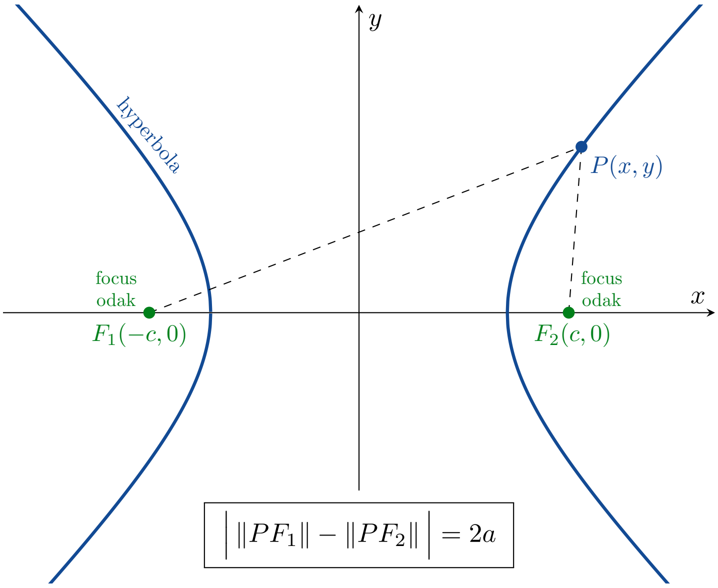


Figure 24.11: A hyperbola with foci at $F_1(-c, 0)$ and $F_2(c, 0)$.
Şekil 24.11:

To describe a hyperbola, we again need two foci. See figure 24.11.

Definition. A point $P(x, y)$ is on the ***hyperbola*** if and only if

$$\Big| \|PF_1\| - \|PF_2\| \Big| = 2a.$$

So

$$\sqrt{(x+c)^2 + y^2} - \sqrt{(x-c)^2 + y^2} = \pm 2a.$$

This rearranges to

$$\frac{x^2}{a^2} + \frac{y^2}{a^2 - c^2} = 1$$

where $c > a > 0$. If we set $b = \sqrt{c^2 - a^2}$, then

$$\boxed{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.}$$

Hiperbolü tanımlamak için, yine iki odak noktasına ihtiyaç var. Bkz. şekil 24.11.

Tanım. Bir $P(x, y)$ noktası bir ***hiperbol*** üzerindedir ancak ve ancak

$$\Big| \|PF_1\| - \|PF_2\| \Big| = 2a.$$

Bundan hareketle,

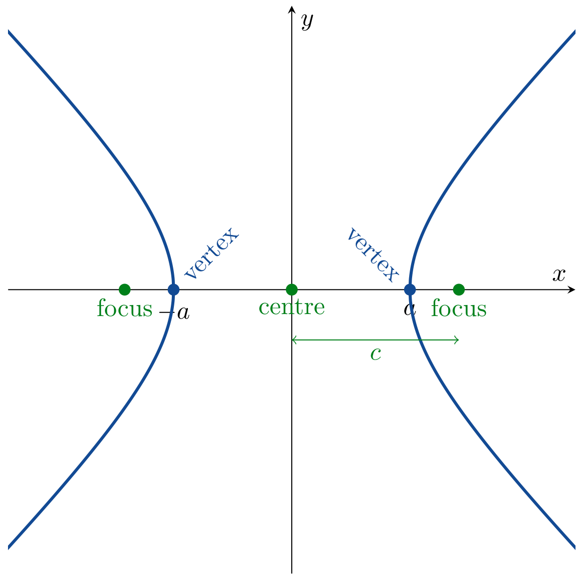
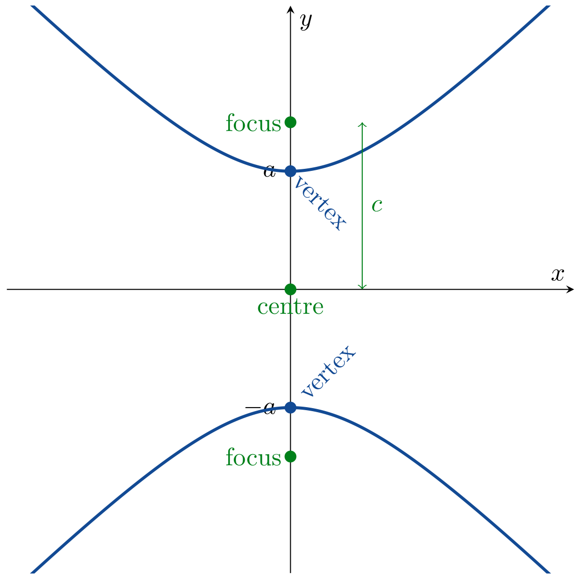
$$\sqrt{(x+c)^2 + y^2} - \sqrt{(x-c)^2 + y^2} = \pm 2a.$$

Düzenlersek,

$$\frac{x^2}{a^2} + \frac{y^2}{a^2 - c^2} = 1$$

buluruz ki burada $c > a > 0$. Şimdi $b = \sqrt{c^2 - a^2}$ dersek, o zaman

$$\boxed{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.}$$

graph			graf
equation	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$	$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$	denklem
centre-to-focus distance	$c = \sqrt{a^2 + b^2}$	$c = \sqrt{a^2 + b^2}$	merkez-odak uzaklığı
foci	$F_1(-c, 0)$ & $F_2(c, 0)$	$F_1(0, -c)$ & $F_2(0, c)$	odaklar
vertices	$(-a, 0)$ & $(a, 0)$	$(0, -a)$ & $(0, a)$	tepe noktaları

Example 24.5. The hyperbola $\frac{x^2}{4} - \frac{y^2}{5} = 1$ has

- $a = 2$ and $b = \sqrt{5}$;
- centre at $(0, 0)$;
- centre-to-focus distance $c = \sqrt{a^2 + b^2} = \sqrt{4 + 5} = 3$;
- foci at $(-3, 0)$ and $(3, 0)$; and
- vertices at $(-2, 0)$ and $(2, 0)$.

Örnek 24.5. Hiperbol olarak $\frac{x^2}{4} - \frac{y^2}{5} = 1$ alırsak,

- $a = 2$ ve $b = \sqrt{5}$;
- merkezi $(0, 0)$;
- merkez-odak uzaklığı $c = \sqrt{a^2 + b^2} = \sqrt{4 + 5} = 3$;
- odakları $(-3, 0)$ and $(3, 0)$; ve
- tepe noktaları da $(-2, 0)$ ve $(2, 0)$.

Example 24.6. The hyperbola $\frac{y^2}{9} - \frac{x^2}{16} = 1$ has

- $a = 3$ and $b = 4$;
- centre at $(0, 0)$;
- centre-to-focus distance $c = \sqrt{a^2 + b^2} = \sqrt{9 + 16} = 5$;
- foci at $(0, -5)$ and $(0, 5)$; and
- vertices at $(0, -3)$ and $(0, 3)$.

Örnek 24.6. $\frac{y^2}{9} - \frac{x^2}{16} = 1$ hiperbolü için

- $a = 3$ ve $b = 4$;
- merkez $(0, 0)$;
- merkez-odak uzaklığı $c = \sqrt{a^2 + b^2} = \sqrt{9 + 16} = 5$;
- odaklar $(0, -5)$ ve $(0, 5)$; ve
- tepe noktaları $(0, -3)$ ve $(0, 3)$.

Reflective Properties

Parabolas, ellipses and hyperbolas are useful in architecture and engineering because of their reflective properties.

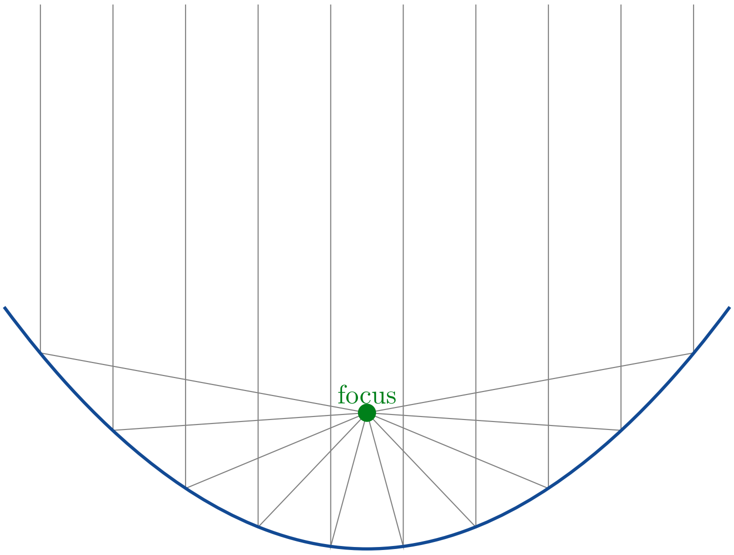


Figure 24.12: Rays originating at the focus of a parabola are reflected out of the parabola as parallel lines.
Şekil 24.12: Parabolün odağından çıkan ışınlar parabolün dışında paralel doğrular olarak yoluna devam ederler

Figure 24.14: A car headlight
Şekil 24.14: Bir araba farı.

Yansıma Özellikleri

Paraboller, elipsler ve hiperboller, yansıma özellikleri nedeniyle mimaride ve mühendislikte kullanışlıdırlar.

Figure 24.13: One of a pair of whispering dishes in San Francisco, USA.
Şekil 24.13: A.B.D. San Fransisko'daki bir çift akustik çanak

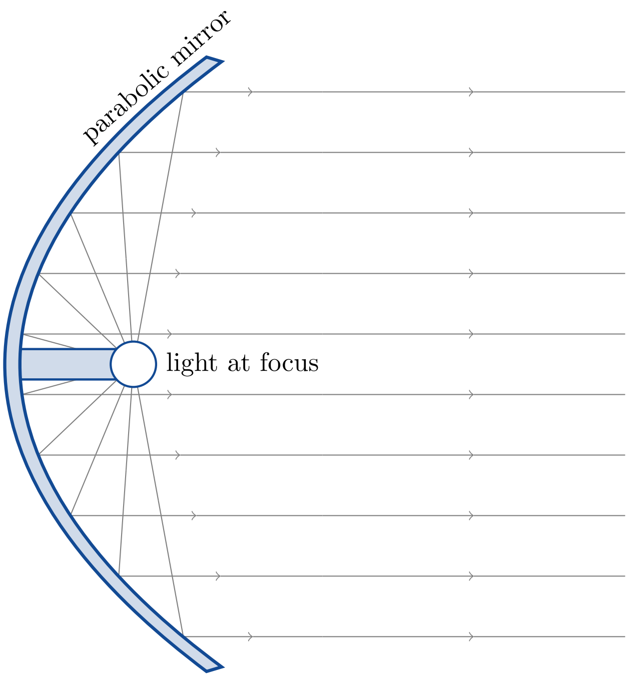


Figure 24.15: A car headlight
Şekil 24.15: Bir araba farı.

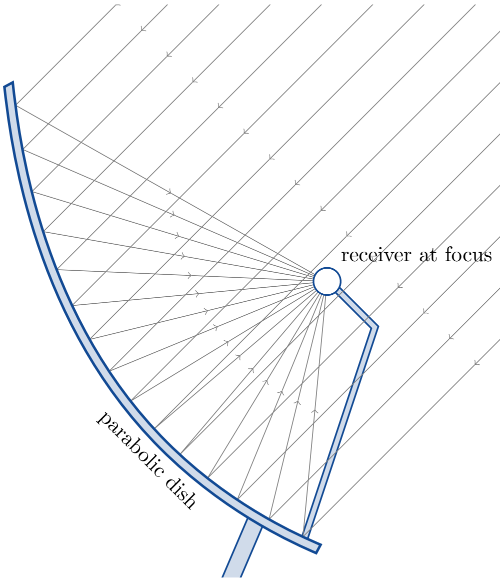


Figure 24.16: A satellite dish.
Şekil 24.16: Bir çanak anten

Figure 24.17: A satellite dish.
Şekil 24.17: Bir çanak anten

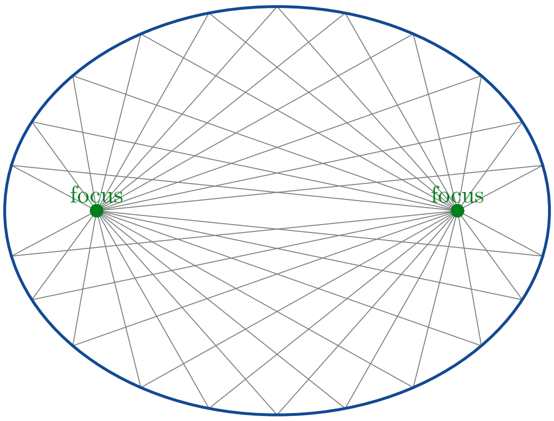


Figure 24.18: Rays originating from one focus of an ellipse are reflected toward the other focus.
Şekil 24.18: Elipsin bir odağından çıkan ışınlar diğer odağa yansıyorlar.

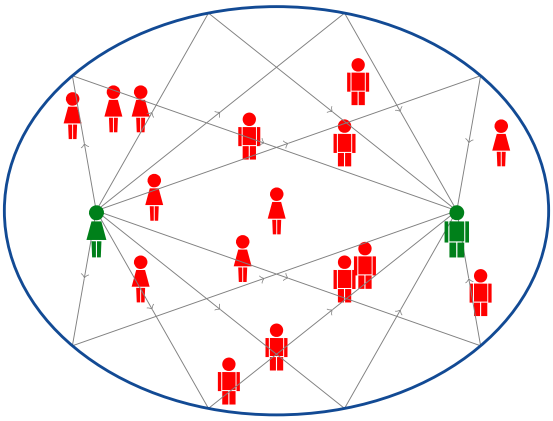


Figure 24.19: A whispering gallery.
Şekil 24.19:

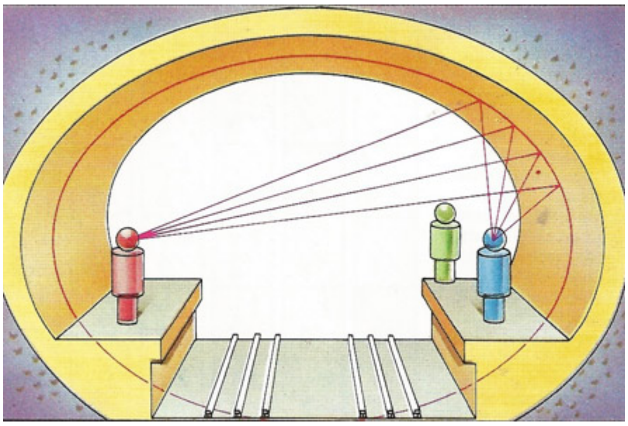


Figure 24.20: A whispering gallery.
Şekil 24.20:

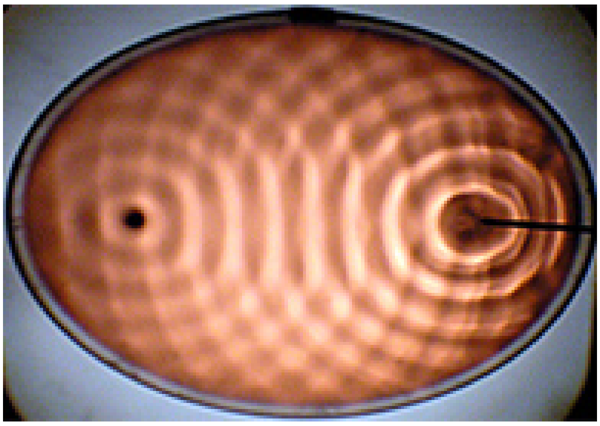


Figure 24.21: A whispering gallery.
Şekil 24.21:

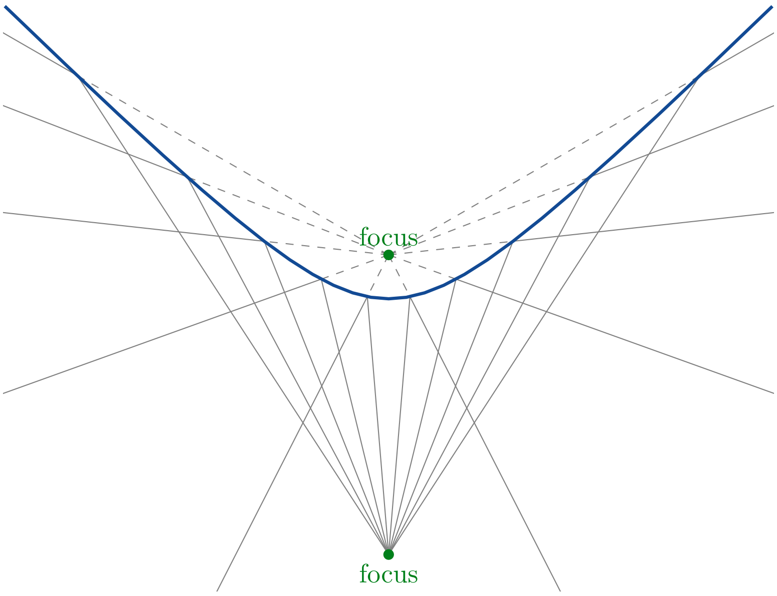


Figure 24.22: One half of a hyperbola. Rays aimed at one focus are reflected to the second focus.
Şekil 24.22: Hiperbolün bir yarısı. Odaklardan birine gelen ışınlar ikinci odağa yansıyorlar.

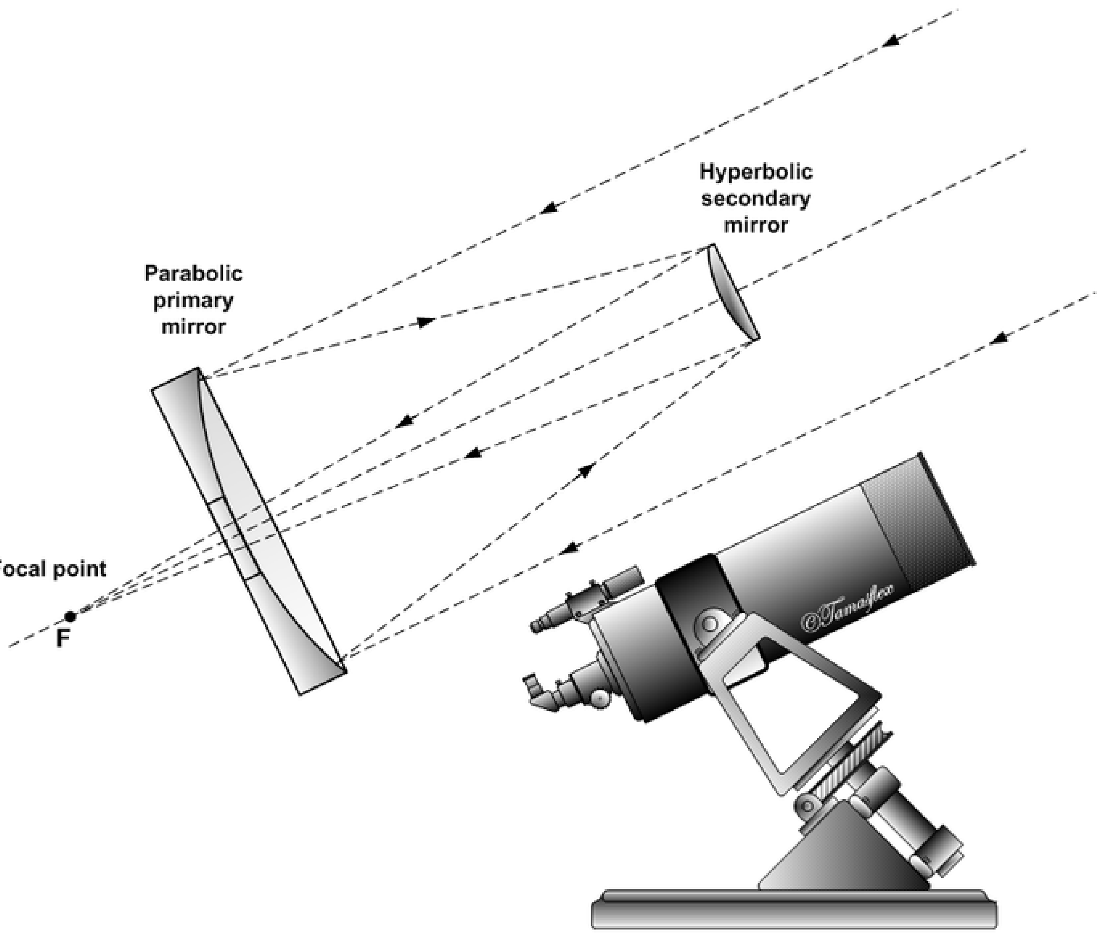


Figure 24.23: A telescope using a parabola and a hyperbola.
Şekil 24.23: Bir parabol ve bir elips kullanılan teleskop

Problems

Problem 24.1.

(a). Find the focus of the parabola $x^2 = -8y$.

(b). Find the foci of the ellipse $7x^2 + 16y^2 = 112$.

Sorular

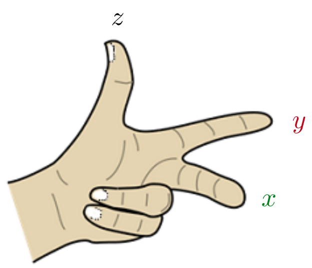
Soru 24.1.

(a). $x^2 = -8y$ parabolünün odağını bulunuz.

(b). $7x^2 + 16y^2 = 112$ elipsinin odaklarını bulunuz.

Three Dimensional Cartesian Coordinates

Üç Boyutlu Kartezyen Koordinatlar



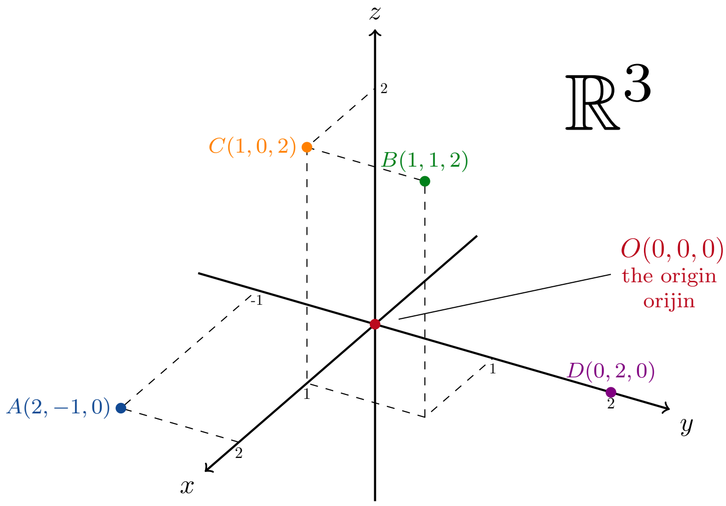


Figure 25.1: The Left-Handed Coordinate System
Şekil 25.1:

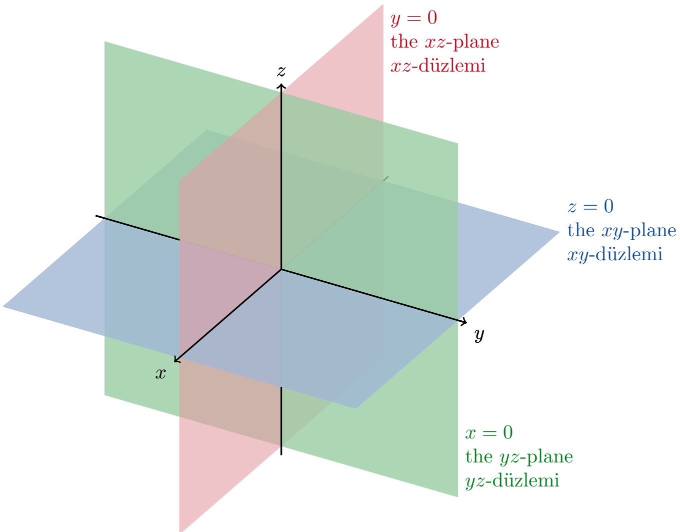


Figure 25.2: The planes $x = 0$, $y = 0$ and $z = 0$.
Şekil 25.2: $x = 0$, $y = 0$ ve $z = 0$ düzlemleri.

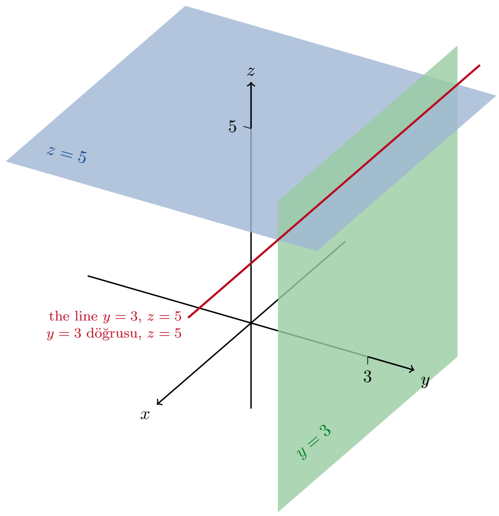


Figure 25.3: The planes $y = 3$ and $z = 5$, and the line $y = 3$, $z = 5$.
Şekil 25.3:

Example 25.1. Which points $P(x, y, z)$ satisfy $x^2 + y^2 = 4$ and $z = 3$?

solution: We know that $z = 3$ is a horizontal plane and we recognise that $x^2 + y^2 = 4$ is the equation of a circle of radius 2. Putting these together, we obtain figure 25.4.

Örnek 25.1. Hangi $P(x, y, z)$ noktaları $x^2+y^2 = 4$ ve $z = 3$'ü sağlar?

çözüm: Biliyoruz ki $z = 3$ yatay bir düzlem ve $x^2 + y^2 = 4$ denklemi 2 yarıçaplı bir çemberdir. Bunları bir araya getirirsek, şekil 25.4'yi elde ederiz.

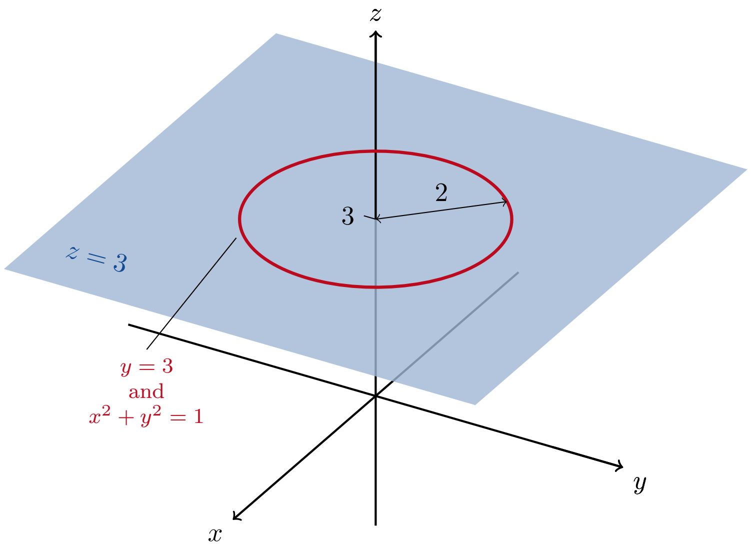


Figure 25.4: The circle $x^2 + y^2 = 4$ in the plane $z = 3$.
Şekil 25.4: $z = 3$ düzlemindeki $x^2 + y^2 = 4$ çemberi.

