Your Name / Adınız - Soyadınız

Your Signature / İmza

Student ID # / Öğrenci No

Professor's Name / Öğretim Üyesi

Your Department / Bölüm

- This exam is closed book.
- Give your answers in exact form (for example $\frac{\pi}{3}$ or $5\sqrt{3}$), except as noted in particular problems.
- Calculators, cell phones are not allowed.
- In order to receive credit, you must **show all of your work**. If you do not indicate the way in which you solved a problem, you may get little or no credit for it, even if your answer is correct. **Show your work in evaluating any limits, derivatives**.
- Place a box around your answer to each question.
- If you need more room, use the backs of the pages and indicate that you have done so.
- Do not ask the invigilator anything.
- This exam has 3 pages plus this cover sheet and 6 problems. Please make sure that your exam is complete.
- Time limit is 90 min.

Problem	Points	Score
1	20	
2	15	
3	15	
4	15	
5	15	
6	20	
Total:	100	

Do not write in the table to the right.

1. Suppose $f(x) = \dfrac{x^2}{x+1}$.

   (a) 4 Points All critical points of f, and the intervals where f is increasing and decreasing;

Solution: p.212, pr.83 Differentiating, we have

$$f'(x) = \frac{x^2+2x}{(x+1)^2}$$

and

$$f''(x) = \frac{2}{(x+1)^3}.$$

Solving $f'(x) = 0$ gives us the critical numbers 0 and -2. This splits the real line into 4 open subintervals, namely, $(-\infty,-2)$, $(-2,-1)$, $(-1,0)$, and $(0,\infty)$. By considering test values on each of these intervals, we see that f is increasing on $(-\infty,-2)\cup(0,\infty)$ and decreasing on $(-2,-1)\cup(-2,0)$.

   (b) 4 Points All inflection points of f, and the open intervals where f is concave up resp. concave down.

Solution: Since $f''(x) = \dfrac{2}{(x+1)^3} \neq 0$ and there is no domain point for f where the second derivative is undefined, there are no points of inflection for the graph of$f$. By looking at the sign for f'', we see that the graph is concave up on $(\infty,-1)\cup(1,+\infty)$ and concave down on $(-1,1)$.

   (c) 4 Points Classify the critical points of f as either local maxima, local minima or neither.

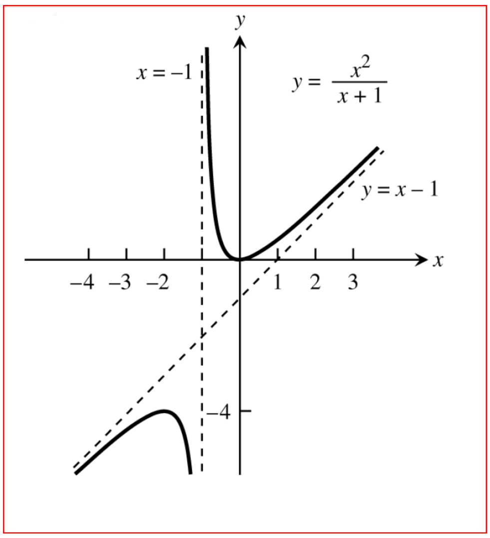
Solution: At $x = 0$, the graph has a local minimum, namely, the point $(0,0)$ is a point of local minimum and at $x = -2$, there is a local maximum and so the point $(-2,-4)$ is the point of local maximum.

   (d) 4 Points Find the asymptotes.

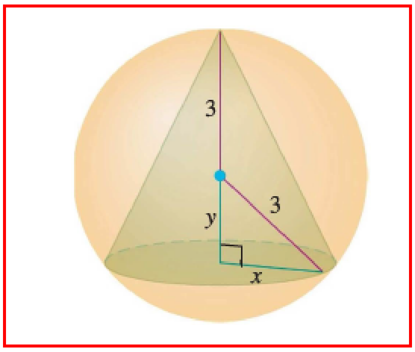
Solution: The vertical line $x = -1$ is a vertical asymptote as $\lim_{x\to -1^{\pm}} \dfrac{x^2}{x+1} = \pm\infty$. Also since by long division $\dfrac{x^2}{x+1} = x-1+\dfrac{1}{x+1}$ and $\lim_{x\to\pm\infty} \dfrac{1}{x+1} = 0$, the line $y = x-1$