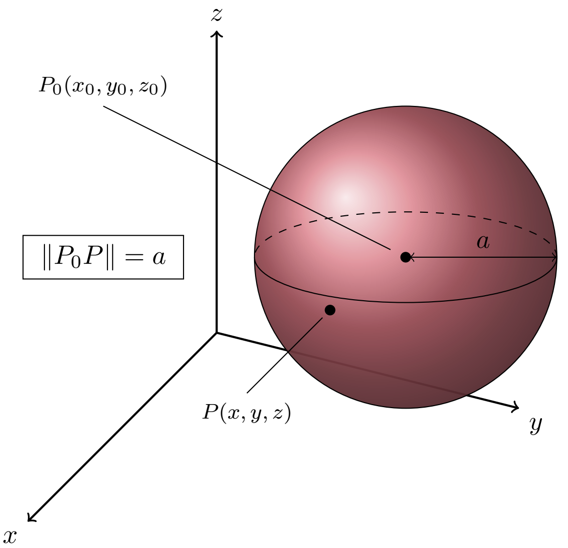


Figure 25.5: The sphere of radius a centred at $P_0(x_0, y_0, z_0)$.
Şekil 25.5: Yarıçapı a ver merkezı $P_0(x_0, y_0, z_0)$ noktası olan küre.

Distance in $\mathbb{R}^3$

Definition. The set

$$\{(x, y, z)|x, y, z \in \mathbb{R}\}$$

is denoted by $\mathbb{R}^3$.

Definition. The ***distance*** between $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ is

$$\|P_1P_2\| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

Example 25.2. The distance between $A(2, 1, 5)$ and $B(-2, 3, 0)$ is

$$\begin{aligned}\|AB\| &= \sqrt{((-2) - 2)^2 + (3 - 1)^2 + (0 - 5)^2}\\ &= \sqrt{16 + 4 + 25} = \sqrt{45}\\ &= 3\sqrt{5} \approx 6.7.\end{aligned}$$

Distance in $\mathbb{R}^3$

Tanım. The set

$$\{(x, y, z)|x, y, z \in \mathbb{R}\}$$

is denoted by $\mathbb{R}^3$.

Tanım. $P_1(x_1, y_1, z_1)$ ve $P_2(x_2, y_2, z_2)$ noktaları arasındaki ***uzaklık***

$$\|P_1P_2\| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

Örnek 25.3. $C(1, 2, 3)$ ve $D(3, 2, 1)$ noktaları arasındaki uzaklık aşağıdaki gibidir;

$$\begin{aligned}\|AB\| &= \sqrt{(3 - 1)^2 + (2 - 2)^2 + (1 - 3)^2}\\ &= \sqrt{4 + 0 + 4} = \sqrt{8}\\ &= 2\sqrt{2} \approx 2.8.\end{aligned}$$

Spheres

See figure 25.5.

Definition. The ***standard equation for a sphere*** of radius a centred at $P_0(x_0, y_0, z_0)$ is

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = a^2.$$

Example 25.4. Find the centre and radius of the sphere

$$x^2 + y^2 + z^2 + 3x - 4z + 1 = 0.$$

solution: We need to put this equation into the standard form. Since $(x - b)^2 = x^2 - 2b + b^2$ we have that

$$\begin{aligned} x^2 + y^2 + z^2 + 3x - 4z + 1 &= 0 \\ (x^2 + 3x) + y^2 + (z^2 - 4z) &= -1 \\ \left(x^2 + 3x + \frac{9}{4}\right) - \frac{9}{4} + y^2 + (z^2 - 4z + 4) - 4 &= -1 \\ \left(x^2 + 3x + \frac{9}{4}\right) + y^2 + (z^2 - 4z + 4) &= -1 + \frac{9}{4} + 4 \\ \left(x + \frac{3}{2}\right)^2 + y^2 + (z - 2)^2 &= \frac{21}{4}. \end{aligned}$$

The centre is at $P_0(x_0, y_0, z_0) = P_0(-\frac{3}{2}, 0, 2)$ and the radius is $a = \sqrt{\frac{21}{4}} = \frac{\sqrt{3}\sqrt{7}}{2}$.

Spheres

Bkz. şekil 25.5.

Tanım. Yarıçapı a ve merkezi $P_0(x_0, y_0, z_0)$ olan ***Bir kürenin standart denklemi***

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = a^2.$$

Örnek 25.5. Verilen kürenin merkez ve yarıçapını bulunuz:

$$x^2 + y^2 + z^2 + 6x - 6y + 6z = 22.$$

çözüm: Bu denklemi standart forma getirmemiz gerek. Şimdi $(x - b)^2 = x^2 - 2b + b^2$ olduğundan

$$\begin{aligned} x^2 + y^2 + z^2 + 6x - 6y + 6z &= 22 \\ (x^2 + 6x) + (y^2 - 6y) + (z^2 + 6z) &= 22 \\ (x^2 + 6x + 9) - 9 + (y^2 - 6y + 9) - 9 + (z^2 + 6z + 9) - 9 &= 22 \\ (x^2 + 6x + 9) + (y^2 - 6y + 9) + (z^2 + 6z + 9) &= 49 \\ (x + 3)^2 + (y - 3)^2 + (z + 3)^2 &= 49 \end{aligned}$$

Merkezi $P_0(x_0, y_0, z_0) = P_0(-3, 3, -3)$ olup yarıçapı $a = \sqrt{49} = 7$.

Problems

Problem 25.1. Find the distance between the following pairs of points.

(a). $P_1(-1, 1, 5)$ and $P_2(2, 5, 0)$.

(b). $A(1, 0, 0)$ and $B(0, 0, 1)$.

(c). $C(10, 5, -8)$ and $D(10, -25, 32)$.

(d). $E(8, 9, 7)$ and $F(2, 2, 3)$.

(e). $G(-4, 2, -4)$ and $O(0, 0, 0)$.

Problem 25.2. Find the centre and the radius of the sphere

$$x^2 + y^2 + z^2 - 6y + 8z = 0.$$

Sorular

Soru 25.1. Aşağısaki nokta çiftleri arasındaki uzaklığı bulunuz.

(a). $P_1(-1, 1, 5)$ ve $P_2(2, 5, 0)$.

(b). $A(1, 0, 0)$ ve $B(0, 0, 1)$.

(c). $C(10, 5, -8)$ ve $D(10, -25, 32)$.

(d). $E(8, 9, 7)$ ve $F(2, 2, 3)$.

(e). $G(-4, 2, -4)$ ve $O(0, 0, 0)$.

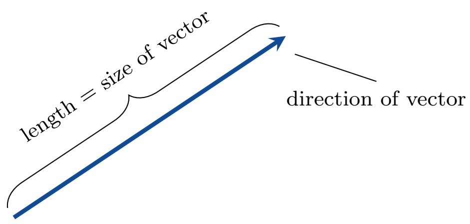
Soru 25.2. Verilen denklemdeki kürenin merkezini ve yarıçapını bulunuz

$$x^2 + y^2 + z^2 - 6y + 8z = 0.$$

26

Vectors

For some quantities (mass, time, distance, ...) we only need a number. For some quantities (velocity, force, ...) we need a number and a direction.



A ***vector*** is an object which has a size (length) and a direction.

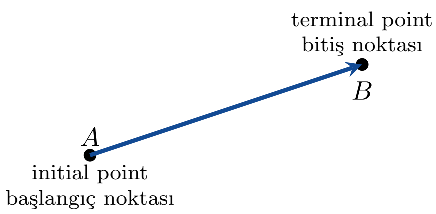


Figure 26.1: The initial point and terminal point of a vector.
Şekil 26.1:

Definition. The vector $\overrightarrow{AB}$ has ***initial point*** A and ***terminal point*** B.

The ***length*** of $\overrightarrow{AB}$ is written $\left\|\overrightarrow{AB}\right\|$.

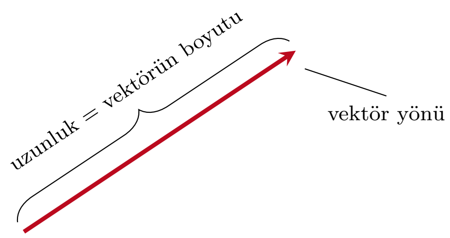
Two vectors are equal if they have the same length and the same direction. In figure 26.2, we can say that

$$\overrightarrow{AB} = \overrightarrow{CD} = \overrightarrow{EF} = \overrightarrow{OP}.$$

Note that $\overrightarrow{AB} \neq \overrightarrow{GH}$ because the lengths are different, and $\overrightarrow{AB} \neq \overrightarrow{IJ}$ because the directions are different.

Vektörler

Bazı büyüklükler (kütle, zaman, mesafe, ldots) sadece bir sayı yeterli oluyor. Ancak bazı büyüklükler için (hız, kuvvet, ldots) bir sayıyla bir de yöne ihtiyacımız var.



Vektör bir büyüklüğü (uzunluğu) ve bir yönü olan nesnedir.

Tanım. $\overrightarrow{AB}$ vektörünün ***başlangıç noktası*** A ve ***bitiş noktası*** B dir.

$\overrightarrow{AB}$'nin ***uzunluğu*** $\left\|\overrightarrow{AB}\right\|$ ile gösterilir.

Two vectors are equal if they have the same length and the same direction. In figure 26.2, we can say that

$$\overrightarrow{AB} = \overrightarrow{CD} = \overrightarrow{EF} = \overrightarrow{OP}.$$

Note that $\overrightarrow{AB} \neq \overrightarrow{GH}$ because the lengths are different, and $\overrightarrow{AB} \neq \overrightarrow{IJ}$ because the directions are different.

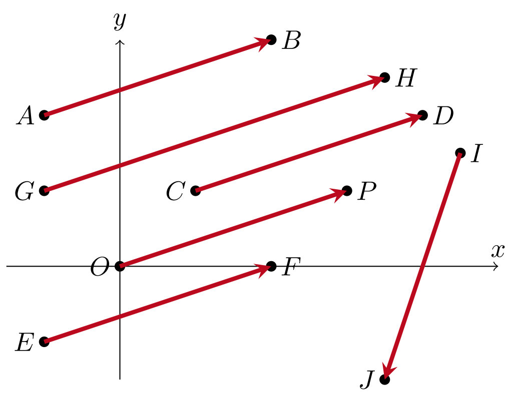


Figure 26.2: Six vectors.
Şekil 26.2: Altı vektör.

Notation

When we use a computer, we use bold letters for vectors: $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$, ... When we use a pen, we use underlined letters for vectors: $\underline{u}$, $\underline{v}$, $\underline{w}$, ...

If we type $a\mathbf{u} + b\mathbf{v}$ or write $a\underline{u} + b\underline{v}$, then

- a and b are numbers; and
- $\mathbf{u}$, $\mathbf{v}$, $\underline{u}$ and $\underline{v}$ are vectors.

Definition. In $\mathbb{R}^2$: If $\mathbf{v}$ has initial point $(0,0)$ and terminal point (v_1, v_2), then the ***component form*** of $\mathbf{v}$ is $\mathbf{v} = (v_1, v_2)$.

In $\mathbb{R}^3$: If $\mathbf{v}$ has initial point $(0,0,0)$ and terminal point (v_1, v_2, v_3), then the ***component form*** of $\mathbf{v}$ is $\mathbf{v} = (v_1, v_2, v_3)$.

Definition. In $\mathbb{R}^2$: The ***norm*** (or ***length***) of $\mathbf{v} = (v_1, v_2)$ is

$$\|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2}$$

In $\mathbb{R}^3$: The ***norm*** of $\mathbf{v} = \overrightarrow{PQ}$ is

$$\begin{aligned}\|\mathbf{v}\| &= \sqrt{v_1^2 + v_2^2 + v_3^2} \\ &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.\end{aligned}$$

The vectors $\mathbf{0} = (0,0)$ and $\mathbf{0} = (0,0,0)$ have norm $\|\mathbf{0}\| = 0$. If $\mathbf{v} \neq 0$, then $\|\mathbf{v}\| > 0$.

Example 26.1. Find (a) the component form; and (b) the norm of the vector with initial point $P(-3, 4, 1)$ and terminal point $Q(-5, 2, 2)$.

solution:
(a) $\mathbf{v} = (v_1, v_2, v_3) = Q - P = (-5, 2, 2) - (-3, 4, 1) = (-2, -2, 1)$.

(b) $\|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2 + v_3^2} = \sqrt{(-2)^2 + (-2)^2 + 1^2} = \sqrt{9} = 3$.

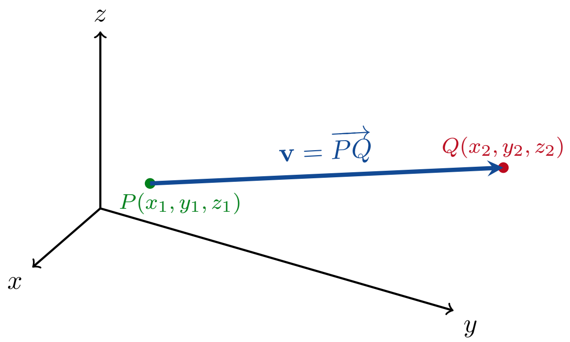


Figure 26.3: The vector $(v_1, v_2, v_3) = \mathbf{v} = (x_2 - x_1, y_2 - y_1, z_2 - z_1)$.
Şekil 26.3:

Notation

When we use a computer, we use bold letters for vectors: $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$, ... When we use a pen, we use underlined letters for vectors: $\underline{u}$, $\underline{v}$, $\underline{w}$, ...

If we type $a\mathbf{u} + b\mathbf{v}$ or write $a\underline{u} + b\underline{v}$, then

- a and b are numbers; and
- $\mathbf{u}$, $\mathbf{v}$, $\underline{u}$ and $\underline{v}$ are vectors.

Tanım. In $\mathbb{R}^2$: If $\mathbf{v}$ has initial point $(0,0)$ and terminal point (v_1, v_2), then the ***component form*** of $\mathbf{v}$ is $\mathbf{v} = (v_1, v_2)$.

In $\mathbb{R}^3$: If $\mathbf{v}$ has initial point $(0,0,0)$ and terminal point (v_1, v_2, v_3), then the ***component form*** of $\mathbf{v}$ is $\mathbf{v} = (v_1, v_2, v_3)$.

Tanım. In $\mathbb{R}^2$: The ***norm*** (or ***length***) of $\mathbf{v} = (v_1, v_2)$ is

$$\|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2}$$

In $\mathbb{R}^3$: The ***norm*** of $\mathbf{v} = \overrightarrow{PQ}$ is

$$\begin{aligned}\|\mathbf{v}\| &= \sqrt{v_1^2 + v_2^2 + v_3^2} \\ &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.\end{aligned}$$

The vectors $\mathbf{0} = (0,0)$ and $\mathbf{0} = (0,0,0)$ have norm $\|\mathbf{0}\| = 0$. If $\mathbf{v} \neq 0$, then $\|\mathbf{v}\| > 0$.

Örnek 26.1. Find (a) the component form; and (b) the norm of the vector with initial point $P(-3, 4, 1)$ and terminal point $Q(-5, 2, 2)$.

çözüm:
(a) $\mathbf{v} = (v_1, v_2, v_3) = Q - P = (-5, 2, 2) - (-3, 4, 1) = (-2, -2, 1)$.

(b) $\|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2 + v_3^2} = \sqrt{(-2)^2 + (-2)^2 + 1^2} = \sqrt{9} = 3$.

Vector Algebra

???????

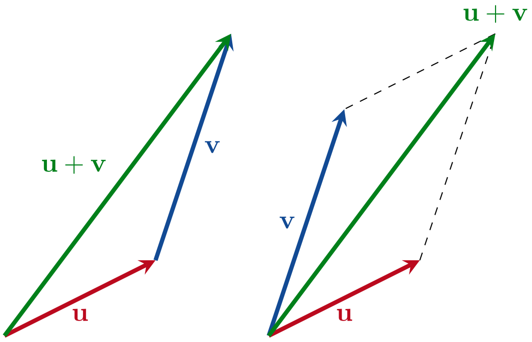


Figure 26.4: **u** + **v** considered in two ways.
Şekil 26.4:

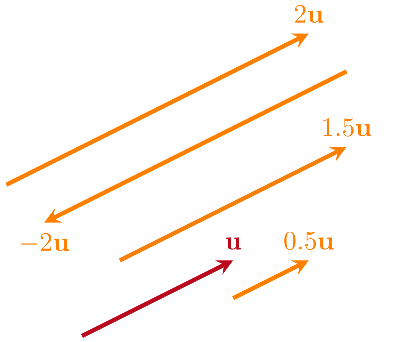


Figure 26.5: Constant multiples of **u**.
Şekil 26.5:

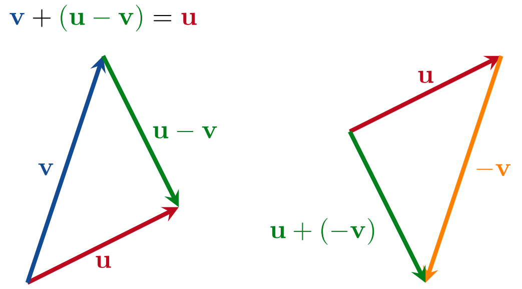


Figure 26.6: **u** − **v** considered in two ways.
Şekil 26.6:

Let $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$ be vectors. Let k be a number. Then

$$\mathbf{u} + \mathbf{v} = (u_1 + v_1, u_2 + v_2, u_3 + v_3)$$

and

$$k\mathbf{u} = (ku_1, ku_2, ku_3).$$

Note that

$$\begin{aligned}\|k\mathbf{u}\| &= \|(ku_1, ku_2, ku_3)\| = \sqrt{(ku_1)^2 + (ku_2)^2 + (ku_3)^2} \\ &= \sqrt{k^2u_1^2 + k^2u_2^2 + k^2u_3^2} = \sqrt{k^2(u_1^2 + u_2^2 + u_3^2)} \\ &= \sqrt{k^2}\sqrt{u_1^2 + u_2^2 + u_3^2} = |k|\,\|\mathbf{u}\|\,.\end{aligned}$$

The vector $-\mathbf{u} = (-1)\mathbf{u}$ has the same length as $\mathbf{u}$, but points in the opposite direction.

Example 26.2. Let $\mathbf{u} = (-1, 3, 1)$ and $\mathbf{v} = (4, 7, 0)$. Find (a) $2\mathbf{u} + 3\mathbf{v}$, (b) $\mathbf{u} - \mathbf{v}$, and (c) $\left\|\frac{1}{2}\mathbf{u}\right\|$.

solution:

(a) $2\mathbf{u} + 3\mathbf{v} = 2(-1, 3, 1) + 3(4, 7, 0) = (-2, 6, 2) + (12, 21, 0) = (10, 27, 2)$;

(b) $\mathbf{u} - \mathbf{v} = (-1, 3, 1) - (4, 7, 0) = (-5, -4, 1)$;

(c) $\left\|\frac{1}{2}\mathbf{u}\right\| = \frac{1}{2}\|\mathbf{u}\| = \frac{1}{2}\sqrt{(-1)^2 + 3^2 + 1^2} = \frac{1}{2}\sqrt{11}$.

Let $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$ be vectors. Let k be a number. Then

$$\mathbf{u} + \mathbf{v} = (u_1 + v_1, u_2 + v_2, u_3 + v_3)$$

and

$$k\mathbf{u} = (ku_1, ku_2, ku_3).$$

Note that

$$\begin{aligned}\|k\mathbf{u}\| &= \|(ku_1, ku_2, ku_3)\| = \sqrt{(ku_1)^2 + (ku_2)^2 + (ku_3)^2} \\ &= \sqrt{k^2u_1^2 + k^2u_2^2 + k^2u_3^2} = \sqrt{k^2(u_1^2 + u_2^2 + u_3^2)} \\ &= \sqrt{k^2}\sqrt{u_1^2 + u_2^2 + u_3^2} = |k|\,\|\mathbf{u}\|\,.\end{aligned}$$

The vector $-\mathbf{u} = (-1)\mathbf{u}$ has the same length as $\mathbf{u}$, but points in the opposite direction.

Örnek 26.2. Let $\mathbf{u} = (-1, 3, 1)$ and $\mathbf{v} = (4, 7, 0)$. Find (a) $2\mathbf{u} + 3\mathbf{v}$, (b) $\mathbf{u} - \mathbf{v}$, and (c) $\left\|\frac{1}{2}\mathbf{u}\right\|$.

çözüm:

(a) $2\mathbf{u} + 3\mathbf{v} = 2(-1, 3, 1) + 3(4, 7, 0) = (-2, 6, 2) + (12, 21, 0) = (10, 27, 2)$;

(b) $\mathbf{u} - \mathbf{v} = (-1, 3, 1) - (4, 7, 0) = (-5, -4, 1)$;

(c) $\left\|\frac{1}{2}\mathbf{u}\right\| = \frac{1}{2}\|\mathbf{u}\| = \frac{1}{2}\sqrt{(-1)^2 + 3^2 + 1^2} = \frac{1}{2}\sqrt{11}$.

Properties of Vector Operations

Let **u**, **v** and **w** be vectors. Let a and b be numbers. Then

(i). $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$;

(ii). $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$;

(iii). $\mathbf{u} + \mathbf{0} = \mathbf{u}$;

(iv). $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$;

(v). $0\mathbf{u} = \mathbf{0}$;

Properties of Vector Operations

Let **u**, **v** and **w** be vectors. Let a and b be numbers. Then

(i). $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$;

(ii). $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$;

(iii). $\mathbf{u} + \mathbf{0} = \mathbf{u}$;

(iv). $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$;

(v). $0\mathbf{u} = \mathbf{0}$;

(vi). $1\mathbf{u} = \mathbf{u}$;

(vii). $a(b\mathbf{u}) = (ab)\mathbf{u}$;

(viii). $a(\mathbf{u}+\mathbf{v}) = a\mathbf{u} + a\mathbf{v}$;

(ix). $(a+b)\mathbf{u} = a\mathbf{u} + b\mathbf{u}$.

Remark. We **can not** multiply vectors. Never never never never write "**uv**".

Unit Vectors

Definition. $\mathbf{u}$ is called a ***unit vector*** $\iff$ $\|\mathbf{u}\| = 1$.

Example 26.3. $\mathbf{u} = (2^{-\frac{1}{2}}, \frac{1}{2}, -\frac{1}{2})$ is a unit vector because

$$\|\mathbf{u}\| = \sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{2}\right)^2 + \left(-\frac{1}{2}\right)^2} = \sqrt{\frac{1}{2} + \frac{1}{4} + \frac{1}{4}} = 1.$$

In $\mathbb{R}^2$: The ***standard unit vectors*** are $\mathbf{i} = (1,0)$ and $\mathbf{j} = (0,1)$.
In $\mathbb{R}^3$: The ***standard unit vectors*** are $\mathbf{i} = (1,0,0)$, $\mathbf{j} = (0,1,0)$ and $\mathbf{k} = (0,0,1)$. Any vector $\mathbf{v} \in \mathbb{R}^3$ can be written

$$\begin{aligned}\mathbf{v} &= (v_1, v_2, v_3) = (v_1,0,0) + (0,v_2,0) + (0,0,v_3)\\ &= v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}.\end{aligned}$$

If $\|\mathbf{v}\| \neq 0$, then $\frac{\mathbf{v}}{\|\mathbf{v}\|}$ is a unit vector because

$$\left\|\frac{\mathbf{v}}{\|\mathbf{v}\|}\right\| = \left\|\frac{1}{\|\mathbf{v}\|}\mathbf{v}\right\| = \frac{1}{\|\mathbf{v}\|}\|\mathbf{v}\| = 1.$$

Clearly $\frac{\mathbf{v}}{\|\mathbf{v}\|}$ and $\mathbf{v}$ point in the same direction.

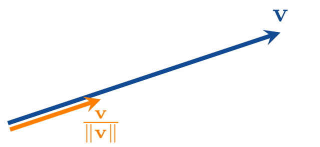


Figure 26.7: $\frac{\mathbf{v}}{\|\mathbf{v}\|}$ is a unit vector pointing in the same direction as $\mathbf{v}$.
Şekil 26.7:

Example 26.4. Find a unit vector $\mathbf{u}$ with the same direction as $\overrightarrow{P_1P_2}$ where $P_(1,0,1)$ and $P_2(3,2,0)$.

solution:
We calculate that $\overrightarrow{P_1P_2} = P_2 - P_1 = (3,2,0) - (1,0,1) = (2,2,-1) = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ and that $\left\|\overrightarrow{P_1P_2}\right\| = \sqrt{2^2 + 2^2 + (-1)^2} = 3$. The required unit vector is

$$\mathbf{u} = \frac{\overrightarrow{P_1P_2}}{\left\|\overrightarrow{P_1P_2}\right\|} = \frac{2\mathbf{i} + 2\mathbf{j} - \mathbf{k}}{3} = \frac{2}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} - \frac{1}{3}\mathbf{k}.$$

(vi). $1\mathbf{u} = \mathbf{u}$;

(vii). $a(b\mathbf{u}) = (ab)\mathbf{u}$;

(viii). $a(\mathbf{u}+\mathbf{v}) = a\mathbf{u} + a\mathbf{v}$;

(ix). $(a+b)\mathbf{u} = a\mathbf{u} + b\mathbf{u}$.

Not. We **can not** multiply vectors. Never never never never write "**uv**".

Unit Vectors

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In $\mathbb{R}^2$: The ***standard unit vectors*** are $\mathbf{i} = (1,0)$ and $\mathbf{j} = (0,1)$.
In $\mathbb{R}^3$: The ***standard unit vectors*** are $\mathbf{i} = (1,0,0)$, $\mathbf{j} = (0,1,0)$ and $\mathbf{k} = (0,0,1)$. Any vector $\mathbf{v} \in \mathbb{R}^3$ can be written

$$\begin{aligned}\mathbf{v} &= (v_1, v_2, v_3) = (v_1,0,0) + (0,v_2,0) + (0,0,v_3)\\ &= v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}.\end{aligned}$$

If $\|\mathbf{v}\| \neq 0$, then $\frac{\mathbf{v}}{\|\mathbf{v}\|}$ is a unit vector because

$$\left\|\frac{\mathbf{v}}{\|\mathbf{v}\|}\right\| = \left\|\frac{1}{\|\mathbf{v}\|}\mathbf{v}\right\| = \frac{1}{\|\mathbf{v}\|}\|\mathbf{v}\| = 1.$$

Clearly $\frac{\mathbf{v}}{\|\mathbf{v}\|}$ and $\mathbf{v}$ point in the same direction.

Örnek 26.4. Find a unit vector $\mathbf{u}$ with the same direction as $\overrightarrow{P_1P_2}$ where $P_(1,0,1)$ and $P_2(3,2,0)$.

çözüm:
We calculate that $\overrightarrow{P_1P_2} = P_2 - P_1 = (3,2,0) - (1,0,1) = (2,2,-1) = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ and that $\left\|\overrightarrow{P_1P_2}\right\| = \sqrt{2^2 + 2^2 + (-1)^2} = 3$. The required unit vector is

$$\mathbf{u} = \frac{\overrightarrow{P_1P_2}}{\left\|\overrightarrow{P_1P_2}\right\|} = \frac{2\mathbf{i} + 2\mathbf{j} - \mathbf{k}}{3} = \frac{2}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} - \frac{1}{3}\mathbf{k}.$$

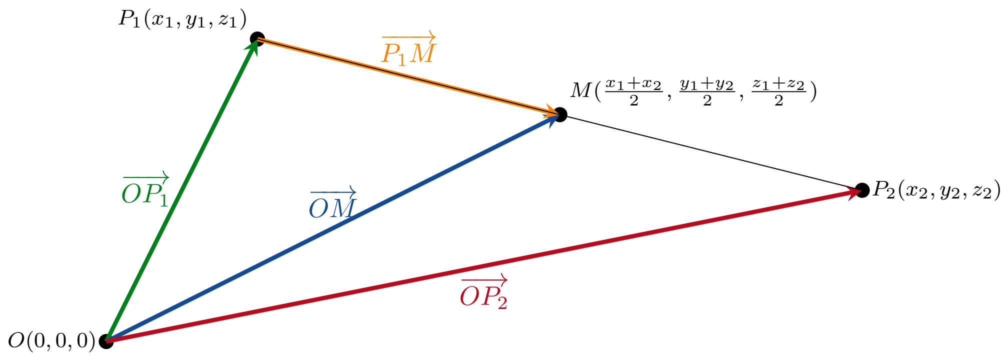


Figure 26.8: Midpoint of a Line Segment.
Şekil 26.8:

Midpoint of a Line Segment

The ***midpoint*** of the line segment from $P_1(x_1, y_1, z_1)$ to $P_2(x_2, y_2, z_2)$ is the point $M(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2})$. To see why, we can calculate that

$$\begin{aligned}\overrightarrow{OM} &= \overrightarrow{OP_1} + \frac{1}{2}\overrightarrow{P_1P_2} = \overrightarrow{OP_1} + \frac{1}{2}\left(\overrightarrow{OP_2} - \overrightarrow{OP_1}\right)\\ &= \frac{1}{2}\left(\overrightarrow{OP_1} + \overrightarrow{OP_2}\right)\\ &= \frac{x_1+x_2}{2}\mathbf{i} + \frac{y_1+y_2}{2}\mathbf{j} + \frac{z_1+z_2}{2}\mathbf{k}.\end{aligned}$$

Example 26.5. Find the midpoint of the line segment from $P_1(3, -2, 0)$ to $P_2(7, 4, 4)$.

solution: $(\frac{3+7}{2}, \frac{-2+4}{2}, \frac{0+4}{2}) = (5, 1, 2)$.

Midpoint of a Line Segment

The ***midpoint*** of the line segment from $P_1(x_1, y_1, z_1)$ to $P_2(x_2, y_2, z_2)$ is the point $M(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2})$. To see why, we can calculate that

$$\begin{aligned}\overrightarrow{OM} &= \overrightarrow{OP_1} + \frac{1}{2}\overrightarrow{P_1P_2} = \overrightarrow{OP_1} + \frac{1}{2}\left(\overrightarrow{OP_2} - \overrightarrow{OP_1}\right)\\ &= \frac{1}{2}\left(\overrightarrow{OP_1} + \overrightarrow{OP_2}\right)\\ &= \frac{x_1+x_2}{2}\mathbf{i} + \frac{y_1+y_2}{2}\mathbf{j} + \frac{z_1+z_2}{2}\mathbf{k}.\end{aligned}$$

Örnek 26.5. Find the midpoint of the line segment from $P_1(3, -2, 0)$ to $P_2(7, 4, 4)$.

çözüm: $(\frac{3+7}{2}, \frac{-2+4}{2}, \frac{0+4}{2}) = (5, 1, 2)$.

Problems

Problem 26.1.

(a). Find $(5\mathbf{a} - 3\mathbf{b})$ if $\mathbf{a} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ and $\mathbf{b} = 2\mathbf{i} + 5\mathbf{k}$.

(b). Find a unit vector which points in the same direction as $\mathbf{v} = 6\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$.

Sorular

Soru 26.1.

(a). Find $(5\mathbf{a} - 3\mathbf{b})$ if $\mathbf{a} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ and $\mathbf{b} = 2\mathbf{i} + 5\mathbf{k}$.

(b). Find a unit vector which points in the same direction as $\mathbf{v} = 6\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$.

27

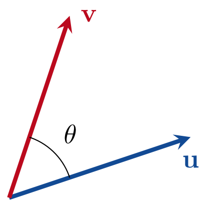
The Dot Product

Definition. In $\mathbb{R}^2$, the ***dot product*** of $\mathbf{u} = (u_1, u_2) = u_1\mathbf{i} + u_2\mathbf{j}$ and $\mathbf{v} = (v_1, v_2) = v_1\mathbf{i} + v_2\mathbf{j}$ is

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2.$$

Definition. In $\mathbb{R}^3$, the ***dot product*** of $\mathbf{u} = (u_1, u_2, u_3) = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ and $\mathbf{v} = (v_1, v_2, v_3) = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ is

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + u_3 v_3.$$



Theorem 27.1. *The angle between* $\mathbf{u}$ *and* $\mathbf{v}$ *is*

$$\theta = \cos^{-1}\left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \, \|\mathbf{v}\|}\right).$$

Example 27.1.

$$\begin{aligned}(1, -2, 1) \cdot (-6, 2, -3) &= (1 \times -6) + (-2 \times 2) + (1 \times -3) \\ &= -6 - 4 + 3 = -7.\end{aligned}$$

Example 27.2.

$$\begin{aligned}(\tfrac{1}{2}\mathbf{i} + 3\mathbf{j} + \mathbf{k}) \cdot (4\mathbf{i} - \mathbf{j} + 2\mathbf{k}) &= (\tfrac{1}{2} \times 4) + (3 \times -1) + (1 \times 2) \\ &= 2 - 3 + 2 = 1.\end{aligned}$$

Example 27.3. Find the angle between $\mathbf{u} = \mathbf{i} - 2\mathbf{j} - 2\mathbf{k}$ and $\mathbf{v} = 6\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$.

solution: Since $\mathbf{u} \cdot \mathbf{v} = (1, -2, -2) \cdot (6, 3, 2) = (1 \times 6) + (-2 \times 3) + (-2 \times 2) = 6 - 6 - 4 = -4$, $\|\mathbf{u}\| = \sqrt{1^2 + (-2)^2 + (-2)^2} = \sqrt{9} = 3$ and $\|\mathbf{v}\| = \sqrt{6^2 + 3^2 + 2^2} = \sqrt{49} = 7$, we have that

$$\theta = \cos^{-1}\left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \, \|\mathbf{v}\|}\right) = \cos^{-1}\left(-\frac{4}{21}\right) \approx 1.76 \text{ radians} \approx 98.5°.$$

Example 27.4. If $A(0, 0)$, $B(3, 5)$ and $C(5, 2)$, find $\theta = \measuredangle ACB$.

solution: θ is the angle between $\overrightarrow{CA}$ and $\overrightarrow{CB}$. We calcu-

Nokta Çarpım

Tanım. In $\mathbb{R}^2$, the ***dot product*** of $\mathbf{u} = (u_1, u_2) = u_1\mathbf{i} + u_2\mathbf{j}$ and $\mathbf{v} = (v_1, v_2) = v_1\mathbf{i} + v_2\mathbf{j}$ is

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2.$$

Tanım. In $\mathbb{R}^3$, the ***dot product*** of $\mathbf{u} = (u_1, u_2, u_3) = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ and $\mathbf{v} = (v_1, v_2, v_3) = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ is

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + u_3 v_3.$$

Teorem 27.1. *The angle between* $\mathbf{u}$ *and* $\mathbf{v}$ *is*

$$\theta = \cos^{-1}\left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \, \|\mathbf{v}\|}\right).$$

Örnek 27.1.

$$\begin{aligned}(1, -2, 1) \cdot (-6, 2, -3) &= (1 \times -6) + (-2 \times 2) + (1 \times -3) \\ &= -6 - 4 + 3 = -7.\end{aligned}$$

Örnek 27.2.

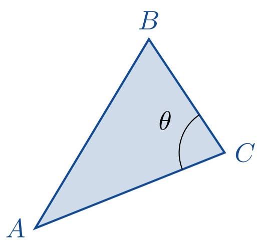
$$\begin{aligned}(\tfrac{1}{2}\mathbf{i} + 3\mathbf{j} + \mathbf{k}) \cdot (4\mathbf{i} - \mathbf{j} + 2\mathbf{k}) &= (\tfrac{1}{2} \times 4) + (3 \times -1) + (1 \times 2) \\ &= 2 - 3 + 2 = 1.\end{aligned}$$

Örnek 27.3. Find the angle between $\mathbf{u} = \mathbf{i} - 2\mathbf{j} - 2\mathbf{k}$ and $\mathbf{v} = 6\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$.

çözüm: Since $\mathbf{u} \cdot \mathbf{v} = (1, -2, -2) \cdot (6, 3, 2) = (1 \times 6) + (-2 \times 3) + (-2 \times 2) = 6 - 6 - 4 = -4$, $\|\mathbf{u}\| = \sqrt{1^2 + (-2)^2 + (-2)^2} = \sqrt{9} = 3$ and $\|\mathbf{v}\| = \sqrt{6^2 + 3^2 + 2^2} = \sqrt{49} = 7$, we have that

$$\theta = \cos^{-1}\left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \, \|\mathbf{v}\|}\right) = \cos^{-1}\left(-\frac{4}{21}\right) \approx 1.76 \text{ radians} \approx 98.5°.$$

Örnek 27.4. If $A(0, 0)$, $B(3, 5)$ and $C(5, 2)$, find $\theta = \measuredangle ACB$.



late that $\overrightarrow{CA} = A - C = (0,0) - (5,2) = (-5,-2)$, $\overrightarrow{CB} = B - C = (3,5) - (5,2) = (-2,3)$, $\overrightarrow{CA} \cdot \overrightarrow{CB} = (-5,-2) \cdot (-2,3) = 4$, $\left\|\overrightarrow{CA}\right\| = \sqrt{(-5)^2 + (-2)^2} = \sqrt{29}$ and $\left\|\overrightarrow{CB}\right\| = \sqrt{(-2)^2 + 3^2} = \sqrt{13}$. Therefore

$$\theta = \cos^{-1}\left(\frac{\overrightarrow{CA} \cdot \overrightarrow{CB}}{\left\|\overrightarrow{CA}\right\| \left\|\overrightarrow{CB}\right\|}\right) = \cos^{-1}\left(\frac{4}{\sqrt{29}\sqrt{13}}\right)$$
$$\approx 78.1^\circ \approx 1.36 \text{ radians.}$$

Definition. $\mathbf{u}$ and $\mathbf{v}$ are ***orthogonal*** $\iff \mathbf{u} \cdot \mathbf{v} = 0$.

Remark. Note that

$$\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \|\mathbf{v}\| \cos\theta$$

by Theorem 27.1. Therefore

$$\mathbf{u} \text{ and } \mathbf{v} \text{ are orthogonal} \iff \begin{array}{c} \mathbf{u} = \mathbf{0}, \\ \mathbf{v} = \mathbf{0} \\ \text{or } \theta = 90^\circ. \end{array}$$

Example 27.5. $\mathbf{u} = (3,-2)$ and $\mathbf{v} = (4,6)$ are orthogonal because $\mathbf{u} \cdot \mathbf{v} = (3,-2) \cdot (4,6) = (3 \times 4) + (-2 \times 6) = 12 - 12 = 0$.

Example 27.6. $\mathbf{u} = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ and $\mathbf{v} = 2\mathbf{j} + 4\mathbf{k}$ are orthogonal because $\mathbf{u} \cdot \mathbf{v} = (3 \times 0) + (-2 \times 2) + (1 \times 4) = 0 - 4 + 4 = 0$.

Example 27.7. $\mathbf{0}$ is orthogonal to every vector $\mathbf{u}$ because $\mathbf{0} \cdot \mathbf{u} = (0,0,0) \cdot (u_1, u_2, u_3) = 0u_1 + 0u_2 + 0u_3 = 0$.

Properties of the Dot Product

Let $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ be vectors. Let k be a number. Then

(i). $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$;

(ii). $(k\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (k\mathbf{v}) = k(\mathbf{u} \cdot \mathbf{v})$;

(iii). $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \cdot \mathbf{v}) + (\mathbf{u} \cdot \mathbf{w})$;

(iv). $\mathbf{u} \cdot \mathbf{u} = \|\mathbf{u}\|^2$; and

(v). $\mathbf{0} \cdot \mathbf{u} = 0$.

çözüm: θ is the angle between $\overrightarrow{CA}$ and $\overrightarrow{CB}$. We calculate that $\overrightarrow{CA} = A - C = (0,0) - (5,2) = (-5,-2)$, $\overrightarrow{CB} = B - C = (3,5) - (5,2) = (-2,3)$, $\overrightarrow{CA} \cdot \overrightarrow{CB} = (-5,-2) \cdot (-2,3) = 4$, $\left\|\overrightarrow{CA}\right\| = \sqrt{(-5)^2 + (-2)^2} = \sqrt{29}$ and $\left\|\overrightarrow{CB}\right\| = \sqrt{(-2)^2 + 3^2} = \sqrt{13}$. Therefore

$$\theta = \cos^{-1}\left(\frac{\overrightarrow{CA} \cdot \overrightarrow{CB}}{\left\|\overrightarrow{CA}\right\| \left\|\overrightarrow{CB}\right\|}\right) = \cos^{-1}\left(\frac{4}{\sqrt{29}\sqrt{13}}\right)$$
$$\approx 78.1^\circ \approx 1.36 \text{ radians.}$$

Tanım. $\mathbf{u}$ and $\mathbf{v}$ are ***orthogonal*** $\iff \mathbf{u} \cdot \mathbf{v} = 0$.

Not. Note that

$$\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \|\mathbf{v}\| \cos\theta$$

by Theorem 27.1. Therefore

$$\mathbf{u} \text{ and } \mathbf{v} \text{ are orthogonal} \iff \begin{array}{c} \mathbf{u} = \mathbf{0}, \\ \mathbf{v} = \mathbf{0} \\ \text{or } \theta = 90^\circ. \end{array}$$

Örnek 27.5. $\mathbf{u} = (3,-2)$ and $\mathbf{v} = (4,6)$ are orthogonal because $\mathbf{u} \cdot \mathbf{v} = (3,-2) \cdot (4,6) = (3 \times 4) + (-2 \times 6) = 12 - 12 = 0$.

Örnek 27.6. $\mathbf{u} = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ and $\mathbf{v} = 2\mathbf{j} + 4\mathbf{k}$ are orthogonal because $\mathbf{u} \cdot \mathbf{v} = (3 \times 0) + (-2 \times 2) + (1 \times 4) = 0 - 4 + 4 = 0$.

Örnek 27.7. $\mathbf{0}$ is orthogonal to every vector $\mathbf{u}$ because $\mathbf{0} \cdot \mathbf{u} = (0,0,0) \cdot (u_1, u_2, u_3) = 0u_1 + 0u_2 + 0u_3 = 0$.

Properties of the Dot Product

Let $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ be vectors. Let k be a number. Then

(i). $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$;

(ii). $(k\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (k\mathbf{v}) = k(\mathbf{u} \cdot \mathbf{v})$;

(iii). $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \cdot \mathbf{v}) + (\mathbf{u} \cdot \mathbf{w})$;

(iv). $\mathbf{u} \cdot \mathbf{u} = \|\mathbf{u}\|^2$; and

(v). $\mathbf{0} \cdot \mathbf{u} = 0$.

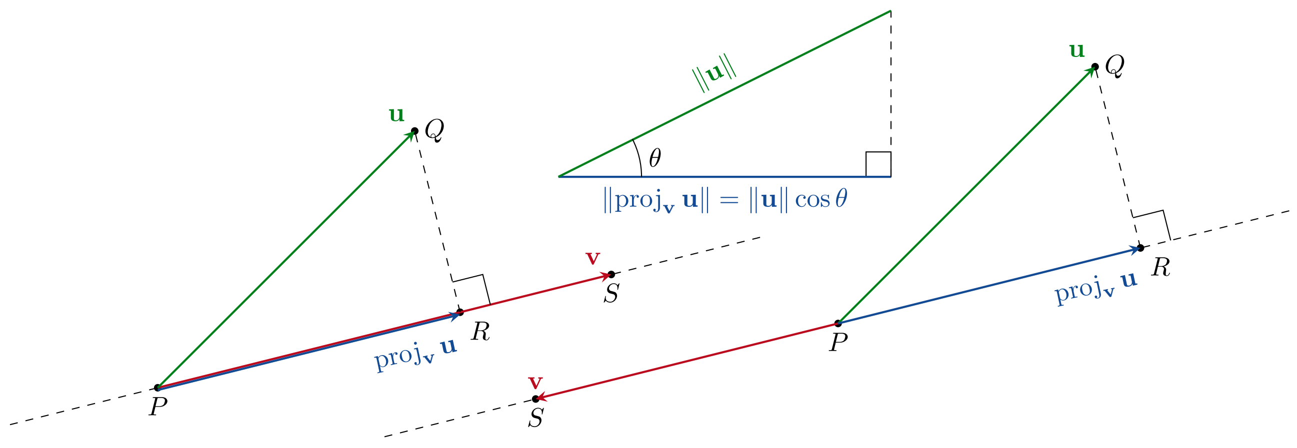


Figure 27.1: Vector Projections
Şekil 27.1: Vektör İzdüşümleri

Vector Projections

See figure 27.1.

Definition. The ***vector projection*** of $\mathbf{u}$ onto $\mathbf{v}$ is the vector

$$\operatorname{proj}_{\mathbf{v}} \mathbf{u} = \overrightarrow{PR}.$$

Now

$$\begin{aligned}
\operatorname{proj}_{\mathbf{v}} \mathbf{u} &= \left(\text{length of } \operatorname{proj}_{\mathbf{v}} \mathbf{u}\right) \begin{pmatrix}\text{a unit vector in the} \\ \text{same direction as } \mathbf{v}\end{pmatrix} \\
&= \|\operatorname{proj}_{\mathbf{v}} \mathbf{u}\| \left(\frac{\mathbf{v}}{\|\mathbf{v}\|}\right) \\
&= \|\mathbf{u}\| (\cos\theta) \left(\frac{\mathbf{v}}{\|\mathbf{v}\|}\right) \\
&= \left(\frac{\|\mathbf{u}\|\,\|\mathbf{v}\| \cos\theta}{\|\mathbf{v}\|^2}\right) \mathbf{v} \\
&= \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2}\right) \mathbf{v}.
\end{aligned}$$

Since this is an important formula, we write it as a theorem.

Theorem 27.2. *The **vector projection** of $\mathbf{u}$ onto $\mathbf{v}$ is*

$$\operatorname{proj}_{\mathbf{v}} \mathbf{u} = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2}\right) \mathbf{v}.$$

Example 27.8. Find the vector projection of $\mathbf{u} = 6\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$ onto $\mathbf{v} = \mathbf{i} - 2\mathbf{j} - 2\mathbf{k}$.

solution:

$$\begin{aligned}
\operatorname{proj}_{\mathbf{v}} \mathbf{u} &= \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2}\right) \mathbf{v} = \left(\frac{6-6-4}{1+4+4}\right) (\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}) \\
&= -\frac{4}{9}\mathbf{i} + \frac{8}{9}\mathbf{j} + \frac{8}{9}\mathbf{k}.
\end{aligned}$$

Example 27.9. Find the vector projection of $\mathbf{F} = 5\mathbf{i} + 2\mathbf{j}$ onto $\mathbf{v} = \mathbf{i} - 3\mathbf{j}$.

Vector Projections

See figure 27.1.

Tanım. The ***vector projection*** of $\mathbf{u}$ onto $\mathbf{v}$ is the vector

$$\operatorname{proj}_{\mathbf{v}} \mathbf{u} = \overrightarrow{PR}.$$

Now

$$\begin{aligned}
\operatorname{proj}_{\mathbf{v}} \mathbf{u} &= \left(\text{length of } \operatorname{proj}_{\mathbf{v}} \mathbf{u}\right) \begin{pmatrix}\text{a unit vector in the} \\ \text{same direction as } \mathbf{v}\end{pmatrix} \\
&= \|\operatorname{proj}_{\mathbf{v}} \mathbf{u}\| \left(\frac{\mathbf{v}}{\|\mathbf{v}\|}\right) \\
&= \|\mathbf{u}\| (\cos\theta) \left(\frac{\mathbf{v}}{\|\mathbf{v}\|}\right) \\
&= \left(\frac{\|\mathbf{u}\|\,\|\mathbf{v}\| \cos\theta}{\|\mathbf{v}\|^2}\right) \mathbf{v} \\
&= \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2}\right) \mathbf{v}.
\end{aligned}$$

Since this is an important formula, we write it as a theorem.

Teorem 27.2. *The **vector projection** of $\mathbf{u}$ onto $\mathbf{v}$ is*

$$\operatorname{proj}_{\mathbf{v}} \mathbf{u} = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2}\right) \mathbf{v}.$$

Örnek 27.8. Find the vector projection of $\mathbf{u} = 6\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$ onto $\mathbf{v} = \mathbf{i} - 2\mathbf{j} - 2\mathbf{k}$.

çözüm:

$$\begin{aligned}
\operatorname{proj}_{\mathbf{v}} \mathbf{u} &= \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2}\right) \mathbf{v} = \left(\frac{6-6-4}{1+4+4}\right) (\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}) \\
&= -\frac{4}{9}\mathbf{i} + \frac{8}{9}\mathbf{j} + \frac{8}{9}\mathbf{k}.
\end{aligned}$$

Örnek 27.9. Find the vector projection of $\mathbf{F} = 5\mathbf{i} + 2\mathbf{j}$ onto $\mathbf{v} = \mathbf{i} - 3\mathbf{j}$.

solution:

$$\text{proj}_{\mathbf{v}}\,\mathbf{F} = \left(\frac{\mathbf{F}\cdot\mathbf{v}}{\|\mathbf{v}\|^2}\right)\mathbf{v} = \left(\frac{5-6}{1+9}\right)(\mathbf{i}-3\mathbf{j})$$
$$= -\frac{1}{10}\mathbf{i} + \frac{3}{10}\mathbf{j}.$$

Problems

Problem 27.1. Let $\mathbf{u} = 2\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ and $\mathbf{v} = 2\mathbf{i} + 10\mathbf{j} - 11\mathbf{k}$. Let θ denote the angle between $\mathbf{u}$ and $\mathbf{v}$.

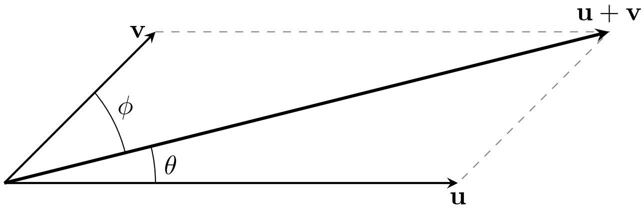
(a). Find $\mathbf{v}\cdot\mathbf{u}$.

(b). Find $\|\mathbf{u}\|$ and $\|\mathbf{v}\|$.

(c). Find $\cos\theta$.

(d). Find $\text{proj}_{\mathbf{v}}\,\mathbf{u}$.

Problem 27.2. Let $\mathbf{u}$ and $\mathbf{v}$ be vectors. Let θ denote the angle between $\mathbf{u}$ and $\mathbf{u}+\mathbf{v}$; and let ϕ denote the angle between $\mathbf{u}+\mathbf{v}$ and $\mathbf{v}$.



(a). Show that if $\|\mathbf{u}\| = \|\mathbf{v}\|$, then $\mathbf{u}\cdot(\mathbf{u}+\mathbf{v}) = (\mathbf{u}+\mathbf{v})\cdot\mathbf{v}$

(b). Show that if $\|\mathbf{u}\| = \|\mathbf{v}\|$, then $\theta = \phi$.

çözüm:

$$\text{proj}_{\mathbf{v}}\,\mathbf{F} = \left(\frac{\mathbf{F}\cdot\mathbf{v}}{\|\mathbf{v}\|^2}\right)\mathbf{v} = \left(\frac{5-6}{1+9}\right)(\mathbf{i}-3\mathbf{j})$$
$$= -\frac{1}{10}\mathbf{i} + \frac{3}{10}\mathbf{j}.$$

Sorular

Soru 27.1. Let $\mathbf{u} = 2\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ and $\mathbf{v} = 2\mathbf{i} + 10\mathbf{j} - 11\mathbf{k}$. Let θ denote the angle between $\mathbf{u}$ and $\mathbf{v}$.

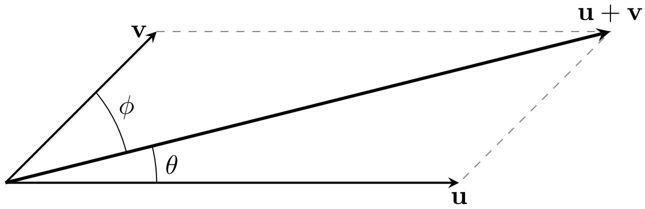
(a). Find $\mathbf{v}\cdot\mathbf{u}$.

(b). Find $\|\mathbf{u}\|$ and $\|\mathbf{v}\|$.

(c). Find $\cos\theta$.

(d). Find $\text{proj}_{\mathbf{v}}\,\mathbf{u}$.

Soru 27.2. Let $\mathbf{u}$ and $\mathbf{v}$ be vectors. Let θ denote the angle between $\mathbf{u}$ and $\mathbf{u}+\mathbf{v}$; and let ϕ denote the angle between $\mathbf{u}+\mathbf{v}$ and $\mathbf{v}$.



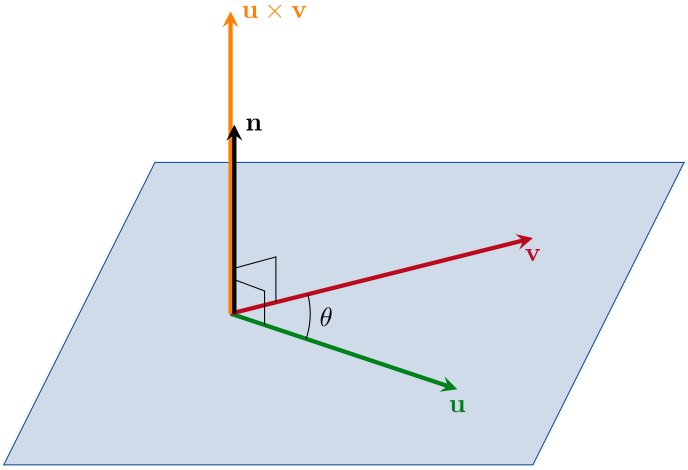
(a). Show that if $\|\mathbf{u}\| = \|\mathbf{v}\|$, then $\mathbf{u}\cdot(\mathbf{u}+\mathbf{v}) = (\mathbf{u}+\mathbf{v})\cdot\mathbf{v}$

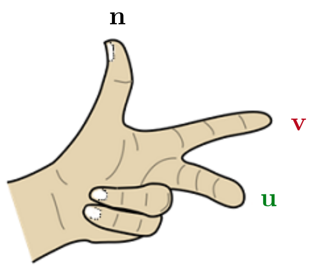
(b). Show that if $\|\mathbf{u}\| = \|\mathbf{v}\|$, then $\theta = \phi$.

28

The Cross Product

Vektörel Çarpım





Let $\mathbf{n}$ be a unit vector which satisfies

(i). $\mathbf{n}$ is orthogonal to $\mathbf{u}$ $\left(\overset{\mathbf{n}}{\uparrow}\!\llcorner\!\rightarrow \mathbf{u} \right)$;

(ii). $\mathbf{n}$ is orthogonal to $\mathbf{v}$ $\left(\overset{\mathbf{n}}{\uparrow}\!\llcorner\!\rightarrow \mathbf{v} \right)$; and

(iii). the direction of $\mathbf{n}$ is chosen using the left-hand rule.

Definition. The ***cross product*** of $\mathbf{u}$ and $\mathbf{v}$ is

$$\mathbf{u} \times \mathbf{v} = \|\mathbf{u}\| \, \|\mathbf{v}\| \, (\sin\theta)\mathbf{n}.$$

Remark.

- $\mathbf{u} \cdot \mathbf{v}$ is a number.
- $\mathbf{u} \times \mathbf{v}$ is a vector.

Remark.

$$\left(\begin{matrix} \mathbf{u} \text{ and } \mathbf{v} \\ \text{are parallel} \end{matrix} \right) \iff \theta = 0^\circ \text{ or } 180^\circ$$
$$\implies \sin\theta = 0 \implies \mathbf{u} \times \mathbf{v} = \mathbf{0}.$$

Let $\mathbf{n}$ be a unit vector which satisfies

(i). $\mathbf{n}$ is orthogonal to $\mathbf{u}$ $\left(\overset{\mathbf{n}}{\uparrow}\!\llcorner\!\rightarrow \mathbf{u} \right)$;

(ii). $\mathbf{n}$ is orthogonal to $\mathbf{v}$ $\left(\overset{\mathbf{n}}{\uparrow}\!\llcorner\!\rightarrow \mathbf{v} \right)$; and

(iii). the direction of $\mathbf{n}$ is chosen using the left-hand rule.

Tanım. The ***cross product*** of $\mathbf{u}$ and $\mathbf{v}$ is

$$\mathbf{u} \times \mathbf{v} = \|\mathbf{u}\| \, \|\mathbf{v}\| \, (\sin\theta)\mathbf{n}.$$

Not.

- $\mathbf{u} \cdot \mathbf{v}$ is a number.
- $\mathbf{u} \times \mathbf{v}$ is a vector.

Not.

$$\left(\begin{matrix} \mathbf{u} \text{ and } \mathbf{v} \\ \text{are parallel} \end{matrix} \right) \iff \theta = 0^\circ \text{ or } 180^\circ$$
$$\implies \sin\theta = 0 \implies \mathbf{u} \times \mathbf{v} = \mathbf{0}.$$

Properties of the Cross Product

Let $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ be vectors. Let r and s be numbers. Then

(i). $(r\mathbf{u}) \times (s\mathbf{v}) = (rs)(\mathbf{u} \times \mathbf{v})$;

(ii). $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w})$;

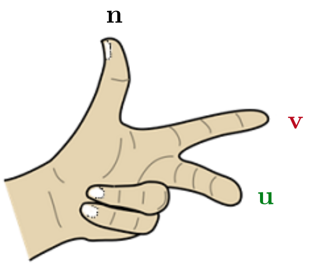
(iii). $\mathbf{v} \times \mathbf{u} = -\mathbf{u} \times \mathbf{v}$;

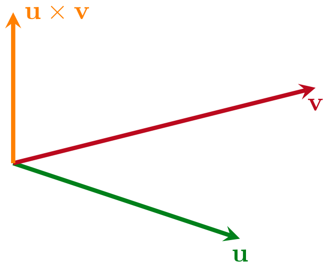
(iv). $(\mathbf{v} + \mathbf{w}) \times \mathbf{u} = (\mathbf{v} \times \mathbf{u}) + (\mathbf{w} \times \mathbf{u})$;

(v). $\mathbf{0} \times \mathbf{u} = \mathbf{0}$; and

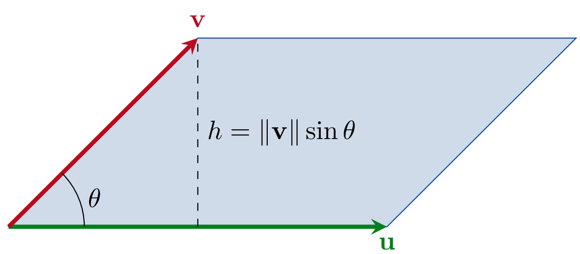
(vi). $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w}$.

Property (iii)

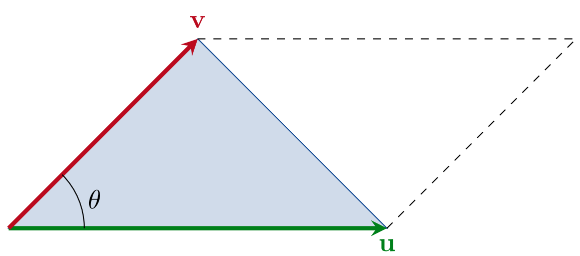




Area of a Parallelogram



Area of a Triangle



Properties of the Cross Product

Let $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ be vectors. Let r and s be numbers. Then

(i). $(r\mathbf{u}) \times (s\mathbf{v}) = (rs)(\mathbf{u} \times \mathbf{v})$;

(ii). $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w})$;

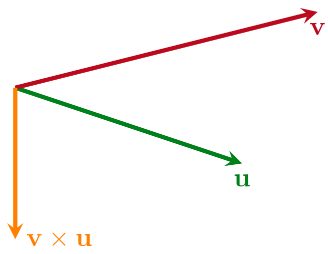
(iii). $\mathbf{v} \times \mathbf{u} = -\mathbf{u} \times \mathbf{v}$;

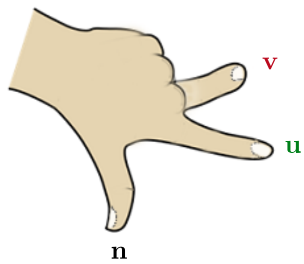
(iv). $(\mathbf{v} + \mathbf{w}) \times \mathbf{u} = (\mathbf{v} \times \mathbf{u}) + (\mathbf{w} \times \mathbf{u})$;

(v). $\mathbf{0} \times \mathbf{u} = \mathbf{0}$; and

(vi). $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w}$.

Özellik (iii)





Paralelkenarın Alanı

$$\text{area} = (\text{base})\,(\text{height}) = \|\mathbf{u}\|\,\|\mathbf{v}\| \sin\theta = \|\mathbf{u} \times \mathbf{v}\|\,.$$

$$\text{alan} = (\text{taban})\,(\text{yükseklik}) = \|\mathbf{u}\|\,\|\mathbf{v}\| \sin\theta = \|\mathbf{u} \times \mathbf{v}\|\,.$$

Üçgenin Alanı

$$\begin{aligned}\text{area of triangle} &= \tfrac{1}{2}\,(\text{area of parallelogram}) \\ &= \tfrac{1}{2}\,\|\mathbf{u} \times \mathbf{v}\|\,.\end{aligned}$$

$$\begin{aligned}\text{üçgenin alanı} &= \tfrac{1}{2}\,(\text{paralelkenarın alanı}) \\ &= \tfrac{1}{2}\,\|\mathbf{u} \times \mathbf{v}\|\,.\end{aligned}$$

A Formula for $\mathbf{u} \times \mathbf{v}$

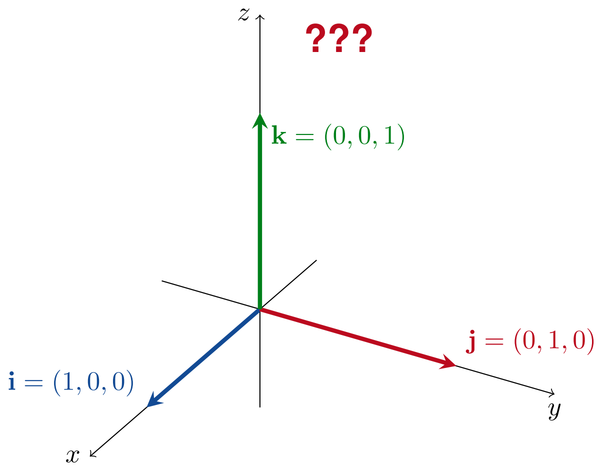


Figure 28.1: The standard unit vectors in $\mathbb{R}^3$.
Şekil 28.1:

Note first that

$$\mathbf{i} \times \mathbf{i} = \|\mathbf{i}\| \, \|\mathbf{i}\| \sin 0° \mathbf{n} = \mathbf{0}.$$

Similarly $\mathbf{j} \times \mathbf{j} = \mathbf{0}$ and $\mathbf{k} \times \mathbf{k} = \mathbf{0}$ also.

Next note that $\mathbf{i} \times \mathbf{j}$ must point in the same direction at $\mathbf{k}$ by the left-hand rule. Thus

$$\mathbf{i} \times \mathbf{j} = \|\mathbf{i}\| \, \|\mathbf{j}\| \sin 90° \mathbf{k} = \mathbf{k}.$$

We then immediately also have

$$\mathbf{j} \times \mathbf{i} = -(\mathbf{i} \times \mathbf{j}) = -\mathbf{k}.$$

It is left for you to check that

$$\mathbf{j} \times \mathbf{k} = \mathbf{i}, \qquad \mathbf{k} \times \mathbf{j} = -\mathbf{i}, \qquad \mathbf{i} \times \mathbf{k} = -\mathbf{j} \quad \text{and} \quad \mathbf{k} \times \mathbf{i} = \mathbf{j}.$$

Now suppose that $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ and $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$. Then we can calculate that

Note first that

$$\mathbf{i} \times \mathbf{i} = \|\mathbf{i}\| \, \|\mathbf{i}\| \sin 0° \mathbf{n} = \mathbf{0}.$$

Similarly $\mathbf{j} \times \mathbf{j} = \mathbf{0}$ and $\mathbf{k} \times \mathbf{k} = \mathbf{0}$ also.

Next note that $\mathbf{i} \times \mathbf{j}$ must point in the same direction at $\mathbf{k}$ by the left-hand rule. Thus

$$\mathbf{i} \times \mathbf{j} = \|\mathbf{i}\| \, \|\mathbf{j}\| \sin 90° \mathbf{k} = \mathbf{k}.$$

We then immediately also have

$$\mathbf{j} \times \mathbf{i} = -(\mathbf{i} \times \mathbf{j}) = -\mathbf{k}.$$

It is left for you to check that

$$\mathbf{j} \times \mathbf{k} = \mathbf{i}, \qquad \mathbf{k} \times \mathbf{j} = -\mathbf{i}, \qquad \mathbf{i} \times \mathbf{k} = -\mathbf{j} \quad \text{and} \quad \mathbf{k} \times \mathbf{i} = \mathbf{j}.$$

Now suppose that $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ and $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$. Then we can calculate that

$$\begin{aligned} \mathbf{u} \times \mathbf{v} &= (u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}) \times (v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}) \\ &= u_1v_1\mathbf{i} \times \mathbf{i} + u_1v_2\mathbf{i} \times \mathbf{j} + u_1v_3\mathbf{i} \times \mathbf{k} + u_2v_1\mathbf{j} \times \mathbf{i} + u_2v_2\mathbf{j} \times \mathbf{j} + u_2v_3\mathbf{j} \times \mathbf{k} + u_3v_1\mathbf{k} \times \mathbf{i} + u_3v_2\mathbf{k} \times \mathbf{j} + u_3v_3\mathbf{k} \times \mathbf{k} \\ &= (u_2v_3 - u_3v_2)\mathbf{i} - (u_1v_3 - u_3v_1)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}. \end{aligned}$$

Theorem 28.1. *If $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ and $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$, then*

$$\boxed{\mathbf{u} \times \mathbf{v} = (u_2v_3 - u_3v_2)\mathbf{i} - (u_1v_3 - u_3v_1)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}}$$

If you studied matrices and determinants at high school, then you may prefer to use the following symbolic determinant formula instead.

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}.$$

Example 28.1. Find $\mathbf{u} \times \mathbf{v}$ and $\mathbf{v} \times \mathbf{u}$ if $\mathbf{u} = 2\mathbf{i} + \mathbf{j} + \mathbf{k}$ and

Teorem 28.1. *If $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ and $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$, then*

$$\boxed{\mathbf{u} \times \mathbf{v} = (u_2v_3 - u_3v_2)\mathbf{i} - (u_1v_3 - u_3v_1)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}}$$

If you studied matrices and determinants at high school, then you may prefer to use the following symbolic determinant formula instead.

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}.$$

Örnek 28.1. Find $\mathbf{u} \times \mathbf{v}$ and $\mathbf{v} \times \mathbf{u}$ if $\mathbf{u} = 2\mathbf{i} + \mathbf{j} + \mathbf{k}$ and

$\mathbf{v} = -4\mathbf{i} + 3\mathbf{j} + \mathbf{k}$.

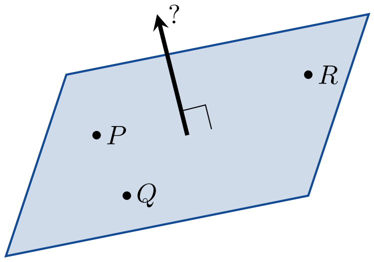
solution:

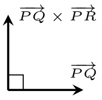
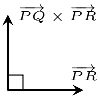
$$\mathbf{u} \times \mathbf{v} = (1-3)\mathbf{i} - (2 - -4)\mathbf{j} + (6 - -4)\mathbf{k} = -2\mathbf{i} - 6\mathbf{j} + 10\mathbf{k}$$

and

$$\mathbf{v} \times \mathbf{u} = -\mathbf{u} \times \mathbf{v} = 2\mathbf{i} + 6\mathbf{j} - 10\mathbf{k}.$$

Example 28.2. Find a vector perpendicular to the plane containing the three points $P(1,-1,0)$, $Q(2,1,-1)$ and $R(-1,1,2)$.



solution: The vector $\overrightarrow{PQ} \times \overrightarrow{PR}$ is perpendicular to the plane because  and . We calculate that

$$\begin{aligned}
\overrightarrow{PQ} &= Q - P = (2,1,-1) - (1,-1,0) \\
&= (2-1, 1+1, -1-0) = \mathbf{i} + 2\mathbf{j} - \mathbf{k} \\
\overrightarrow{PR} &= R - P = (-1,1,2) - (1,-1,0) \\
&= (-1-1, 1+1, 2-0) = -2\mathbf{i} + 2\mathbf{j} + 2\mathbf{k} \\
\overrightarrow{PQ} \times \overrightarrow{PR} &= (4+2)\mathbf{i} - (2-2)\mathbf{j} + (2+4)\mathbf{k} = 6\mathbf{i} + 6\mathbf{k}.
\end{aligned}$$

Example 28.3. Find the area of triangle PQR.

solution: The area of the triangle is

$$\begin{aligned}
\text{area} &= \frac{1}{2}\left\|\overrightarrow{PQ} \times \overrightarrow{PR}\right\| = \frac{1}{2}\|6\mathbf{i} + 6\mathbf{k}\| \\
&= \frac{1}{2}\sqrt{6^2 + 0^2 + 6^2} = 3\sqrt{2}.
\end{aligned}$$

Example 28.4. Find a unit vector perpendicular to the plane containing P, Q and R.

solution: We know that $\overrightarrow{PQ} \times \overrightarrow{PR}$ is perpendicular to the plane. We just need to normalise this vector to find a unit vector.

$$\mathbf{n} = \frac{\overrightarrow{PQ} \times \overrightarrow{PR}}{\left\|\overrightarrow{PQ} \times \overrightarrow{PR}\right\|} = \frac{6\mathbf{i} + 6\mathbf{k}}{6\sqrt{2}} = \frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{k}.$$

Example 28.5. A triangle is inscribed inside a cube of side 2 as shown in figure 28.2. Use the cross product to find the area of the triangle.

solution: First we draw coordinate axes and assign coordinates to the vertices of the triangle. See figure 28.3. Then we can calculate

$$\overrightarrow{AB} = B - A = (2,2,0) - (2,0,0) = (0,2,0) = 2\mathbf{j}$$

$\mathbf{v} = -4\mathbf{i} + 3\mathbf{j} + \mathbf{k}$.

çözüm:

$$\mathbf{u} \times \mathbf{v} = (1-3)\mathbf{i} - (2 - -4)\mathbf{j} + (6 - -4)\mathbf{k} = -2\mathbf{i} - 6\mathbf{j} + 10\mathbf{k}$$

and

$$\mathbf{v} \times \mathbf{u} = -\mathbf{u} \times \mathbf{v} = 2\mathbf{i} + 6\mathbf{j} - 10\mathbf{k}.$$

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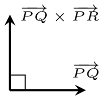
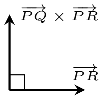
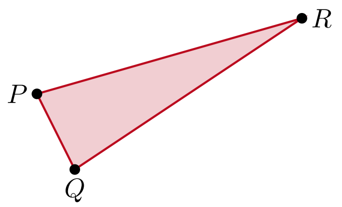
çözüm: The vector $\overrightarrow{PQ} \times \overrightarrow{PR}$ is perpendicular to the plane because  and . We calculate that

$$\begin{aligned}
\overrightarrow{PQ} &= Q - P = (2,1,-1) - (1,-1,0) \\
&= (2-1, 1+1, -1-0) = \mathbf{i} + 2\mathbf{j} - \mathbf{k} \\
\overrightarrow{PR} &= R - P = (-1,1,2) - (1,-1,0) \\
&= (-1-1, 1+1, 2-0) = -2\mathbf{i} + 2\mathbf{j} + 2\mathbf{k} \\
\overrightarrow{PQ} \times \overrightarrow{PR} &= (4+2)\mathbf{i} - (2-2)\mathbf{j} + (2+4)\mathbf{k} = 6\mathbf{i} + 6\mathbf{k}.
\end{aligned}$$

Örnek 28.3. Find the area of triangle PQR.



çözüm: The area of the triangle is

$$\begin{aligned}
\text{area} &= \frac{1}{2}\left\|\overrightarrow{PQ} \times \overrightarrow{PR}\right\| = \frac{1}{2}\|6\mathbf{i} + 6\mathbf{k}\| \\
&= \frac{1}{2}\sqrt{6^2 + 0^2 + 6^2} = 3\sqrt{2}.
\end{aligned}$$

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çözüm: We know that $\overrightarrow{PQ} \times \overrightarrow{PR}$ is perpendicular to the plane. We just need to normalise this vector to find a unit vector.

$$\mathbf{n} = \frac{\overrightarrow{PQ} \times \overrightarrow{PR}}{\left\|\overrightarrow{PQ} \times \overrightarrow{PR}\right\|} = \frac{6\mathbf{i} + 6\mathbf{k}}{6\sqrt{2}} = \frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{k}.$$

Örnek 28.5. A triangle is inscribed inside a cube of side 2 as shown in figure 28.2. Use the cross product to find the area of the triangle.

çözüm: First we draw coordinate axes and assign coordinates to the vertices of the triangle. See figure 28.3. Then we can calculate

$$\overrightarrow{AB} = B - A = (2,2,0) - (2,0,0) = (0,2,0) = 2\mathbf{j}$$

and

$$\overrightarrow{AC} = C - A = (0,0,2) - (2,0,0) = (-2,0,2) = -2\mathbf{i} + 2\mathbf{k}.$$

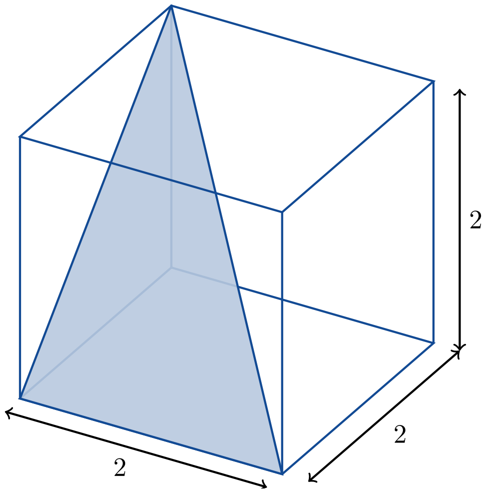

Figure 28.2: A triangle inscribed inside a cube of side 2.
Şekil 28.2:

and

$$\overrightarrow{AC} = C - A = (0,0,2) - (2,0,0) = (-2,0,2) = -2\mathbf{i} + 2\mathbf{k}.$$

It follows that

$$\overrightarrow{AB} \times \overrightarrow{AC} = (2\mathbf{j}) \times (-2I \times 2\mathbf{k}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 2 & 0 \\ -2 & 0 & 2 \end{vmatrix}$$
$$= \mathbf{i}(4-0) - \mathbf{j}(0-0) + \mathbf{k}(0 - -4) = 4\mathbf{i} + 4\mathbf{k}.$$

Therefore

$$\text{area of triangle} = \frac{1}{2}\left\|\overrightarrow{AB} \times \overrightarrow{AC}\right\| = \frac{1}{2}\sqrt{4^2 + 0^2 + 4^2}$$
$$= \frac{1}{2}\sqrt{32} = \frac{1}{2}\sqrt{4}\sqrt{8} = \sqrt{8} = 2\sqrt{2}.$$

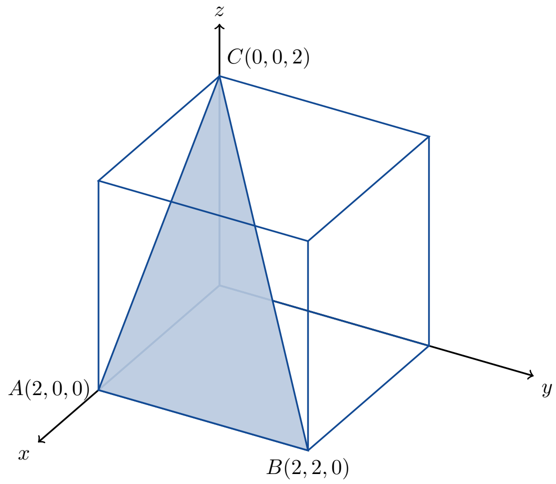

Figure 28.3: A triangle inscribed inside a cube of side 2.
Şekil 28.3:

It follows that

$$\overrightarrow{AB} \times \overrightarrow{AC} = (2\mathbf{j}) \times (-2I \times 2\mathbf{k}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 2 & 0 \\ -2 & 0 & 2 \end{vmatrix}$$
$$= \mathbf{i}(4-0) - \mathbf{j}(0-0) + \mathbf{k}(0 - -4) = 4\mathbf{i} + 4\mathbf{k}.$$

Therefore

$$\text{area of triangle} = \frac{1}{2}\left\|\overrightarrow{AB} \times \overrightarrow{AC}\right\| = \frac{1}{2}\sqrt{4^2 + 0^2 + 4^2}$$
$$= \frac{1}{2}\sqrt{32} = \frac{1}{2}\sqrt{4}\sqrt{8} = \sqrt{8} = 2\sqrt{2}.$$

The Triple Scalar Product

Definition. The ***triple scalar product*** of $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ is

$$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}.$$

The Triple Scalar Product

Tanım. The ***triple scalar product*** of $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ is

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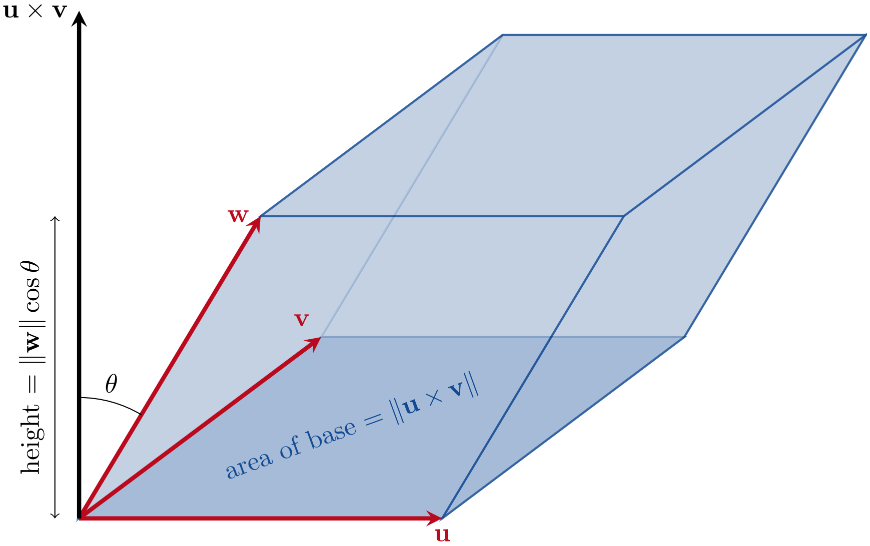
The Volume of a Parallelepiped



Paralelyüzlünün Hacmi

$$\begin{aligned} \text{volume} &= \big(\text{area of base}\big)\big(\text{height}\big) \\ &= \|\mathbf{u} \times \mathbf{v}\| \, \|\mathbf{w}\| \cos\theta \\ &= |(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}| \end{aligned}$$

$$\begin{aligned} \text{hacim} &= \big(\text{taban alanı}\big)\big(\text{yükseklik}\big) \\ &= \|\mathbf{u} \times \mathbf{v}\| \, \|\mathbf{w}\| \cos\theta \\ &= |(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}| \end{aligned}$$

One Final Comment

We can do the dot product in both $\mathbb{R}^2$ and $\mathbb{R}^3$. But we can only do the cross product in $\mathbb{R}^3$. There is no cross product in $\mathbb{R}^2$.

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Problems

Problem 28.1.

(a). Find the area of the triangle with vertices at $A(0,0,0)$, $B(-1,1,-1)$ and $C(3,0,3)$.

(b). Find a unit vector which is perpendicular to the plane containing A, B and C.

Problem 28.2. Let $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ be vectors. Which of the following make sense? Give reasons for your answers.

(a). $1 \cdot \mathbf{u}$.

(b). $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$

(c). $\mathbf{u} \times (\mathbf{v} \cdot \mathbf{w})$

(d). $\mathbf{u} \times (\mathbf{v} \times \mathbf{w})$

(e). $\mathbf{u} \cdot (\mathbf{v} \cdot \mathbf{w})$

Problem 28.3.

(a). Calculate $(\mathbf{i} + \mathbf{j} - \mathbf{k}) \times (\mathbf{0})$.

(b). Calculate $(\frac{3}{2}\mathbf{i} - \frac{1}{2}\mathbf{j} + \mathbf{k}) \times (\mathbf{i} + \mathbf{j} + 2\mathbf{k})$.

Problem 28.4. Use the cross product to calculate the area of the triangles shown in figures 28.4 and 28.5.

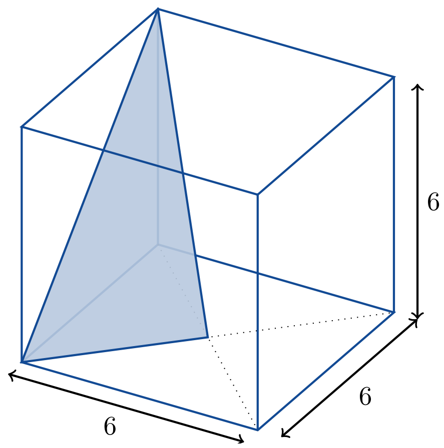

Figure 28.4: Another triangle inscribed inside a cube.
Şekil 28.4:

Problem 28.5. Calculate the triple scalar product of $\mathbf{u} = 2\mathbf{i} + \mathbf{j}$, $\mathbf{v} = 2\mathbf{i} - \mathbf{j} + \mathbf{k}$ and $\mathbf{w} = \mathbf{i} + 2\mathbf{k}$.

Sorular

Soru 28.1.

(a). Find the area of the triangle with vertices at $A(0,0,0)$, $B(-1,1,-1)$ and $C(3,0,3)$.

(b). Find a unit vector which is perpendicular to the plane containing A, B and C.

Soru 28.2. Let $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ be vectors. Which of the following make sense? Give reasons for your answers.

(a). $1 \cdot \mathbf{u}$.

(b). $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$

(c). $\mathbf{u} \times (\mathbf{v} \cdot \mathbf{w})$

(d). $\mathbf{u} \times (\mathbf{v} \times \mathbf{w})$

(e). $\mathbf{u} \cdot (\mathbf{v} \cdot \mathbf{w})$

Soru 28.3.

(a). Calculate $(\mathbf{i} + \mathbf{j} - \mathbf{k}) \times (\mathbf{0})$.

(b). Calculate $(\frac{3}{2}\mathbf{i} - \frac{1}{2}\mathbf{j} + \mathbf{k}) \times (\mathbf{i} + \mathbf{j} + 2\mathbf{k})$.

Soru 28.4. Use the cross product to calculate the area of the triangles shown in figures 28.4 and 28.5.

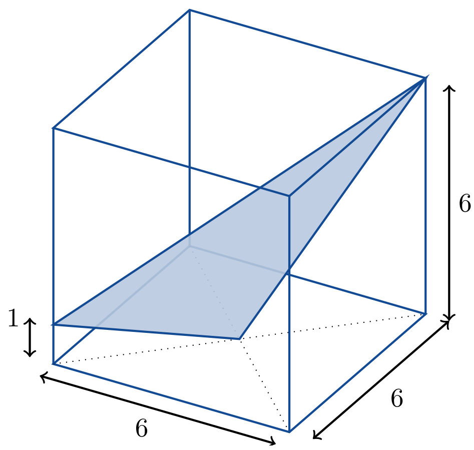

Figure 28.5: Yet another triangle inscribed inside a cube.
Şekil 28.5:

Soru 28.5. Calculate the triple scalar product of $\mathbf{u} = 2\mathbf{i} + \mathbf{j}$, $\mathbf{v} = 2\mathbf{i} - \mathbf{j} + \mathbf{k}$ and $\mathbf{w} = \mathbf{i} + 2\mathbf{k}$.

29 Lines

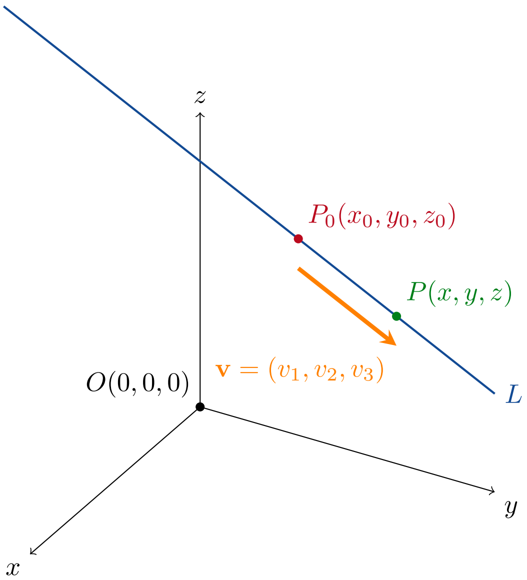

Figure 29.1: A line in $\mathbb{R}^3$ passing through the point P_0 parallel to $\mathbf{v}$.
Şekil 29.1:

Lines

To describe a line in $\mathbb{R}^3$, we need

- a point $P_0(x_0, y_0, z_0)$ which the line passes through; and
- a vector $\mathbf{v}$ which gives the direction of the line.

Let $\mathbf{r}_0 = \overrightarrow{OP_0}$ and $\mathbf{r} = \overrightarrow{OP}$.

Definition. The ***line*** L ***passing through*** $P_0(x_0, y_0, z_0)$ ***parallel to*** $\mathbf{v} = (v_1, v_2, v_3)$ has the vector equation

$$\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}, \qquad -\infty < t < \infty.$$

This equation is equivalent to

$$(x, y, z) = (x_0, y_0, z_0) + t(v_1, v_2, v_3)$$

or to the set of three equations

$$x = x_0 + tv_1, \qquad y = y_0 + tv_2, \qquad z = z_0 + tv_3.$$

Doğrular

Doğrular

To describe a line in $\mathbb{R}^3$, we need

- a point $P_0(x_0, y_0, z_0)$ which the line passes through; and
- a vector $\mathbf{v}$ which gives the direction of the line.

Let $\mathbf{r}_0 = \overrightarrow{OP_0}$ and $\mathbf{r} = \overrightarrow{OP}$.

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$$(x, y, z) = (x_0, y_0, z_0) + t(v_1, v_2, v_3)$$

or to the set of three equations

$$x = x_0 + tv_1, \qquad y = y_0 + tv_2, \qquad z = z_0 + tv_3.$$

Tanım. The ***parametric equations*** for the line L passing through $P_0(x_0, y_0, z_0)$ parallel to $\mathbf{v} = (v_1, v_2, v_3)$ are

$$x = x_0 + tv_1, \qquad y = y_0 + tv_2, \qquad z = z_0 + tv_3.$$

Örnek 29.1. Find parametric equations for the line passing through $P_0(-2, 0, 4)$ parallel to $\mathbf{v} = 2\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$.

çözüm: We can write

$$x = -2 + 2t, \qquad y = 4t, \qquad z = 4 - 2t.$$

See figure 29.3

Örnek 29.2. Find parametric equations for the line passing through $P(-3, 2, -3)$ and $Q(1, -1, 4)$.

çözüm: Choose $P_0 = P$ and $\mathbf{v} = \overrightarrow{PQ} = (4, -3, 7) = 4\mathbf{i} - 3\mathbf{j} + 7\mathbf{k}$. Then we can write

$$x = -3 + 4t, \qquad y = 2 - 3t, \qquad z = -3 + 7t.$$

Tanım. The vector equation

$$\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}, \qquad a < t < b$$

denotes a ***line segment***.

Definition. The ***parametric equations*** for the line L passing through $P_0(x_0, y_0, z_0)$ parallel to $\mathbf{v} = (v_1, v_2, v_3)$ are

$$x = x_0 + tv_1, \qquad y = y_0 + tv_2, \qquad z = z_0 + tv_3.$$

Example 29.1. Find parametric equations for the line passing through $P_0(-2, 0, 4)$ parallel to $\mathbf{v} = 2\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$.

solution: We can write

$$x = -2 + 2t, \qquad y = 4t, \qquad z = 4 - 2t.$$

See figure 29.3

Example 29.2. Find parametric equations for the line passing through $P(-3, 2, -3)$ and $Q(1, -1, 4)$.

solution: Choose $P_0 = P$ and $\mathbf{v} = \overrightarrow{PQ} = (4, -3, 7) = 4\mathbf{i} - 3\mathbf{j} + 7\mathbf{k}$. Then we can write

$$x = -3 + 4t, \qquad y = 2 - 3t, \qquad z = -3 + 7t.$$

Definition. The vector equation

$$\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}, \qquad a < t < b$$

denotes a ***line segment***.

Example 29.3. Parametrise the line segment joining $P(-3, 2, -3)$ and $Q(1, -1, 4)$.

solution: We know that $x = -3+4t$, $y = 2-3t$ and $z = -3+7t$. The line passes through P then $t = 0$ and passed through Q when $t = 1$. Therefore

$$x = -3 + 4t, \qquad y = 2 - 3t, \qquad z = -3 + 7t, \qquad 0 < t < 1$$

denotes the line segment from P to Q. See figure 29.2.

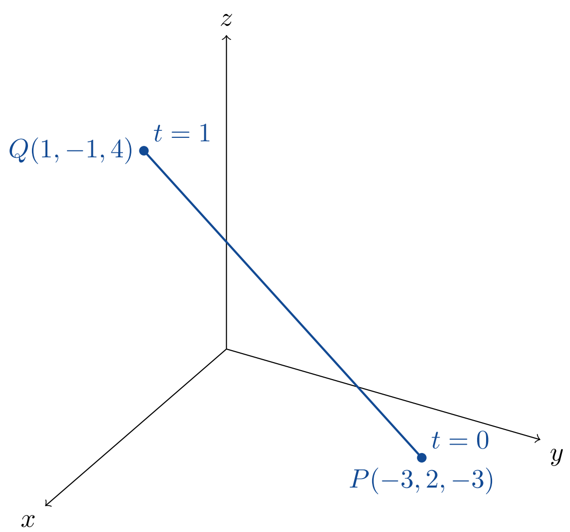

Figure 29.2: The line segment $\mathbb{R}^3$ joining $P(-3, 2, -3)$ and $Q(1, -1, 4)$.
Şekil 29.2:

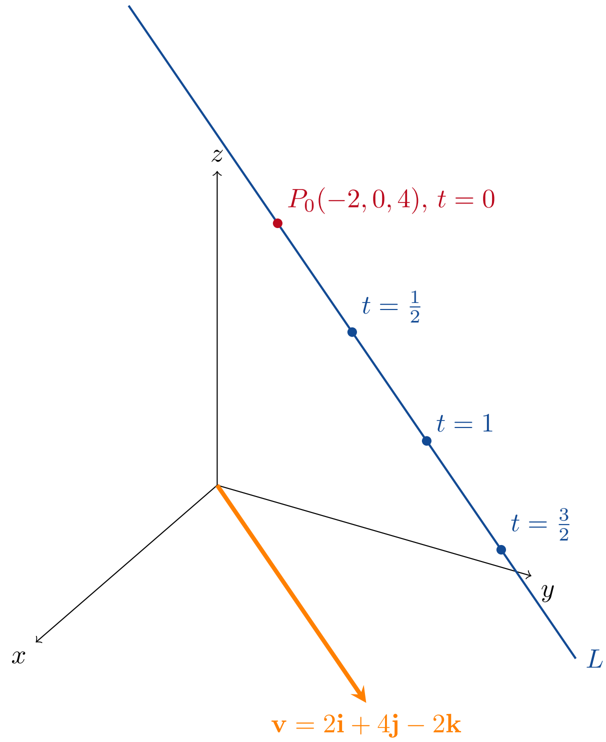

Figure 29.3: A line in $\mathbb{R}^3$ passing through the point P_0 parallel to $\mathbf{v}$.
Şekil 29.3:

Örnek 29.3. Parametrise the line segment joining $P(-3, 2, -3)$ and $Q(1, -1, 4)$.

çözüm: We know that $x = -3+4t$, $y = 2-3t$ and $z = -3+7t$. The line passes through P then $t = 0$ and passed through Q when $t = 1$. Therefore

$$x = -3 + 4t, \qquad y = 2 - 3t, \qquad z = -3 + 7t, \qquad 0 < t < 1$$

denotes the line segment from P to Q. See figure 29.2.

The Distance from a Point to a Line

Let d be the shortest distance from the point S to the line L as shown in figure 29.4. We can see from this figure that

$$d = \left\| \overrightarrow{PS} \right\| \sin\theta.$$

But remember that $\overrightarrow{PS} \times \mathbf{v} = \left\| \overrightarrow{PS} \right\| \|\mathbf{v}\| \sin\theta \; \mathbf{n}$. Therefore

$$\boxed{d = \frac{\left\| \overrightarrow{PS} \times \mathbf{v} \right\|}{\|\mathbf{v}\|}.}$$

Example 29.4. Find the distance from the point $S(1, 1, 5)$ to the line

$$x = 1 + t, \qquad y = 3 - t, \qquad z = 2t.$$

solution: The line passes through the point $P(1, 3, 0)$ in the direction $\mathbf{v} = \mathbf{i} - \mathbf{j} + 2\mathbf{k}$. Thus

$$\overrightarrow{PS} = S - P = (1, 1, 5) - (1, 3, 0) = (0, -2, 5) = -2\mathbf{j} + 5\mathbf{k}$$

and

$$\overrightarrow{PS} \times \mathbf{v} = (-4 + 5)\mathbf{i} - (0 - 5)\mathbf{j} + (0 + 2)\mathbf{k} = \mathbf{i} + 5\mathbf{j} + 2\mathbf{k}.$$

Therefore

$$d = \frac{\left\| \overrightarrow{PS} \times \mathbf{v} \right\|}{\|\mathbf{v}\|} = \frac{\sqrt{1^2 + 5^2 + 2^2}}{\sqrt{1^2 + 1^2 + 2^2}} = \frac{\sqrt{30}}{\sqrt{6}} = \sqrt{5}.$$

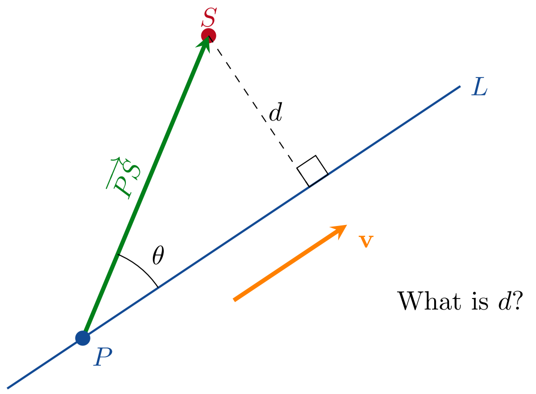

Figure 29.4: The distance from a point S to a line L.
Şekil 29.4:

Points of Intersection

Definition. Two lines intersect at a point P if and only if P lies on both lines.

Example 29.5. Do the following two lines intersect? Is yes, where?

Line 1: $x = 7 - t$, $y = 3 + 3t$, $z = 2t$.

Line 2: $x = -1 + 2s$, $y = 3s$, $z = 1 + s$.

solution: The two lines intersect if and only if there exist $s, t \in$

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$$x = 1 + t, \qquad y = 3 - t, \qquad z = 2t.$$

çözüm: The line passes through the point $P(1, 3, 0)$ in the direction $\mathbf{v} = \mathbf{i} - \mathbf{j} + 2\mathbf{k}$. Thus

$$\overrightarrow{PS} = S - P = (1, 1, 5) - (1, 3, 0) = (0, -2, 5) = -2\mathbf{j} + 5\mathbf{k}$$

and

$$\overrightarrow{PS} \times \mathbf{v} = (-4 + 5)\mathbf{i} - (0 - 5)\mathbf{j} + (0 + 2)\mathbf{k} = \mathbf{i} + 5\mathbf{j} + 2\mathbf{k}.$$

Therefore

$$d = \frac{\left\| \overrightarrow{PS} \times \mathbf{v} \right\|}{\|\mathbf{v}\|} = \frac{\sqrt{1^2 + 5^2 + 2^2}}{\sqrt{1^2 + 1^2 + 2^2}} = \frac{\sqrt{30}}{\sqrt{6}} = \sqrt{5}.$$

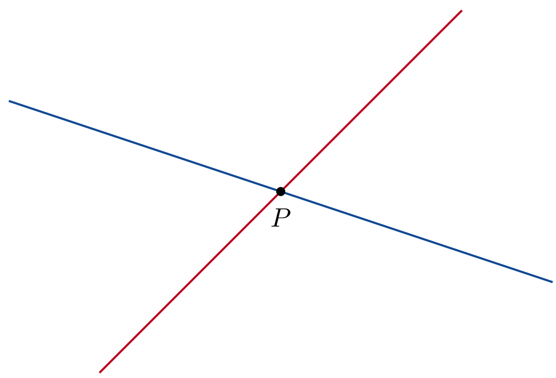

Figure 29.5: Intersecting lines.
Şekil 29.5:

Points of Intersection

Tanım. Two lines intersect at a point P if and only if P lies on both lines.

Örnek 29.5. Do the following two lines intersect? Is yes, where?

Doğru 1: $x = 7 - t$, $y = 3 + 3t$, $z = 2t$.

Doğru 2: $x = -1 + 2s$, $y = 3s$, $z = 1 + s$.

çözüm: The two lines intersect if and only if there exist $s, t \in \mathbb{R}$

$\mathbb{R}$ such that

$$\begin{aligned} 7-t = x = -1+2s &\implies t = 8-2s \\ 3+3t = y = 3s &\implies s = t+1 \\ 2t = z = 1+s & \end{aligned}$$

The first equation tells us that $t = 8-2s$. Putting this into the second equation gives $s = t+1 = (8-2s)+1 = 9-2s$ which implies that $s = 3$ and $t = 2$. We must check the third equation: $2t = 2 \times 2 = 4 = 1+3 = 1+s$. Because the third equation is also true, we know that they two lines intersect at $P(5,9,4)$.

Example 29.6. Do the following two lines intersect? If yes, where?

Line 1: $x = 1+t$, $y = 3t$, $z = 3+3t$.

Line 2: $x = -1+2s$, $y = 3s$, $z = 1+s$.

solution: Can we find $s,t \in \mathbb{R}$ such that

$$\begin{aligned} 1+t = x = -1+2s & \\ 3t = y = 3s &\implies s = t \\ 3+3t = z = 1+s & \end{aligned}$$

are all true?

The second equation gives $s = t$. Thus $1+t = -1+2t \implies 2+t = 2t \implies t = 2$. However $3+3t = 1+t \implies 2+2t = 0 \implies t = -2 \neq 2$. Therefore it is not possible to find an s and a t. Hence the lines do not intersect.

The Distance Between Two Lines

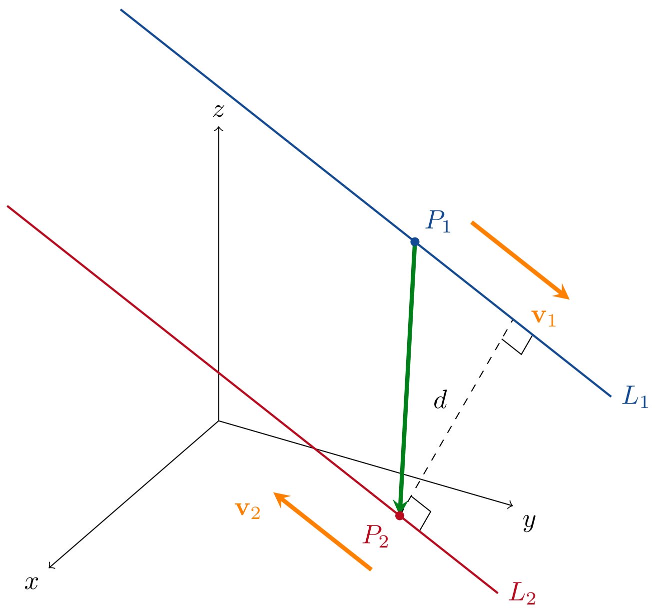

Figure 29.6: The distance between parallel lines.
Şekil 29.6:

If two lines intersect, then clearly the distance between them is zero. If they do not intersect then there are two possi-

such that

$$\begin{aligned} 7-t = x = -1+2s &\implies t = 8-2s \\ 3+3t = y = 3s &\implies s = t+1 \\ 2t = z = 1+s & \end{aligned}$$

The first equation tells us that $t = 8-2s$. Putting this into the second equation gives $s = t+1 = (8-2s)+1 = 9-2s$ which implies that $s = 3$ and $t = 2$. We must check the third equation: $2t = 2 \times 2 = 4 = 1+3 = 1+s$. Because the third equation is also true, we know that they two lines intersect at $P(5,9,4)$.

Örnek 29.6. Do the following two lines intersect? If yes, where?

Doğru 1: $x = 1+t$, $y = 3t$, $z = 3+3t$.

Doğru 2: $x = -1+2s$, $y = 3s$, $z = 1+s$.

çözüm: Can we find $s,t \in \mathbb{R}$ such that

$$\begin{aligned} 1+t = x = -1+2s & \\ 3t = y = 3s &\implies s = t \\ 3+3t = z = 1+s & \end{aligned}$$

are all true?

The second equation gives $s = t$. Thus $1+t = -1+2t \implies 2+t = 2t \implies t = 2$. However $3+3t = 1+t \implies 2+2t = 0 \implies t = -2 \neq 2$. Therefore it is not possible to find an s and a t. Hence the lines do not intersect.

The Distance Between Two Lines

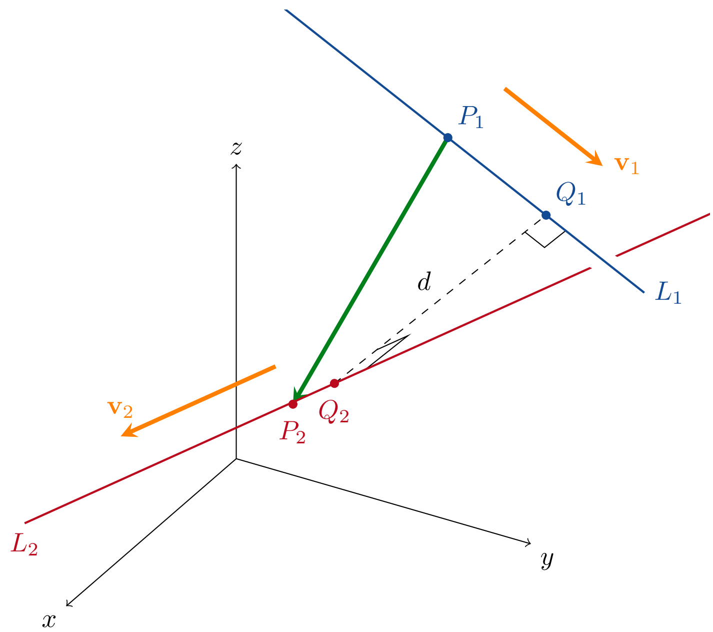

Figure 29.7: The distance between skew lines.
Şekil 29.7:

If two lines intersect, then clearly the distance between bilities, either

- the lines are parallel ($\mathbf{v}_1 = k\mathbf{v}_2$ for some $k \in \mathbb{R}$); or
- the lines are skew ($\mathbf{v}_1 \neq k\mathbf{v}_2$ for all $k \in \mathbb{R}$).

Parallel Lines

First we will consider parallel lines. We can see from figure 29.6 on page 148 that the distance between the two parallel lines is the same as the distance between P_2 and the line L_1. Hence

$$\boxed{d = \frac{\left\| \overrightarrow{P_1P_2} \times \mathbf{v}_1 \right\|}{\|\mathbf{v}_1\|}.}$$

Skew Lines

Now consider skew lines. See figure 29.7 on page 148. Let $\mathbf{n} = \mathbf{v}_1 \times \mathbf{v}_2$. Then $\mathbf{n}$ is orthogonal to both $\mathbf{v}_1$ and $\mathbf{v}_2$. So

$$d = \left\| \overrightarrow{Q_1Q_2} \right\| = \left\| \text{proj}_{\mathbf{n}} \overrightarrow{P_1P_2} \right\| = \frac{\left| \overrightarrow{P_1P_2} \cdot \mathbf{n} \right|}{\|\mathbf{n}\|}.$$

Thus

$$\boxed{d = \frac{\left| \overrightarrow{P_1P_2} \cdot (\mathbf{v}_1 \times \mathbf{v}_2) \right|}{\|\mathbf{v}_1 \times \mathbf{v}_2\|}.}$$

Example 29.7. Find the distance between the following two lines.
line 1: $x = 0$, $y = -t$, $z = t$,
line 2: $x = 1 + 2s$, $y = s$, $z = -3s$.

solution: We have that $P_1(0,0,0)$, $\mathbf{v}_1 = -\mathbf{j} + \mathbf{k}$, $P_2(1,0,0)$ and $\mathbf{v}_2 = 2\mathbf{i} + \mathbf{j} - 3\mathbf{k}$. Since

$$\mathbf{v}_1 \times \mathbf{v}_2 = 2\mathbf{i} + 2\mathbf{j} + 2\mathbf{k} \neq \mathbf{0},$$

the lines are skew. (Recall that we have $\mathbf{v}_1 \times \mathbf{v}_2 = \mathbf{0}$ for parallel vectors.) Moreover note that $\overrightarrow{P_1P_2} = \mathbf{i}$. Then we calculate that

$$\begin{aligned} d &= \frac{\left| \overrightarrow{P_1P_2} \cdot (\mathbf{v}_1 \times \mathbf{v}_2) \right|}{\|\mathbf{v}_1 \times \mathbf{v}_2\|} = \frac{|(\mathbf{i}) \cdot (2\mathbf{i} + 2\mathbf{j} + 2\mathbf{k})|}{\|2\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}\|} \\ &= \frac{|2 + 0 + 0|}{\sqrt{2^2 + 2^2 + 2^2}} = \frac{1}{\sqrt{3}}. \end{aligned}$$

them is zero. If they do not intersect then there are two possibilities, either

- the lines are parallel ($\mathbf{v}_1 = k\mathbf{v}_2$ for some $k \in \mathbb{R}$); or
- the lines are skew ($\mathbf{v}_1 \neq k\mathbf{v}_2$ for all $k \in \mathbb{R}$).

Parallel Lines

First we will consider parallel lines. We can see from figure 29.6 on page 148 that the distance between the two parallel lines is the same as the distance between P_2 and the line L_1. Hence

$$\boxed{d = \frac{\left\| \overrightarrow{P_1P_2} \times \mathbf{v}_1 \right\|}{\|\mathbf{v}_1\|}.}$$

Skew Lines

Now consider skew lines. See figure 29.7 on page 148. Let $\mathbf{n} = \mathbf{v}_1 \times \mathbf{v}_2$. Then $\mathbf{n}$ is orthogonal to both $\mathbf{v}_1$ and $\mathbf{v}_2$. So

$$d = \left\| \overrightarrow{Q_1Q_2} \right\| = \left\| \text{proj}_{\mathbf{n}} \overrightarrow{P_1P_2} \right\| = \frac{\left| \overrightarrow{P_1P_2} \cdot \mathbf{n} \right|}{\|\mathbf{n}\|}.$$

Thus

$$\boxed{d = \frac{\left| \overrightarrow{P_1P_2} \cdot (\mathbf{v}_1 \times \mathbf{v}_2) \right|}{\|\mathbf{v}_1 \times \mathbf{v}_2\|}.}$$

Örnek 29.7. Find the distance between the following two lines.
doğru 1: $x = 0$, $y = -t$, $z = t$,
doğru 2: $x = 1 + 2s$, $y = s$, $z = -3s$.

çözüm: We have that $P_1(0,0,0)$, $\mathbf{v}_1 = -\mathbf{j} + \mathbf{k}$, $P_2(1,0,0)$ and $\mathbf{v}_2 = 2\mathbf{i} + \mathbf{j} - 3\mathbf{k}$. Since

$$\mathbf{v}_1 \times \mathbf{v}_2 = 2\mathbf{i} + 2\mathbf{j} + 2\mathbf{k} \neq \mathbf{0},$$

the lines are skew. (Recall that we have $\mathbf{v}_1 \times \mathbf{v}_2 = \mathbf{0}$ for parallel vectors.) Moreover note that $\overrightarrow{P_1P_2} = \mathbf{i}$. Then we calculate that

$$\begin{aligned} d &= \frac{\left| \overrightarrow{P_1P_2} \cdot (\mathbf{v}_1 \times \mathbf{v}_2) \right|}{\|\mathbf{v}_1 \times \mathbf{v}_2\|} = \frac{|(\mathbf{i}) \cdot (2\mathbf{i} + 2\mathbf{j} + 2\mathbf{k})|}{\|2\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}\|} \\ &= \frac{|2 + 0 + 0|}{\sqrt{2^2 + 2^2 + 2^2}} = \frac{1}{\sqrt{3}}. \end{aligned}$$

Problems

Problem 29.1. Find parametric equations for the line through $P(3, -4, -1)$ which is parallel to the vector $\mathbf{v} = \mathbf{i} + \mathbf{j} - \mathbf{k}$.

Problem 29.2. Find parametric equations for the line through the points $P(1, 2, -1)$ and $Q(-1, 0, 1)$.

Problem 29.3. Find parametric equations for the line through the point $P(2, 3, 0)$ which is perpendicular to the vectors $\mathbf{u} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ and $\mathbf{v} = 3\mathbf{i} + 4\mathbf{j} + 5\mathbf{k}$.

Problem 29.4. Find the distance from the point $S(-1, 4, 3)$ to the line $x = 10 + 4t$, $y = -3$, $z = 4t$.

Problem 29.5. Find the distance between the line $x = 10+4t$, $y = -3$, $z = 4t$ and the line $x = 10 - 4ts$, $y = 0$, $z = 2 - 4s$.

Problem 29.6. Find the distance between the line $x = 10+4t$, $y = -t$, $z = 4t$ and the line $x = 10 - 4s$, $y = 1$, $z = 2 - 4s$.

Sorular

Soru 29.1. Find parametric equations for the line through $P(3, -4, -1)$ which is parallel to the vector $\mathbf{v} = \mathbf{i} + \mathbf{j} - \mathbf{k}$.

Soru 29.2. Find parametric equations for the line through the points $P(1, 2, -1)$ and $Q(-1, 0, 1)$.

Soru 29.3. Find parametric equations for the line through the point $P(2, 3, 0)$ which is perpendicular to the vectors $\mathbf{u} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ and $\mathbf{v} = 3\mathbf{i} + 4\mathbf{j} + 5\mathbf{k}$.

Soru 29.4. Find the distance from the point $S(-1, 4, 3)$ to the line $x = 10 + 4t$, $y = -3$, $z = 4t$.

Soru 29.5. Find the distance between the line $x = 10 + 4t$, $y = -3$, $z = 4t$ and the line $x = 10 - 4ts$, $y = 0$, $z = 2 - 4s$.

Soru 29.6. Find the distance between the line $x = 10 + 4t$, $y = -t$, $z = 4t$ and the line $x = 10 - 4s$, $y = 1$, $z = 2 - 4s$.

30

Planes

To describe a plane, we need

- a point $P_0(x_0, y_0, z_0)$ which the plane passes through; and
- a vector $\mathbf{n} = A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$ which is perpendicular to the plane.

The vector $\mathbf{n}$ is said to be ***normal*** to the plane.

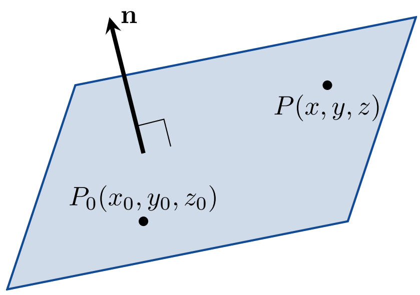

Figure 30.1: A plane passing through the point $P_0(x_0, y_0, z_0)$ with normal vector $\mathbf{n} = A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$.
Şekil 30.1:

Definition. The plane passing through the point $P_0(x_0, y_0, z_0)$ with normal vector $\mathbf{n} = A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$ has the vector equation

$$\mathbf{n} \cdot \overrightarrow{P_0P} = 0.$$

Writing this equation coordinates, we have

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$$

or

$$\boxed{Ax + By + Cz = D}$$

where $D = Ax_0 + By_0 + Cz_0$ is a constant.

Example 30.1. Find an equation for the plane passing through $P_0(-3, 0, 7)$ normal to $\mathbf{n} = 5\mathbf{i} + 2\mathbf{j} - \mathbf{k}$.

solution:

$$\begin{aligned} 5(x - (-3)) + 2(y - 0) + (-1)(z - 7) &= 0 \\ 5x - 15 + 2y - z + 7 &= 0 \\ 5x + 2y - z &= -22. \end{aligned}$$

Düzlemler

To describe a plane, we need

- a point $P_0(x_0, y_0, z_0)$ which the plane passes through; and
- a vector $\mathbf{n} = A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$ which is perpendicular to the plane.

The vector $\mathbf{n}$ is said to be ***normal*** to the plane.

Tanım. The plane passing through the point $P_0(x_0, y_0, z_0)$ with normal vector $\mathbf{n} = A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$ has the vector equation

$$\mathbf{n} \cdot \overrightarrow{P_0P} = 0.$$

Writing this equation coordinates, we have

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$$

or

$$\boxed{Ax + By + Cz = D}$$

where $D = Ax_0 + By_0 + Cz_0$ is a constant.

Örnek 30.1. Find an equation for the plane passing through $P_0(-3, 0, 7)$ normal to $\mathbf{n} = 5\mathbf{i} + 2\mathbf{j} - \mathbf{k}$.

çözüm:

$$\begin{aligned} 5(x - (-3)) + 2(y - 0) + (-1)(z - 7) &= 0 \\ 5x - 15 + 2y - z + 7 &= 0 \\ 5x + 2y - z &= -22. \end{aligned}$$

Not. The vector $\mathbf{n} = A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$ is normal to the plane $Ax + By + Cz = D$.

Örnek 30.2. Find a vector normal to the plane $x+2y+3z = 4$.

çözüm: We can immediately write down $\mathbf{n} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$.

Örnek 30.3. Find an equation for the plane containing the points $E(0, 0, 1)$, $F(2, 0, 0)$ and $G(0, 3, 0)$.

çözüm: First we need to find a vector normal to the plane. Since $\overrightarrow{EF} = 2\mathbf{i} - \mathbf{k}$ and $\overrightarrow{EG} = 3\mathbf{j} - \mathbf{k}$, we have that

$$\begin{aligned} \mathbf{n} = \overrightarrow{EF} \times \overrightarrow{EG} &= (0 - -3)\mathbf{i} - (-2 - 0)\mathbf{j} + (6 - 0)\mathbf{k} \\ &= 3\mathbf{i} + 2\mathbf{j} + 6\mathbf{k} \end{aligned}$$

Remark. The vector $\mathbf{n} = A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$ is normal to the plane $Ax + By + Cz = D$.

Example 30.2. Find a vector normal to the plane $x+2y+3z = 4$.

solution: We can immediately write down $\mathbf{n} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$.

Example 30.3. Find an equation for the plane containing the points $E(0, 0, 1)$, $F(2, 0, 0)$ and $G(0, 3, 0)$.

solution: First we need to find a vector normal to the plane. Since $\overrightarrow{EF} = 2\mathbf{i} - \mathbf{k}$ and $\overrightarrow{EG} = 3\mathbf{j} - \mathbf{k}$, we have that

$$\begin{aligned} \mathbf{n} &= \overrightarrow{EF} \times \overrightarrow{EG} = (0 - -3)\mathbf{i} - (-2 - 0)\mathbf{j} + (6 - 0)\mathbf{k} \\ &= 3\mathbf{i} + 2\mathbf{j} + 6\mathbf{k} \end{aligned}$$

is normal to the plane. See figure 30.2. Using $P_0 = E(0, 0, 1)$, the equation for the plane is

$$\begin{aligned} 3(x - 0) + 2(y - 0) + 6(z - 1) &= 0 \\ 3x + 2y + 6z &= 6. \end{aligned}$$

Lines of Intersection

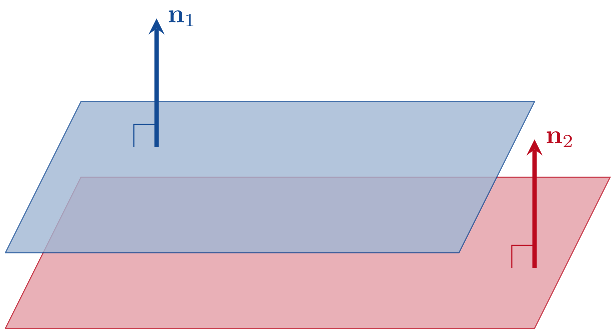

Figure 30.3: Two planes are parallel $\iff$ $\mathbf{n}_1 = k\mathbf{n}_2$ for some $k \in \mathbb{R}$.
Şekil 30.3:

Example 30.4. Find a vector parallel of the line of intersection of the planes $3x - 6y - 2z = 15$ and $2x + y - 2z = 5$.

solution: We can immediately write down $\mathbf{n}_1 = 3\mathbf{i} - 6\mathbf{j} - 2\mathbf{k}$ and $\mathbf{n}_2 = 2\mathbf{i}+\mathbf{j}-2\mathbf{k}$. A vector parallel to the line of intersection is

$$\mathbf{n}_1 \times \mathbf{n}_2 = (12 + 2)\mathbf{i} - (-6 + 4)\mathbf{j} + (3 + 12)\mathbf{k} = 14\mathbf{i} + 2\mathbf{j} + 15\mathbf{k}.$$

Example 30.5. Find the point where the line $x = \frac{8}{3} + 2t$, $y = -2t$, $z = 1 + t$ intersects the plane $3x + 2y + 6z = 6$.

solution: We calculate that

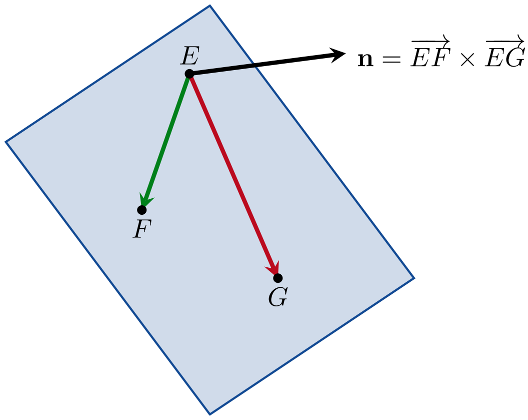

Figure 30.2: The vector $\mathbf{n}$ is perpendicular to the plane containing E, F and G.
Şekil 30.2:

is normal to the plane. See figure 30.2. Using $P_0 = E(0, 0, 1)$, the equation for the plane is

$$\begin{aligned} 3(x - 0) + 2(y - 0) + 6(z - 1) &= 0 \\ 3x + 2y + 6z &= 6. \end{aligned}$$

Kesişim Doğruları

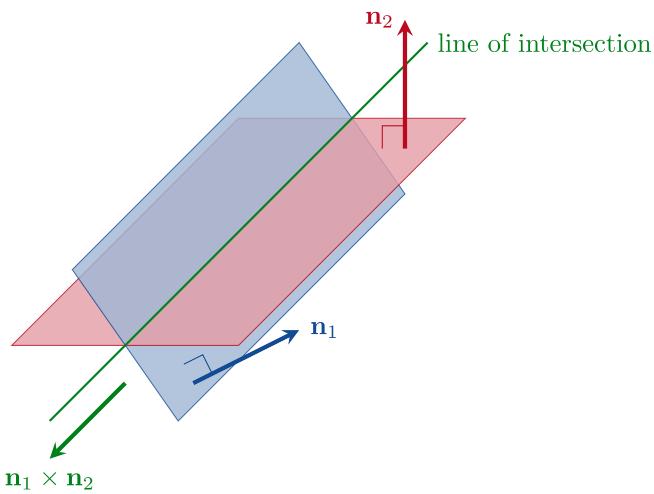

Figure 30.4: Two planes intersect in a line $\iff$ $\mathbf{n}_1 \neq k\mathbf{n}_2$ for all $k \in \mathbb{R}$.
Şekil 30.4:

Örnek 30.4. Find a vector parallel of the line of intersection of the planes $3x - 6y - 2z = 15$ and $2x + y - 2z = 5$.

çözüm: We can immediately write down $\mathbf{n}_1 = 3\mathbf{i} - 6\mathbf{j} - 2\mathbf{k}$ and $\mathbf{n}_2 = 2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$. A vector parallel to the line of intersection is

$$\mathbf{n}_1 \times \mathbf{n}_2 = (12 + 2)\mathbf{i} - (-6 + 4)\mathbf{j} + (3 + 12)\mathbf{k} = 14\mathbf{i} + 2\mathbf{j} + 15\mathbf{k}.$$

Örnek 30.5. Find the point where the line $x = \frac{8}{3}+2t$, $y = -2t$,

$$\begin{aligned} 3x + 2y + 6z &= 6 \\ 3(\frac{8}{3} + 2t) + 2(-2t) + 6(1+t) &= 6 \\ 8 + 6t - 4t + 6 + 6t &= 6 \\ 8t &= -8 \\ y &= -1. \end{aligned}$$

The point of intersection is

$$P(x, y, z)|_{t=-1} = P\left(\frac{8}{3} + 2t, -2t, 1+t\right)\bigg|_{t=-1} = P\left(\frac{2}{3}, 2, 0\right).$$

$z = 1 + t$ intersects the plane $3x + 2y + 6z = 6$.

çözüm: We calculate that

$$\begin{aligned} 3x + 2y + 6z &= 6 \\ 3(\frac{8}{3} + 2t) + 2(-2t) + 6(1+t) &= 6 \\ 8 + 6t - 4t + 6 + 6t &= 6 \\ 8t &= -8 \\ y &= -1. \end{aligned}$$

The point of intersection is

$$P(x, y, z)|_{t=-1} = P\left(\frac{8}{3} + 2t, -2t, 1+t\right)\bigg|_{t=-1} = P\left(\frac{2}{3}, 2, 0\right).$$

The Distance from a Point to a Plane

Bir Noktadan Bir Düzleme Olan Uzaklık

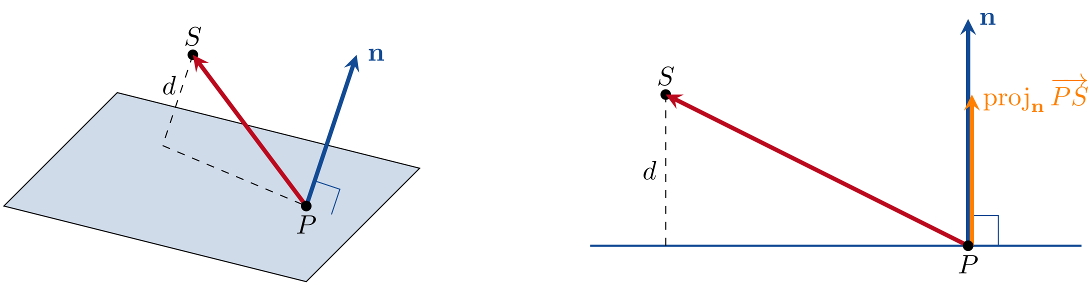

Figure 30.5: The distance from a Point to a Place.
Şekil 30.5:

We can see from figure 30.5 that $d = \left\|\text{proj}_{\mathbf{n}} \overrightarrow{PS}\right\|$. Therefore the distance from a point S to a plane containing the point P is

$$\boxed{d = \frac{\left|\overrightarrow{PS} \cdot \mathbf{n}\right|}{\|\mathbf{n}\|}.}$$

Example 30.6. Find the distance from the point $S(1, 2, 3)$ to the plane $x + 2y + 3z = 4$.

solution: First we need a point in the plane. Setting $y = 0$ and $z = 0$ we must have $x = 4 - 2y - 3z = 4$. Therefore $P(4, 0, 0)$ is in the plane. Clearly $\mathbf{n} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$.

Therefore the required distance is

$$\begin{aligned} d = \frac{\left|\overrightarrow{PS} \cdot \mathbf{n}\right|}{\|\mathbf{n}\|} &= \frac{|(-3\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) \cdot (\mathbf{i} + 2\mathbf{j} + 3\mathbf{k})|}{\|\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}\|} \\ &= \frac{|-3 + 4 + 9|}{\sqrt{1^2 + 2^2 + 3^2}} = \frac{10}{\sqrt{14}}. \end{aligned}$$

We can see from figure 30.5 that $d = \left\|\text{proj}_{\mathbf{n}} \overrightarrow{PS}\right\|$. Therefore the distance from a point S to a plane containing the point P is

$$\boxed{d = \frac{\left|\overrightarrow{PS} \cdot \mathbf{n}\right|}{\|\mathbf{n}\|}.}$$

Örnek 30.6. Find the distance from the point $S(1, 2, 3)$ to the plane $x + 2y + 3z = 4$.

çözüm: First we need a point in the plane. Setting $y = 0$ and $z = 0$ we must have $x = 4 - 2y - 3z = 4$. Therefore $P(4, 0, 0)$ is in the plane. Clearly $\mathbf{n} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$.

Therefore the required distance is

$$\begin{aligned} d = \frac{\left|\overrightarrow{PS} \cdot \mathbf{n}\right|}{\|\mathbf{n}\|} &= \frac{|(-3\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) \cdot (\mathbf{i} + 2\mathbf{j} + 3\mathbf{k})|}{\|\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}\|} \\ &= \frac{|-3 + 4 + 9|}{\sqrt{1^2 + 2^2 + 3^2}} = \frac{10}{\sqrt{14}}. \end{aligned}$$

Angles Between Planes

There are two possible angles that can be measured between planes. We are interested in the smaller angle. See figure 30.6.

Definition. The angle between two planes is defined to be equal to whichever of the following angles is smaller

- the angle between $\mathbf{n}_1$ and $\mathbf{n}_2$;
- $180°$ minus the angle between $\mathbf{n}_1$ and $\mathbf{n}_2$.

The angle between two planes will always be between $0°$ and $90°$.

Example 30.7. Find the angle between the planes $3x - 6y - 2z = 15$ and $-2x - y + 2z = 5$.

solution: We have normal vectors $\mathbf{n}_1 = 3\mathbf{i} - 6\mathbf{j} - 2\mathbf{k}$ and $\mathbf{n}_2 = -2\mathbf{i} - \mathbf{j} + 2\mathbf{k}$. The angle between $\mathbf{n}_1$ and $\mathbf{n}_2$ is

$$\theta = \cos^{-1}\left(\frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{\|\mathbf{n}_1\|\,\|\mathbf{n}_2\|}\right) = \cos^{-1}\left(\frac{-4}{21}\right) \approx 101°.$$

Because $101° > 90°$, the angle between the two planes is approximately $180 - 101° = 79°$.

Düzlemler Arasındaki Açı

There are two possible angles that can be measured between planes. We are interested in the smaller angle. See figure 30.6.

Tanım. The angle between two planes is defined to be equal to whichever of the following angles is smaller

- the angle between $\mathbf{n}_1$ and $\mathbf{n}_2$;
- $180°$ minus the angle between $\mathbf{n}_1$ and $\mathbf{n}_2$.

The angle between two planes will always be between $0°$ and $90°$.

Örnek 30.7. Find the angle between the planes $3x - 6y - 2z = 15$ and $-2x - y + 2z = 5$.

çözüm: We have normal vectors $\mathbf{n}_1 = 3\mathbf{i} - 6\mathbf{j} - 2\mathbf{k}$ and $\mathbf{n}_2 = -2\mathbf{i} - \mathbf{j} + 2\mathbf{k}$. The angle between $\mathbf{n}_1$ and $\mathbf{n}_2$ is

$$\theta = \cos^{-1}\left(\frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{\|\mathbf{n}_1\|\,\|\mathbf{n}_2\|}\right) = \cos^{-1}\left(\frac{-4}{21}\right) \approx 101°.$$

Because $101° > 90°$, the angle between the two planes is approximately $180 - 101° = 79°$.

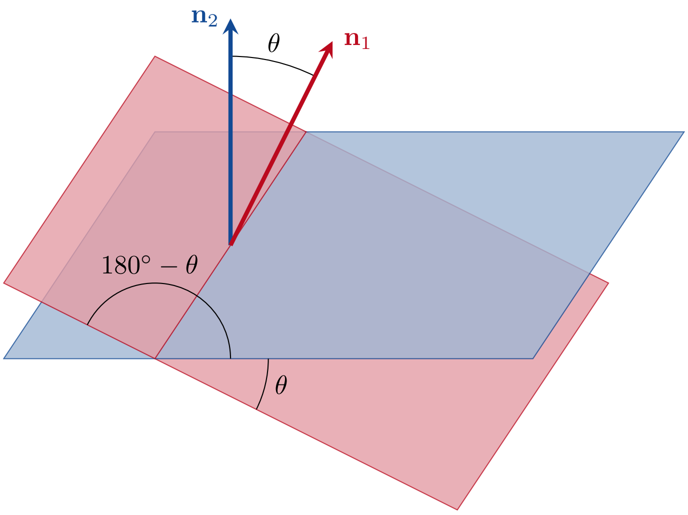

Figure 30.6: The angle between two planes is either θ or $(180° - \theta)$, whichever is smaller.
Şekil 30.6:

Problems

Problem 30.1. Find an equation for the plane passing through the points $E(2,4,5)$, $F(1,5,7)$ and $G(-1,6,8)$.

Problem 30.2. Let $O(0,0,0)$ be the origin. Find an equation for the plane through the point $A(1,-2,1)$ which is perpendicular to the vector $\overrightarrow{OA}$.

Problem 30.3. Find the point where the line $x=2$, $y=3+2t$, $z=-2-2t$ intersects the plane $6x+3y-4z=-12$.

Problem 30.4. Find the point where the line intersects the plane.

(a). Line: $x=1-t$, $y=3t$, $z=1+t$,
Plane: $2x-y+3z=6$.

(b). Line: $x=2$, $y=3+2t$, $z=-2-2t$,
Plane: $6x+3y-4z=-12$.

Problem 30.5. Find parametric equations for the line in which the following pairs of planes intersect.

(a). Plane 1: $x+y+z=1$,
Plane 2: $x+y=2$.

(b). Plane 1: $3x-6y-2z=1$,
Plane 2: $2x+y-2z=2$.

Problem 30.6. Find the distance from the point $S(1,0,-1)$ to the plane $-4x+y+z=4$.

Problem 30.7. Find a formula for the distance between two planes.

Sorular

Soru 30.1. Find an equation for the plane passing through the points $E(2,4,5)$, $F(1,5,7)$ and $G(-1,6,8)$.

Soru 30.2. Let $O(0,0,0)$ be the origin. Find an equation for the plane through the point $A(1,-2,1)$ which is perpendicular to the vector $\overrightarrow{OA}$.

Soru 30.3. Find the point where the line $x=2$, $y=3+2t$, $z=-2-2t$ intersects the plane $6x+3y-4z=-12$.

Soru 30.4. Find the point where the line intersects the plane.

(a). Line: $x=1-t$, $y=3t$, $z=1+t$,
Plane: $2x-y+3z=6$.

(b). Line: $x=2$, $y=3+2t$, $z=-2-2t$,
Plane: $6x+3y-4z=-12$.

Soru 30.5. Find parametric equations for the line in which the following pairs of planes intersect.

(a). Plane 1: $x+y+z=1$,
Plane 2: $x+y=2$.

(b). Plane 1: $3x-6y-2z=1$,
Plane 2: $2x+y-2z=2$.

Soru 30.6. Find the distance from the point $S(1,0,-1)$ to the plane $-4x+y+z=4$.

Soru 30.7. Find a formula for the distance between two planes.

31

Projections

Recall that in chapter 27 we defined the projection of a vector **u** onto a vector **v** to be

$$\mathrm{proj}_{\mathbf{v}}\,\mathbf{u} = \left(\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{v}\|^2}\right)\mathbf{v}.$$

Projection of a Vector onto a Line

Definition. Let L be the line passing through the point P in the direction $\mathbf{v}$. The projection of a vector $\mathbf{u}$ onto the line L is

$$\mathrm{proj}_L\,\mathbf{u} = \mathrm{proj}_{\mathbf{v}}\,\mathbf{u}.$$

Example 31.1. Find the projection of the vector $\mathbf{u} = 2\mathbf{i} - \mathbf{j} + 3\mathbf{k}$ onto the line $x = 1 + 2t$, $y = 2 - t$, $z = 4 - 4t$.

solution: . Clearly $\mathbf{v} = 2\mathbf{i} - \mathbf{j} - 4\mathbf{k}$ is parallel to the line. Thus

$$\begin{aligned}
\mathrm{proj}_L\,\mathbf{u} = \mathrm{proj}_{\mathbf{v}}\,\mathbf{u} &= \left(\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{v}\|^2}\right)\mathbf{v} \\
&= \left(\frac{4+1-12}{2^2+(-1)^2+(-4)^2}\right)(2\mathbf{i}-\mathbf{j}-4\mathbf{k}) \\
&= \left(\frac{-7}{21}\right)(2\mathbf{i}-\mathbf{j}-4\mathbf{k}) \\
&= -\frac{1}{3}(2\mathbf{i}-\mathbf{j}-4\mathbf{k}) \\
&= -\frac{2}{3}\mathbf{i} + \frac{1}{3}\mathbf{j} + \frac{4}{3}\mathbf{k}.
\end{aligned}$$

İzdüşümler

Recall that in chapter 27 we defined the projection of a vector **u** onto a vector **v** to be

$$\mathrm{proj}_{\mathbf{v}}\,\mathbf{u} = \left(\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{v}\|^2}\right)\mathbf{v}.$$

Projection of a Vector onto a Line

Tanım. Let L be the line passing through the point P in the direction $\mathbf{v}$. The projection of a vector $\mathbf{u}$ onto the line L is

$$\mathrm{proj}_L\,\mathbf{u} = \mathrm{proj}_{\mathbf{v}}\,\mathbf{u}.$$

Örnek 31.1. Find the projection of the vector $\mathbf{u} = 2\mathbf{i} - \mathbf{j} + 3\mathbf{k}$ onto the line $x = 1 + 2t$, $y = 2 - t$, $z = 4 - 4t$.

çözüm: . Clearly $\mathbf{v} = 2\mathbf{i} - \mathbf{j} - 4\mathbf{k}$ is parallel to the line. Thus

$$\begin{aligned}
\mathrm{proj}_L\,\mathbf{u} = \mathrm{proj}_{\mathbf{v}}\,\mathbf{u} &= \left(\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{v}\|^2}\right)\mathbf{v} \\
&= \left(\frac{4+1-12}{2^2+(-1)^2+(-4)^2}\right)(2\mathbf{i}-\mathbf{j}-4\mathbf{k}) \\
&= \left(\frac{-7}{21}\right)(2\mathbf{i}-\mathbf{j}-4\mathbf{k}) \\
&= -\frac{1}{3}(2\mathbf{i}-\mathbf{j}-4\mathbf{k}) \\
&= -\frac{2}{3}\mathbf{i} + \frac{1}{3}\mathbf{j} + \frac{4}{3}\mathbf{k}.
\end{aligned}$$

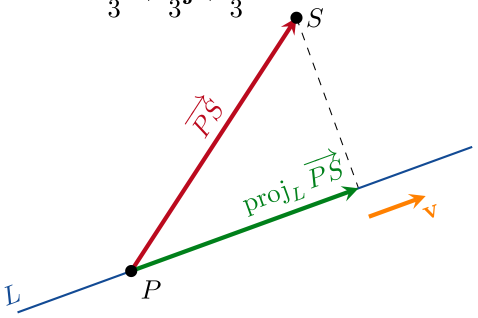

Figure 31.1: Projection of a vector onto a line.
Şekil 31.1:

Projection of a Vector onto a Plane

Definition. The ***projection*** of a vector $\mathbf{u}$ onto a plane with normal vector $\mathbf{n}$ is

$$\operatorname{proj}_{\text{plane}} \mathbf{u} = \mathbf{u} - \operatorname{proj}_{\mathbf{n}} \mathbf{u} = \mathbf{u} - \left(\frac{\mathbf{u} \cdot \mathbf{n}}{\|\mathbf{n}\|^2}\right) \mathbf{n}.$$

See figure 31.2.

Example 31.2. Find the projection of the vector $\mathbf{u} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ onto the plane $3x - y + 2z = 7$.

solution: Clearly $\mathbf{n} = 3\mathbf{i} - \mathbf{j} + 2\mathbf{k}$ and

$$\begin{aligned}\operatorname{proj}_{\mathbf{n}} \mathbf{u} &= \left(\frac{\mathbf{u} \cdot \mathbf{n}}{\|\mathbf{n}\|^2}\right) \mathbf{n} = \left(\frac{3 - 2 + 6}{3^2 + (-1)^2 + 2^2}\right)(3\mathbf{i} - \mathbf{j} + 2\mathbf{k}) \\ &= \frac{1}{2}(3\mathbf{i} - \mathbf{j} + 2\mathbf{k}) = \frac{3}{2}\mathbf{i} - \frac{1}{2}\mathbf{j} + \mathbf{k}.\end{aligned}$$

Therefore

$$\begin{aligned}\operatorname{proj}_{\text{plane}} \mathbf{u} &= \mathbf{u} - \operatorname{proj}_{\mathbf{n}} \mathbf{u} \\ &= (\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) - \left(\frac{3}{2}\mathbf{i} - \frac{1}{2}\mathbf{j} + \mathbf{k}\right) \\ &= -\frac{1}{2}\mathbf{i} + \frac{5}{2}\mathbf{j} + 2\mathbf{k}.\end{aligned}$$

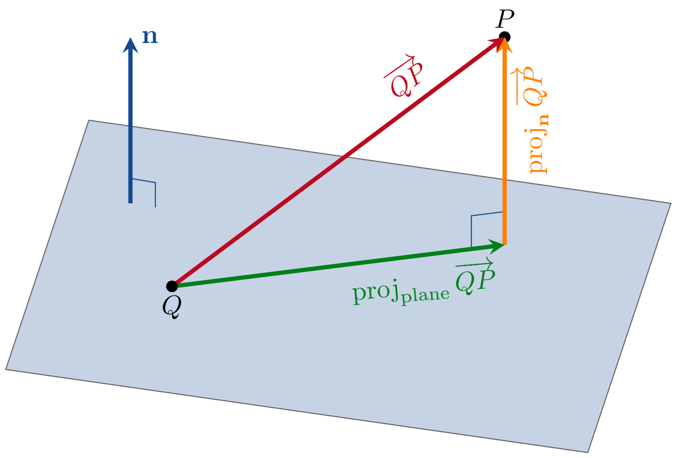

Figure 31.2: The projection of a vector onto a plane.
Şekil 31.2:

Projection of a Vector onto a Plane

Tanım. The ***projection*** of a vector $\mathbf{u}$ onto a plane with normal vector $\mathbf{n}$ is

$$\operatorname{proj}_{\text{düzlem}} \mathbf{u} = \mathbf{u} - \operatorname{proj}_{\mathbf{n}} \mathbf{u} = \mathbf{u} - \left(\frac{\mathbf{u} \cdot \mathbf{n}}{\|\mathbf{n}\|^2}\right) \mathbf{n}.$$

See figure 31.2.

Örnek 31.2. Find the projection of the vector $\mathbf{u} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ onto the plane $3x - y + 2z = 7$.

çözüm: Clearly $\mathbf{n} = 3\mathbf{i} - \mathbf{j} + 2\mathbf{k}$ and

$$\begin{aligned}\operatorname{proj}_{\mathbf{n}} \mathbf{u} &= \left(\frac{\mathbf{u} \cdot \mathbf{n}}{\|\mathbf{n}\|^2}\right) \mathbf{n} = \left(\frac{3 - 2 + 6}{3^2 + (-1)^2 + 2^2}\right)(3\mathbf{i} - \mathbf{j} + 2\mathbf{k}) \\ &= \frac{1}{2}(3\mathbf{i} - \mathbf{j} + 2\mathbf{k}) = \frac{3}{2}\mathbf{i} - \frac{1}{2}\mathbf{j} + \mathbf{k}.\end{aligned}$$

Therefore

$$\begin{aligned}\operatorname{proj}_{\text{düzlem}} \mathbf{u} &= \mathbf{u} - \operatorname{proj}_{\mathbf{n}} \mathbf{u} \\ &= (\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) - \left(\frac{3}{2}\mathbf{i} - \frac{1}{2}\mathbf{j} + \mathbf{k}\right) \\ &= -\frac{1}{2}\mathbf{i} + \frac{5}{2}\mathbf{j} + 2\mathbf{k}.\end{aligned}$$

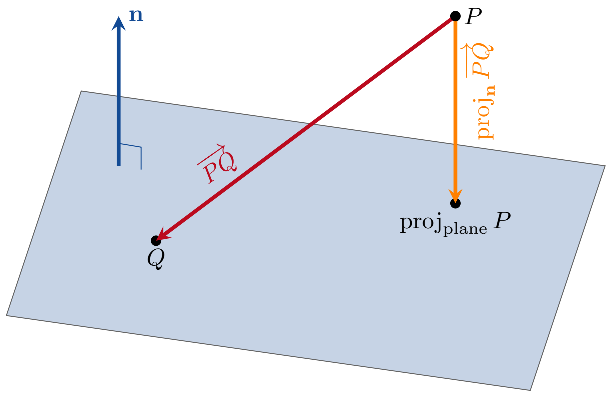

Figure 31.3: The projection of a point onto a plane.
Şekil 31.3:

Projection of a Point on a Plane

Definition. Let P be a point and let $Ax + By + Cz = D$ be a plane. Let Q be a point on the plane and let $\mathbf{n} = A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$ denote a vector normal to the plane.

The projection of the point P onto this plane is

$$\operatorname{proj}_{\text{plane}} P = P + \operatorname{proj}_{\mathbf{n}} \overrightarrow{PQ}$$

as shown in figure 31.3

Example 31.3. Find the projection of the point $P(1, 2, -4)$ on the plane $2x + y + 4z = 2$.

solution: Note first that $\mathbf{n} = 2\mathbf{i} + \mathbf{j} + 4\mathbf{k}$ and that the point $Q(1, 0, 0)$ lies on the plane. Since

$$\overrightarrow{PQ} = Q - P = (1, 0, 0) - (1, 2, -4) = (0, -2, 4) = -2\mathbf{j} + 4\mathbf{k},$$

we have

$$\begin{aligned}
\operatorname{proj}_{\mathbf{n}} \overrightarrow{PQ} &= \left(\frac{\overrightarrow{PQ} \cdot \mathbf{n}}{\|\mathbf{n}\|^2} \right) \mathbf{n} \\
&= \left(\frac{0 - 2 + 16}{2^2 + 1^2 + 4^2} \right) (2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}) \\
&= \left(\frac{14}{21} \right) (2\mathbf{i} + \mathbf{j} + 4\mathbf{k}) \\
&= \frac{2}{3} (2\mathbf{i} + \mathbf{j} + 4\mathbf{k}) \\
&= \frac{4}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} + \frac{8}{3}\mathbf{k}.
\end{aligned}$$

Therefore

$$\begin{aligned}
\operatorname{proj}_{\text{plane}} P &= P + \operatorname{proj}_{\mathbf{n}} \overrightarrow{PQ} \\
&= (1, 2, -4) + \left(\frac{4}{3}, \frac{2}{3}, \frac{8}{3} \right) \\
&= \left(\frac{7}{3}, \frac{8}{3}, -\frac{4}{3} \right).
\end{aligned}$$

We should double check that this point is on the plane.

$$2x+y+4z = 2\left(\frac{7}{3}\right)+\left(\frac{8}{3}\right)+4\left(-\frac{4}{3}\right) = \frac{14}{3}+\frac{8}{3}-\frac{16}{3} = \frac{6}{3} = 2 \checkmark$$

Projection of a Point on a Plane

Tanım. Let P be a point and let $Ax + By + Cz = D$ be a plane. Let Q be a point on the plane and let $\mathbf{n} = A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$ denote a vector normal to the plane.

The projection of the point P onto this plane is

$$\operatorname{proj}_{\text{düzlem}} P = P + \operatorname{proj}_{\mathbf{n}} \overrightarrow{PQ}$$

as shown in figure 31.3

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çözüm: Note first that $\mathbf{n} = 2\mathbf{i} + \mathbf{j} + 4\mathbf{k}$ and that the point $Q(1, 0, 0)$ lies on the plane. Since

$$\overrightarrow{PQ} = Q - P = (1, 0, 0) - (1, 2, -4) = (0, -2, 4) = -2\mathbf{j} + 4\mathbf{k},$$

we have

$$\begin{aligned}
\operatorname{proj}_{\mathbf{n}} \overrightarrow{PQ} &= \left(\frac{\overrightarrow{PQ} \cdot \mathbf{n}}{\|\mathbf{n}\|^2} \right) \mathbf{n} \\
&= \left(\frac{0 - 2 + 16}{2^2 + 1^2 + 4^2} \right) (2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}) \\
&= \left(\frac{14}{21} \right) (2\mathbf{i} + \mathbf{j} + 4\mathbf{k}) \\
&= \frac{2}{3} (2\mathbf{i} + \mathbf{j} + 4\mathbf{k}) \\
&= \frac{4}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} + \frac{8}{3}\mathbf{k}.
\end{aligned}$$

Therefore

$$\begin{aligned}
\operatorname{proj}_{\text{düzlem}} P &= P + \operatorname{proj}_{\mathbf{n}} \overrightarrow{PQ} \\
&= (1, 2, -4) + \left(\frac{4}{3}, \frac{2}{3}, \frac{8}{3} \right) \\
&= \left(\frac{7}{3}, \frac{8}{3}, -\frac{4}{3} \right).
\end{aligned}$$

We should double check that this point is on the plane.

$$2x+y+4z = 2\left(\frac{7}{3}\right)+\left(\frac{8}{3}\right)+4\left(-\frac{4}{3}\right) = \frac{14}{3}+\frac{8}{3}-\frac{16}{3} = \frac{6}{3} = 2 \checkmark$$

Projection of a Line onto a Plane

Let L be a line passing through the point P in the direction $\mathbf{v}$. Let $Ax + By + Cz = D$ be a plane with normal vector $\mathbf{n} = A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$.

There are three cases to consider:

(i). The line is orthogonal to the plane ($\mathbf{v} \times \mathbf{n} = \mathbf{0}$);

(ii). The line is parallel to the plane ($\mathbf{v} \cdot \mathbf{n} = 0$); and

(iii). The line is not parallel to the plane and is not orthogonal to the plane ($\mathbf{v} \cdot \mathbf{n} \neq 0$ and $\mathbf{v} \times \mathbf{n} \neq \mathbf{0}$).

Projection of a Line onto a Plane

Let L be a line passing through the point P in the direction $\mathbf{v}$. Let $Ax + By + Cz = D$ be a plane with normal vector $\mathbf{n} = A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$.

There are three cases to consider:

(i). The line is orthogonal to the plane ($\mathbf{v} \times \mathbf{n} = \mathbf{0}$);

(ii). The line is parallel to the plane ($\mathbf{v} \cdot \mathbf{n} = 0$); and

(iii). The line is not parallel to the plane and is not orthogonal to the plane ($\mathbf{v} \cdot \mathbf{n} \neq 0$ and $\mathbf{v} \times \mathbf{n} \neq \mathbf{0}$).

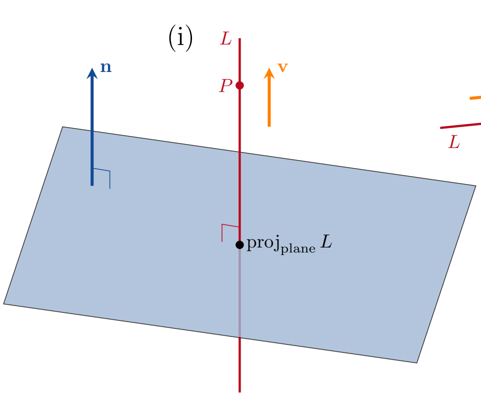

Figure 31.4: The projection of a line onto an orthogonal plane.
Şekil 31.4:

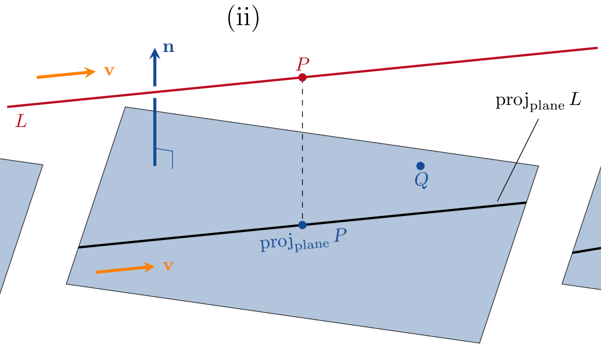

Figure 31.5: The projection of a line onto a parallel plane.
Şekil 31.5:

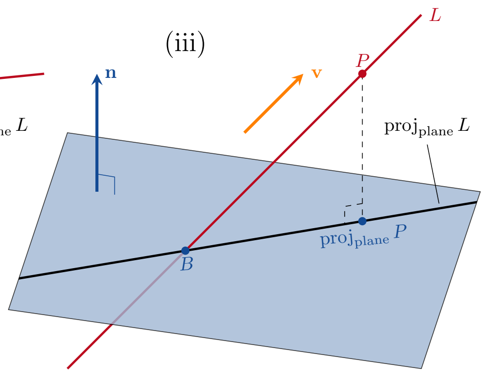

Figure 31.6: The projection of a line onto a plane with neither orthogonal nor parallel to the line.
Şekil 31.6:

A Line Orthogonal to a Plane ($\mathbf{v} \times \mathbf{n} = \mathbf{0}$)

This is the easiest case: The projection of the line onto the plane is just the point where they intersect. See figure 31.4. Therefore

$$\operatorname{proj}_{\text{plane}} L = \operatorname{proj}_{\text{plane}} P.$$

A Line Parallel to a Plane ($\mathbf{v} \cdot \mathbf{n} = 0$)

From figure 31.5, we can see that

$$\operatorname{proj}_{\text{plane}} L = \left(\begin{array}{c} \text{the line passing through the point} \\ \operatorname{proj}_{\text{plane}} P \text{ in the direction } \mathbf{v}. \end{array} \right)$$

A Line which is Neither Parallel nor Orthogonal to the Plane

See figure 31.6. If $\mathbf{v} \cdot \mathbf{n} \neq 0$, then the line must intersect the plane at some point B. Assuming $B \neq P$, we have

$$\operatorname{proj}_{\text{plane}} L = \left(\begin{array}{c} \text{the line passing through the} \\ \text{points } B \text{ and } \operatorname{proj}_{\text{plane}} P. \end{array} \right)$$

A Line Orthogonal to a Plane ($\mathbf{v} \times \mathbf{n} = \mathbf{0}$)

This is the easiest case: The projection of the line onto the plane is just the point where they intersect. See figure 31.4. Therefore

$$\operatorname{proj}_{\text{düzlem}} L = \operatorname{proj}_{\text{düzlem}} P.$$

A Line Parallel to a Plane ($\mathbf{v} \cdot \mathbf{n} = 0$)

From figure 31.5, we can see that

$$\operatorname{proj}_{\text{düzlem}} L = \left(\begin{array}{c} \text{the line passing through the point} \\ \operatorname{proj}_{\text{düzlem}} P \text{ in the direction } \mathbf{v}. \end{array} \right)$$

A Line which is Neither Parallel nor Orthogonal to the Plane

See figure 31.6. If $\mathbf{v} \cdot \mathbf{n} \neq 0$, then the line must intersect the plane at some point B. Assuming $B \neq P$, we have

$$\operatorname{proj}_{\text{düzlem}} L = \left(\begin{array}{c} \text{the line passing through the} \\ \text{points } B \text{ and } \operatorname{proj}_{\text{düzlem}} P. \end{array} \right)$$

Example 31.4. Find the projection of the line $x = 7 + 6t$, $y = -3 + 15t$, $z = 10 - 12t$ onto the plane $2x + 5y - 4z = 13$.

solution:

Step 1. Find $\mathbf{v}$ and $\mathbf{n}$.

$$\begin{aligned} \mathbf{v} &= 6\mathbf{i} + 15\mathbf{j} - 12\mathbf{k} \\ \mathbf{n} &= 2\mathbf{i} + 5\mathbf{j} - 4\mathbf{k} \end{aligned}$$

Step 2. Does the line intersect the plane?

Since

$$\mathbf{v} \cdot \mathbf{n} = 12 + 75 + 48 = 135 \neq 0,$$

the answer is yes, the line does intersect the plane.

Step 3. Find the point of intersection.

We calculate that

$$\begin{aligned} 13 &= 2x + 5y - 4z \\ &= 2(7 + 6t) + 5(-3 + 15t) - 4(10 - 12t) \\ &= 14 + 12t - 15 + 75t - 40 + 48t \\ &= -41 + 135t \\ 54 &= 135t \\ 2 &= 5t \\ \frac{2}{5} &= t. \end{aligned}$$

Hence the point of intersection is

$$\begin{aligned} B(x, y, z)|_{t=\frac{2}{5}} &= B(7 + 6t, -3 + 15t, 10 - 12t)|_{t=\frac{2}{5}} \\ &= B(9.4, 3, 5.2) \end{aligned}$$

Step 4. Is the line orthogonal to the plane?

Since

$$\mathbf{v} \times \mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 6 & 15 & -12 \\ 2 & 5 & -4 \end{vmatrix} = 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k} = \mathbf{0},$$

the answer is yes, the line is orthogonal to the plane.

Step 5. Find $\text{proj}_{\text{plane}} L$.

The projection of the line on the plane is the point

$$\text{proj}_{\text{plane}} L = B(9.4, 3, 5.2).$$

Example 31.5. Find the projection of the line $x = 1 + 4t$, $y = 2 + 4t$, $z = 3 + 4t$ onto the plane $3x + 4y - 7z = 27$.

solution:

Step 1. Find $\mathbf{v}$ and $\mathbf{n}$.

$$\begin{aligned} \mathbf{v} &= 4\mathbf{i} + 4\mathbf{j} + 4\mathbf{k} \\ \mathbf{n} &= 3\mathbf{i} + 4\mathbf{j} - 7\mathbf{k} \end{aligned}$$

Örnek 31.4. Find the projection of the line $x = 7 + 6t$, $y = -3 + 15t$, $z = 10 - 12t$ onto the plane $2x + 5y - 4z = 13$.

çözüm:

Adım 1. Find $\mathbf{v}$ and $\mathbf{n}$.

$$\begin{aligned} \mathbf{v} &= 6\mathbf{i} + 15\mathbf{j} - 12\mathbf{k} \\ \mathbf{n} &= 2\mathbf{i} + 5\mathbf{j} - 4\mathbf{k} \end{aligned}$$

Adım 2. Does the line intersect the plane?

Since

$$\mathbf{v} \cdot \mathbf{n} = 12 + 75 + 48 = 135 \neq 0,$$

the answer is yes, the line does intersect the plane.

Adım 3. Find the point of intersection.

We calculate that

$$\begin{aligned} 13 &= 2x + 5y - 4z \\ &= 2(7 + 6t) + 5(-3 + 15t) - 4(10 - 12t) \\ &= 14 + 12t - 15 + 75t - 40 + 48t \\ &= -41 + 135t \\ 54 &= 135t \\ 2 &= 5t \\ \frac{2}{5} &= t. \end{aligned}$$

Hence the point of intersection is

$$\begin{aligned} B(x, y, z)|_{t=\frac{2}{5}} &= B(7 + 6t, -3 + 15t, 10 - 12t)|_{t=\frac{2}{5}} \\ &= B(9.4, 3, 5.2) \end{aligned}$$

Adım 4. Is the line orthogonal to the plane?

Since

$$\mathbf{v} \times \mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 6 & 15 & -12 \\ 2 & 5 & -4 \end{vmatrix} = 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k} = \mathbf{0},$$

the answer is yes, the line is orthogonal to the plane.

Adım 5. Find $\text{proj}_{\text{düzlem}} L$.

The projection of the line on the plane is the point

$$\text{proj}_{\text{düzlem}} L = B(9.4, 3, 5.2).$$

Örnek 31.5. Find the projection of the line $x = 1 + 4t$, $y = 2 + 4t$, $z = 3 + 4t$ onto the plane $3x + 4y - 7z = 27$.

çözüm:

Adım 1. Find $\mathbf{v}$ and $\mathbf{n}$.

$$\begin{aligned} \mathbf{v} &= 4\mathbf{i} + 4\mathbf{j} + 4\mathbf{k} \\ \mathbf{n} &= 3\mathbf{i} + 4\mathbf{j} - 7\mathbf{k} \end{aligned}$$

Step 2. Does the line intersect the plane?

Since

$$\mathbf{v} \cdot \mathbf{n} = 12 + 16 - 28 = 0,$$

the line does not intersect the plane. Therefore the line is parallel to the plane.

Step 3. Find a point on $\text{proj}_{\text{plane}} L$.

$P(1,2,3)$ lies on the original line and $Q(9,0,0)$ lies on the plane. So

$$\begin{aligned}\overrightarrow{PQ} &= Q - P = (9,0,0) - (1,2,3) = (8,-2,-3)\\ &= 8\mathbf{i} - 2\mathbf{j} - 3\mathbf{k}\end{aligned}$$

and

$$\begin{aligned}\text{proj}_{\mathbf{n}} \overrightarrow{PQ} &= \left(\frac{\overrightarrow{PQ} \cdot \mathbf{n}}{\|\mathbf{n}\|^2}\right)\mathbf{n} = \left(\frac{24-8+21}{9+16+49}\right)\mathbf{n}\\ &= \left(\frac{37}{74}\right)\mathbf{n} = \frac{1}{2}\mathbf{n}.\end{aligned}$$

Therefore

$$\begin{aligned}\text{proj}_{\text{plane}} P &= P + \text{proj}_{\mathbf{n}} \overrightarrow{PQ}\\ &= (1,2,3) + \left(\frac{3}{2}, 2, -\frac{7}{2}\right)\\ &= \left(\frac{5}{2}, 4, -\frac{1}{2}\right).\end{aligned}$$

We should quickly double check that our $\text{proj}_{\text{plane}} P$ really is on the plane:

$$\begin{aligned}3x + 4y - 7z &= 3\left(\frac{5}{2}\right) + 4(4) - 7\left(-\frac{1}{2}\right)\\ &= \frac{15}{2} + 16 + \frac{7}{2} = 27. \checkmark\end{aligned}$$

Step 4. Find $\text{proj}_{\text{plane}} L$.

The projection of the original line on the plane is the line passing through the point $\text{proj}_{\text{plane}} P = \left(\frac{5}{2}, 4, -\frac{1}{2}\right)$ in the direction $\mathbf{v} = 4\mathbf{i}+4\mathbf{j}+4\mathbf{k}$, which has parametrised equations

$$x = \frac{5}{2} + 4t, \qquad y = 4 + 4t, \qquad z = -\frac{1}{2} + 4t.$$

Example 31.6. Find the projection of the line $x = 15 + 15t$, $y = -12 - 15t$, $z = 17 + 11t$ on the plane $13x - 9y + 16z = 69$.

solution: .

Step 1. Find $\mathbf{v}$ and $\mathbf{n}$.

$$\begin{aligned}\mathbf{v} &= 15\mathbf{i} - 15\mathbf{j} + 11\mathbf{k}\\ \mathbf{n} &= 13\mathbf{i} - 9\mathbf{j} + 16\mathbf{k}\end{aligned}$$

Step 2. Does the line intersect the plane?

Since

$$\mathbf{v} \cdot \mathbf{n} = 506 \neq 0,$$

the line intersects the plane.

Adım 2. Does the line intersect the plane?

Since

$$\mathbf{v} \cdot \mathbf{n} = 12 + 16 - 28 = 0,$$

the line does not intersect the plane. Therefore the line is parallel to the plane.

Adım 3. Find a point on $\text{proj}_{\text{düzlem}} L$.

$P(1,2,3)$ lies on the original line and $Q(9,0,0)$ lies on the plane. So

$$\begin{aligned}\overrightarrow{PQ} &= Q - P = (9,0,0) - (1,2,3) = (8,-2,-3)\\ &= 8\mathbf{i} - 2\mathbf{j} - 3\mathbf{k}\end{aligned}$$

and

$$\begin{aligned}\text{proj}_{\mathbf{n}} \overrightarrow{PQ} &= \left(\frac{\overrightarrow{PQ} \cdot \mathbf{n}}{\|\mathbf{n}\|^2}\right)\mathbf{n} = \left(\frac{24-8+21}{9+16+49}\right)\mathbf{n}\\ &= \left(\frac{37}{74}\right)\mathbf{n} = \frac{1}{2}\mathbf{n}.\end{aligned}$$

Therefore

$$\begin{aligned}\text{proj}_{\text{düzlem}} P &= P + \text{proj}_{\mathbf{n}} \overrightarrow{PQ}\\ &= (1,2,3) + \left(\frac{3}{2}, 2, -\frac{7}{2}\right)\\ &= \left(\frac{5}{2}, 4, -\frac{1}{2}\right).\end{aligned}$$

We should quickly double check that our $\text{proj}_{\text{düzlem}} P$ really is on the plane:

$$\begin{aligned}3x + 4y - 7z &= 3\left(\frac{5}{2}\right) + 4(4) - 7\left(-\frac{1}{2}\right)\\ &= \frac{15}{2} + 16 + \frac{7}{2} = 27. \checkmark\end{aligned}$$

Adım 4. Find $\text{proj}_{\text{düzlem}} L$.

The projection of the original line on the plane is the line passing through the point $\text{proj}_{\text{düzlem}} P = \left(\frac{5}{2}, 4, -\frac{1}{2}\right)$ in the direction $\mathbf{v} = 4\mathbf{i} + 4\mathbf{j} + 4\mathbf{k}$, which has parametrised equations

$$x = \frac{5}{2} + 4t, \qquad y = 4 + 4t, \qquad z = -\frac{1}{2} + 4t.$$

Örnek 31.6. Find the projection of the line $x = 15 + 15t$, $y = -12 - 15t$, $z = 17 + 11t$ on the plane $13x - 9y + 16z = 69$.

çözüm: .

Adım 1. Find $\mathbf{v}$ and $\mathbf{n}$.

$$\begin{aligned}\mathbf{v} &= 15\mathbf{i} - 15\mathbf{j} + 11\mathbf{k}\\ \mathbf{n} &= 13\mathbf{i} - 9\mathbf{j} + 16\mathbf{k}\end{aligned}$$

Adım 2. Does the line intersect the plane?

Since

$$\mathbf{v} \cdot \mathbf{n} = 506 \neq 0,$$

the line intersects the plane.

Step 3. Find the point of intersection.

We calculate that

$$\begin{aligned} 69 &= 13x - 9y + 16z \\ &= 13(15 + 15t) - 9(-12 - 15) + 16(17 + 11t) \\ &= 195 + 195t + 108 + 135t + 272 + 176t \\ &= 575 + 506t \\ -506 &= 506t \\ -1 &= t. \end{aligned}$$

Thus the line intersects the plane at

$$\begin{aligned} B(x, y, z)\big|_{t=-1} &= B(15 + 15t, -12 - 15t, 17 + 11t)\big|_{t=-1} \\ &= B(0, 3, 6). \end{aligned}$$

Step 4. Is the line orthogonal to the plane?

Since

$$\mathbf{v} \times \mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 15 & -15 & 11 \\ 13 & -9 & 16 \end{vmatrix} = -141\mathbf{i} - 97\mathbf{j} + 60\mathbf{k} \neq \mathbf{0},$$

the line is not orthogonal to the plane.

Step 5. Find another point on $\operatorname{proj}_{\text{plane}} L$.

The point $P(15, -12, 17)$ lies on the original line. Since $\overrightarrow{PB} = (-15, 15, -11)$ and

$$\operatorname{proj}_{\mathbf{n}} \overrightarrow{PB} = \left(\frac{\overrightarrow{PB} \cdot \mathbf{n}}{\|\mathbf{n}\|^2} \right) \mathbf{n} = \left(\frac{-506}{506} \right) \mathbf{n} = -\mathbf{n}$$

we have that

$$\begin{aligned} \operatorname{proj}_{\text{plane}} P &= P + \operatorname{proj}_{\mathbf{n}} \overrightarrow{PB} \\ &= (15, -12, 17) + (-13, 9, -16) = (2, -3, 1). \end{aligned}$$

Step 6. Find $\operatorname{proj}_{\text{plane}} L$.

Let

$\mathbf{v}_2 =$ the vector from B to $\operatorname{proj}_{\text{plane}} P = 2\mathbf{i} - 6\mathbf{j} - 5\mathbf{k}$.

Then $\operatorname{proj}_{\text{plane}} L$ is the line passing through $B(0, 3, 6)$ in the direction $\mathbf{v}_2 = 2\mathbf{i} - 6\mathbf{j} - 5\mathbf{k}$ which has parametrised equations

$$x = 2t, \qquad y = 3 - 6t, \qquad z = 6 - 5t.$$

Adım 3. Find the point of intersection.

We calculate that

$$\begin{aligned} 69 &= 13x - 9y + 16z \\ &= 13(15 + 15t) - 9(-12 - 15) + 16(17 + 11t) \\ &= 195 + 195t + 108 + 135t + 272 + 176t \\ &= 575 + 506t \\ -506 &= 506t \\ -1 &= t. \end{aligned}$$

Thus the line intersects the plane at

$$\begin{aligned} B(x, y, z)\big|_{t=-1} &= B(15 + 15t, -12 - 15t, 17 + 11t)\big|_{t=-1} \\ &= B(0, 3, 6). \end{aligned}$$

Adım 4. Is the line orthogonal to the plane?

Since

$$\mathbf{v} \times \mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 15 & -15 & 11 \\ 13 & -9 & 16 \end{vmatrix} = -141\mathbf{i} - 97\mathbf{j} + 60\mathbf{k} \neq \mathbf{0},$$

the line is not orthogonal to the plane.

Adım 5. Find another point on $\operatorname{proj}_{\text{düzlem}} L$.

The point $P(15, -12, 17)$ lies on the original line. Since $\overrightarrow{PB} = (-15, 15, -11)$ and

$$\operatorname{proj}_{\mathbf{n}} \overrightarrow{PB} = \left(\frac{\overrightarrow{PB} \cdot \mathbf{n}}{\|\mathbf{n}\|^2} \right) \mathbf{n} = \left(\frac{-506}{506} \right) \mathbf{n} = -\mathbf{n}$$

we have that

$$\begin{aligned} \operatorname{proj}_{\text{düzlem}} P &= P + \operatorname{proj}_{\mathbf{n}} \overrightarrow{PB} \\ &= (15, -12, 17) + (-13, 9, -16) = (2, -3, 1). \end{aligned}$$

Adım 6. Find $\operatorname{proj}_{\text{düzlem}} L$.

Let

$\mathbf{v}_2 =$ the vector from B to $\operatorname{proj}_{\text{düzlem}} P = 2\mathbf{i} - 6\mathbf{j} - 5\mathbf{k}$.

Then $\operatorname{proj}_{\text{düzlem}} L$ is the line passing through $B(0, 3, 6)$ in the direction $\mathbf{v}_2 = 2\mathbf{i} - 6\mathbf{j} - 5\mathbf{k}$ which has parametrised equations

$$x = 2t, \qquad y = 3 - 6t, \qquad z = 6 - 5t.$$

Problems

Problem 31.1. Find the projection of the vector $\mathbf{u} = -2\mathbf{i} + 5\mathbf{j} - 3\mathbf{k}$ onto the line $x = 2 + t$, $y = 1 - 2t$, $z = 3 + 2t$.

Problem 31.2. Find the projection of the vector $\mathbf{u} = \mathbf{i}+7\mathbf{j}+5\mathbf{k}$ onto the plane $6x + 4z = 100$.

Problem 31.3. Find the projection of the point $P(38, -59, 4)$ onto the plane $10x - 20y + z = 61$.

Problem 31.4. Find the projection of the line $x = -48 - t$, $y = 6 + t$, $z = -13 + 4t$ onto the plane $7x - y + 2z = 10$.

Problem 31.5. Find the projection of the line $x = 2 + 30t$, $y = 29 - 130t$, $z = \frac{104}{5} - 114t$ onto the plane $7y + 5z = 11$.

Problem 31.6. Find the projection of the line $x = -t$, $y = 14 + t$, $z = -\frac{23}{4} - t$ onto the plane $8x - 8y + 8z = 10$.

Problem 31.7. Find a formula for the projection of a point P onto a line L.

Sorular

Soru 31.1. Find the projection of the vector $\mathbf{u} = -2\mathbf{i}+5\mathbf{j}-3\mathbf{k}$ onto the line $x = 2 + t$, $y = 1 - 2t$, $z = 3 + 2t$.

Soru 31.2. Find the projection of the vector $\mathbf{u} = \mathbf{i} + 7\mathbf{j} + 5\mathbf{k}$ onto the plane $6x + 4z = 100$.

Soru 31.3. Find the projection of the point $P(38, -59, 4)$ onto the plane $10x - 20y + z = 61$.

Soru 31.4. Find the projection of the line $x = -48 - t$, $y = 6 + t$, $z = -13 + 4t$ onto the plane $7x - y + 2z = 10$.

Soru 31.5. Find the projection of the line $x = 2 + 30t$, $y = 29 - 130t$, $z = \frac{104}{5} - 114t$ onto the plane $7y + 5z = 11$.

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Soru 31.7. Find a formula for the projection of a point P onto a line L.

32 Quadric Surfaces

Kuadratik Yüzeyler

Definition. A ***quadric surface*** is the graph of

$$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gx + Hy + Iz = J$$

for $A, B, C, D, E, F, G, H, I, J \in \mathbb{R}$.

Tanım. A ***quadric surface*** is the graph of

$$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gx + Hy + Iz = J$$

for $A, B, C, D, E, F, G, H, I, J \in \mathbb{R}$.

We will study the easier equation

$$Ax^2 + By^2 + Cz^2 + Dz = E$$

where $A, B, C, D, E \in \mathbb{R}$ are constants.

We will study the easier equation

$$Ax^2 + By^2 + Cz^2 + Dz = E$$

where $A, B, C, D, E \in \mathbb{R}$ are constants.

Example 32.1.

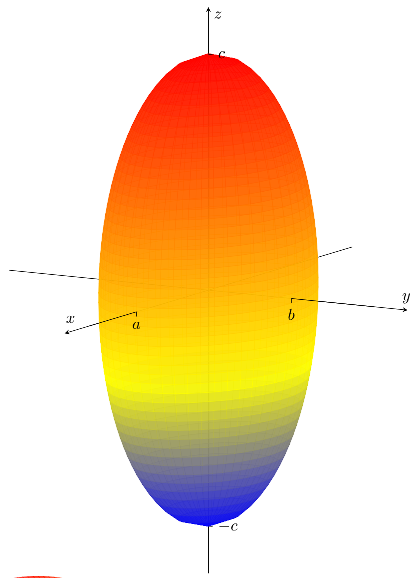
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

is an ***ellipsoid***.

Örnek 32.1.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

bir ***elipsoid***'dir.



Example 32.2.

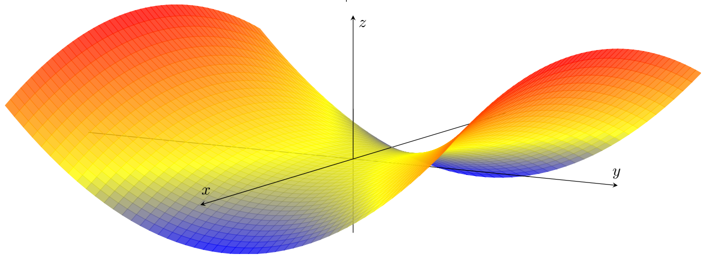
$$\frac{y^2}{b^2} - \frac{x^2}{a^2} = \frac{z}{c}$$

is an ***hyperbolic paraboloid***.

Örnek 32.2.

$$\frac{y^2}{b^2} - \frac{x^2}{a^2} = \frac{z}{c}$$

bir ***hiperbolik paraboloid***'dir.



Example 32.3.

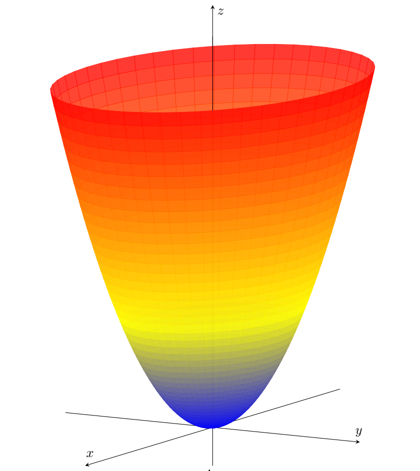
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z}{c}$$

is an ***elliptical paraboloid***.

Örnek 32.3.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z}{c}$$

bir ***eliptik paraboloid***'dir.



Example 32.4.

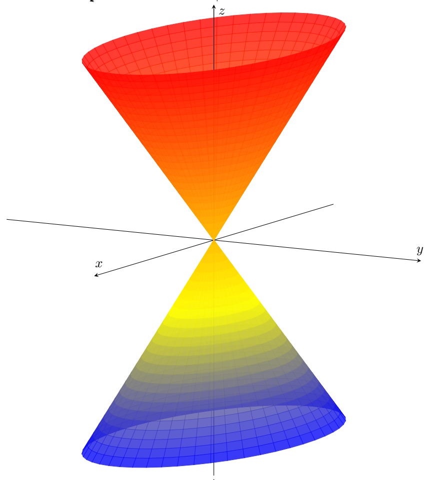
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2}$$

is an ***elliptical cone***.

Örnek 32.4.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2}$$

bir ***eliptik koni***'dir.



Example 32.5.

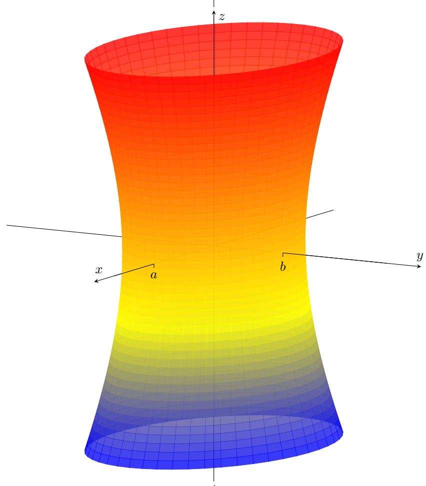
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

is an ***hyperboloid***.

Örnek 32.5.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

bir ***hiperboloid***'dir.



Example 32.6.

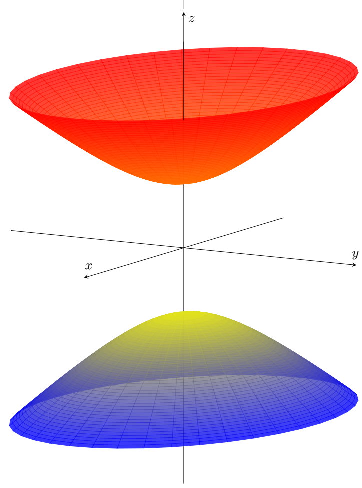
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = -1$$

is an ***hyperboloid***.

Örnek 32.6.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = -1$$

bir ***hiperboloid***'dir.



33

Cylindrical and Spherical Polar Coordinates

Cylindrical Polar Coordinates

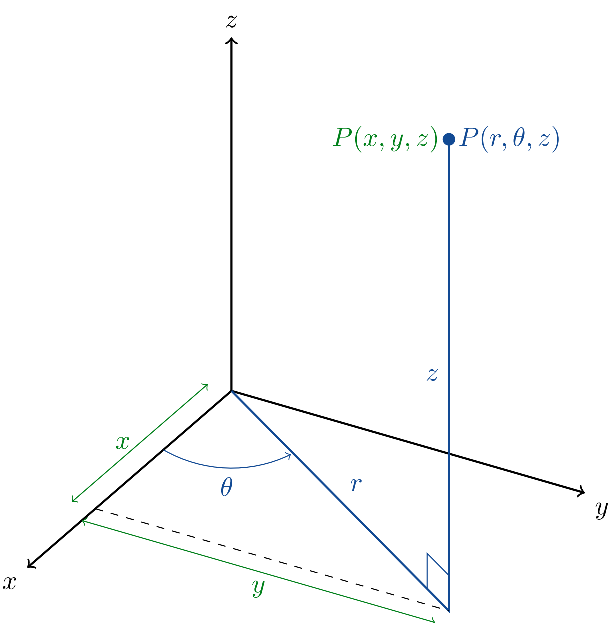


Figure 33.1: Cylindrical Polar Coordinates.
Şekil 33.1: Silindirik Koordinatlar.

Example 33.1. Find cylindrical polar coordinates for the Cartesian coordinates $(x, y, z) = (1, 1, 1)$.

solution:

$$\begin{aligned}(r, \theta, z) &= (\sqrt{x^2 + y^2}, \tan^{-1}\frac{y}{x}, z)\\ &= (\sqrt{1^2 + 1^2}, \tan^{-1} 1, 1) = (\sqrt{2}, 45°, 1).\end{aligned}$$

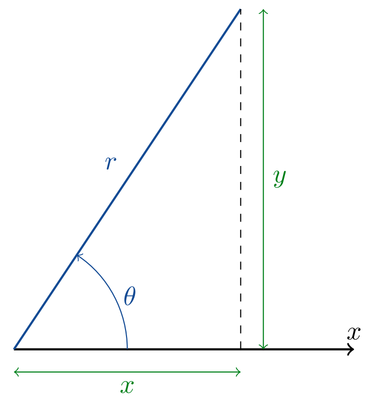
Example 33.2. Convert the cylindrical polar coordinates $(r, \theta, z) = (2, 90°, 2)$ to Cartesian coordinates.

solution:

$$\begin{aligned}(x, y, z) &= (x\cos\theta, y\sin\theta, z)\\ &= (2\cos 90°, 2\sin 90°, 2) = (0, 2, 2).\end{aligned}$$

Silindirik ve Küresel Koordinatlar

Silindirik Koordinatlar



$$\begin{aligned} x &= r\cos\theta & r^2 &= x^2 + y^2\\ y &= r\sin\theta & \tan\theta &= \frac{y}{x}\\ z &= z \end{aligned}$$

Örnek 33.1. Find cylindrical polar coordinates for the Cartesian coordinates $(x, y, z) = (1, 1, 1)$.

çözüm:

$$\begin{aligned}(r, \theta, z) &= (\sqrt{x^2 + y^2}, \tan^{-1}\frac{y}{x}, z)\\ &= (\sqrt{1^2 + 1^2}, \tan^{-1} 1, 1) = (\sqrt{2}, 45°, 1).\end{aligned}$$

Örnek 33.2. Convert the cylindrical polar coordinates $(r, \theta, z) = (2, 90°, 2)$ to Cartesian coordinates.

çözüm:

Example 33.3. Identify the surface described by each of the following cylindrical polar equations.

(a). $r = 5$;

(b). $r^2 + z^2 = 100$;

(c). $z = r$.

solution:

(a). In $\mathbb{R}^2$, we know that $r = 5$ is a circle of radius 5. Since the equation does not contain a z, z can take any value. The surface must be an infinite vertical cylinder of radius 5 centred on the z-axis.

(b). This equation will be easier to identify if we convert the equation into Cartesian coordinates.

$$r^2 + z^2 = 100$$
$$x^2 + y^2 + z^2 = 10^2$$

This is the equation of a sphere of radius 10, centred at the origin.

(c). Converting to Cartesian coordinates, we see that

$$z = r$$
$$z^2 = r^2$$
$$z^2 = x^2 + y^2.$$

From Chapter 32, we know that this is the equation of a cone.

$$(x, y, z) = (x\cos\theta, y\sin\theta, z)$$
$$= (2\cos 90°, 2\sin 90°, 2) = (0, 2, 2).$$

Örnek 33.3. Identify the surface described by each of the following cylindrical polar equations.

(a). $r = 5$;

(b). $r^2 + z^2 = 100$;

(c). $z = r$.

çözüm:

(a). In $\mathbb{R}^2$, we know that $r = 5$ is a circle of radius 5. Since the equation does not contain a z, z can take any value. The surface must be an infinite vertical cylinder of radius 5 centred on the z-axis.

(b). This equation will be easier to identify if we convert the equation into Cartesian coordinates.

$$r^2 + z^2 = 100$$
$$x^2 + y^2 + z^2 = 10^2$$

This is the equation of a sphere of radius 10, centred at the origin.

(c). Converting to Cartesian coordinates, we see that

$$z = r$$
$$z^2 = r^2$$
$$z^2 = x^2 + y^2.$$

From Chapter 32, we know that this is the equation of a cone.

Spherical Polar Coordinates

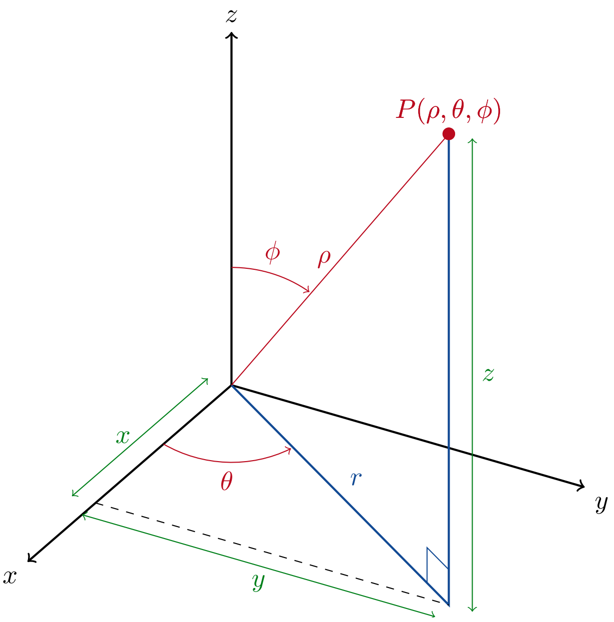


Figure 33.2: Spherical Polar Coordinates.
Şekil 33.2: Küresel Koordinatlar.

Typically, we require that $\rho \geq 0$ and $0 \leq \phi \leq 180°$. As before, θ can be any number.

Example 33.4. Convert the point $P(\sqrt{6}, 45°, \sqrt{2})$ from cylindrical to spherical polar coordinates.

solution: We have that $r = \sqrt{6}$, $\theta = 45°$ and $z = \sqrt{2}$. Therefore

$$\begin{aligned}(\rho,\theta,\phi) &= \left(\sqrt{r^2+z^2}, \theta, \cos^{-1}\frac{z}{\rho}\right)\\ &= \left(\sqrt{6+2}, 45°, \cos^{-1}\frac{\sqrt{2}}{\rho}\right)\\ &= \left(2\sqrt{2}, 45°, \cos^{-1}\frac{\sqrt{2}}{2\sqrt{2}}\right)\\ &= \left(2\sqrt{2}, 45°, \cos^{-1}\frac{1}{2}\right)\\ &= \left(2\sqrt{2}, 45°, 60°\right)\end{aligned}$$

Example 33.5. Convert the point $P(-1, 1, -\sqrt{2})$ from Cartesian to spherical polar coordinates.

solution: First we calculate that

$$\rho = \sqrt{x^2+y^2+z^2} = \sqrt{(-1)^2+1^2+(-\sqrt{2})^2} = \sqrt{4} = 2.$$

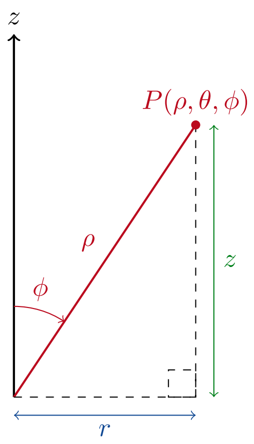
Next we calculate that

$$\phi = \cos^{-1}\frac{z}{\rho} = \cos^{-1}\frac{-\sqrt{2}}{2} = 135°$$

because we want $\phi \in [0, 180°]$. Finally we need a θ.

$$\sin\theta = \frac{y}{\rho\sin\phi} = \frac{1}{2\left(\frac{\sqrt{2}}{2}\right)} = \frac{1}{\sqrt{2}}.$$

Küresel Koordinatlar



$$\begin{aligned} x &= r\cos\theta = \rho\sin\phi\cos\theta & r &= \rho\sin\phi\\ y &= r\sin\theta = \rho\sin\phi\sin\theta & \tan\theta &= \frac{y}{x}\\ z &= \rho\cos\phi & \rho &= \sqrt{x^2+y^2+z^2} = \sqrt{r^2+z^2}\end{aligned}$$

Typically, we require that $\rho \geq 0$ and $0 \leq \phi \leq 180°$. As before, θ can be any number.

Örnek 33.4. Convert the point $P(\sqrt{6}, 45°, \sqrt{2})$ from cylindrical to spherical polar coordinates.

çözüm: We have that $r = \sqrt{6}$, $\theta = 45°$ and $z = \sqrt{2}$. Therefore

$$\begin{aligned}(\rho,\theta,\phi) &= \left(\sqrt{r^2+z^2}, \theta, \cos^{-1}\frac{z}{\rho}\right)\\ &= \left(\sqrt{6+2}, 45°, \cos^{-1}\frac{\sqrt{2}}{\rho}\right)\\ &= \left(2\sqrt{2}, 45°, \cos^{-1}\frac{\sqrt{2}}{2\sqrt{2}}\right)\\ &= \left(2\sqrt{2}, 45°, \cos^{-1}\frac{1}{2}\right)\\ &= \left(2\sqrt{2}, 45°, 60°\right)\end{aligned}$$

Örnek 33.5. Convert the point $P(-1, 1, -\sqrt{2})$ from Cartesian to spherical polar coordinates.

çözüm: First we calculate that

$$\rho = \sqrt{x^2+y^2+z^2} = \sqrt{(-1)^2+1^2+(-\sqrt{2})^2} = \sqrt{4} = 2.$$

Next we calculate that

$$\phi = \cos^{-1}\frac{z}{\rho} = \cos^{-1}\frac{-\sqrt{2}}{2} = 135°$$

because we want $\phi \in [0, 180°]$. Finally we need a θ.

$$\sin\theta = \frac{y}{\rho\sin\phi} = \frac{1}{2\left(\frac{\sqrt{2}}{2}\right)} = \frac{1}{\sqrt{2}}.$$

There are infinitely many θ that satisfy this equation. Two possible θ are $\theta = 45°$ and $\theta = 135°$. Only one of these can be correct. We can see from figure 33.3 that the correct angle must be 135°. Therefore the answer is

$$(\rho, \theta, \phi) = (2, 135°, 135°).$$

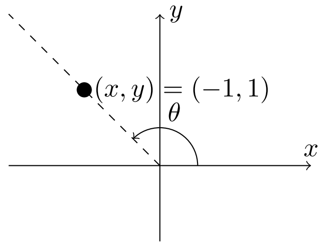


Figure 33.3: The point $(-1, 1)$.
Şekil 33.3:

Example 33.6. Identify the surface described by each of the following spherical polar equations.

(a). $\rho = 5$;

(b). $\phi = 60°$;

(c). $\theta = 120°$;

(d). $\rho \sin \phi = 2$.

solution:

(a). ρ is the distance from the origin to a point. If this is always $= 5$, then we have a surface which is always 5 away from the origin. That sounds like a sphere.

To check, we calculate

$$\begin{aligned} \rho &= 5 \\ \sqrt{x^2 + y^2 + z^2} &= 5 \\ x^2 + y^2 + z^2 &= 5^2. \end{aligned}$$

Yes, this is the equation for a sphere or radius 5 centred at the origin.

(b). The angle between the z-axis and the surface is always 60°. Thinking about this, you should be able to understand that this is the equation for a cone.

(c). This is a vertical plane passing through the origin.

(d). We will convert the equation first into cylindrical polar coordinates, then into Cartesian coordinates.

$$\begin{aligned} \rho \sin \phi &= 2 \\ r &= 2 \\ r^2 &= 4 \\ x^2 + y^2 &= 2^2 \end{aligned}$$

This is the equation for a cylinder of radius 2 centred on the z-axis.

There are infinitely many θ that satisfy this equation. Two possible θ are $\theta = 45°$ and $\theta = 135°$. Only one of these can be correct. We can see from figure 33.3 that the correct angle must be 135°. Therefore the answer is

$$(\rho, \theta, \phi) = (2, 135°, 135°).$$

Örnek 33.6. Identify the surface described by each of the following spherical polar equations.

(a). $\rho = 5$;

(b). $\phi = 60°$;

(c). $\theta = 120°$;

(d). $\rho \sin \phi = 2$.

çözüm:

(a). ρ is the distance from the origin to a point. If this is always $= 5$, then we have a surface which is always 5 away from the origin. That sounds like a sphere.

To check, we calculate

$$\begin{aligned} \rho &= 5 \\ \sqrt{x^2 + y^2 + z^2} &= 5 \\ x^2 + y^2 + z^2 &= 5^2. \end{aligned}$$

Yes, this is the equation for a sphere or radius 5 centred at the origin.

(b). The angle between the z-axis and the surface is always 60°. Thinking about this, you should be able to understand that this is the equation for a cone.

(c). This is a vertical plane passing through the origin.

(d). We will convert the equation first into cylindrical polar coordinates, then into Cartesian coordinates.

$$\begin{aligned} \rho \sin \phi &= 2 \\ r &= 2 \\ r^2 &= 4 \\ x^2 + y^2 &= 2^2 \end{aligned}$$

This is the equation for a cylinder of radius 2 centred on the z-axis.

Summary

- $r = a$ is a cylinder of radius a centred on the z-axis.
- $\theta = b$ is a vertical plane passing through the origin that makes an angle of b with the positive x-axis.
- $\rho = c$ is a sphere of radius c centred at the origin.
- $\phi = d$ is a cone that makes an angle of d with the positive z-axis.

Problems

Problem 33.1.

(a). Convert the Cartesian coordinates $(x, y, z) = (1, 2, 3)$ into cylindrical polar coordinates.

(b). Convert the Cartesian coordinates $(x, y, z) = (1, 2, 3)$ into spherical polar coordinates.

(c). Convert the Cartesian coordinates $(x, y, z) = (0, -1, 0)$ into cylindrical polar coordinates.

(d). Convert the Cartesian coordinates $(x, y, z) = (0, -1, 0)$ into spherical polar coordinates.

(e). Convert the cylindrical polar coordinates $(r, \theta, z) = (\sqrt{2}, 45°, 1)$ into Cartesian coordinates.

(f). Convert the cylindrical polar coordinates $(r, \theta, z) = (\sqrt{2}, 45°, 1)$ into spherical polar coordinates.

(g). Convert the spherical polar coordinates $(\rho, \theta, \phi) = (10, 60°, 45°)$ into Cartesian coordinates.

(h). Convert the spherical polar coordinates $(\rho, \theta, \phi) = (10, 60°, 45°)$ into cylindrical polar coordinates.

Problem 33.2. Sketch the following surfaces.

(a). $r = 1$,

(b). $r = 2$,

(c). $\rho = 3$,

(d). $\rho = \frac{1}{2}$,

(e). $\theta = 60°$,

(f). $\theta = 135°$,

(g). $\theta = 240°$,

(h). $\phi = 30°$,

(i). $\phi = 135°$.

Summary

- $r = a$ is a cylinder of radius a centred on the z-axis.
- $\theta = b$ is a vertical plane passing through the origin that makes an angle of b with the positive x-axis.
- $\rho = c$ is a sphere of radius c centred at the origin.
- $\phi = d$ is a cone that makes an angle of d with the positive z-axis.

Sorular

Soru 33.1.

(a). Convert the Cartesian coordinates $(x, y, z) = (1, 2, 3)$ into cylindrical polar coordinates.

(b). Convert the Cartesian coordinates $(x, y, z) = (1, 2, 3)$ into spherical polar coordinates.

(c). Convert the Cartesian coordinates $(x, y, z) = (0, -1, 0)$ into cylindrical polar coordinates.

(d). Convert the Cartesian coordinates $(x, y, z) = (0, -1, 0)$ into spherical polar coordinates.

(e). Convert the cylindrical polar coordinates $(r, \theta, z) = (\sqrt{2}, 45°, 1)$ into Cartesian coordinates.

(f). Convert the cylindrical polar coordinates $(r, \theta, z) = (\sqrt{2}, 45°, 1)$ into spherical polar coordinates.

(g). Convert the spherical polar coordinates $(\rho, \theta, \phi) = (10, 60°, 45°)$ into Cartesian coordinates.

(h). Convert the spherical polar coordinates $(\rho, \theta, \phi) = (10, 60°, 45°)$ into cylindrical polar coordinates.

Soru 33.2. Sketch the following surfaces.

(a). $r = 1$,

(b). $r = 2$,

(c). $\rho = 3$,

(d). $\rho = \frac{1}{2}$,

(e). $\theta = 60°$,

(f). $\theta = 135°$,

(g). $\theta = 240°$,

(h). $\phi = 30°$,

(i). $\phi = 135°$.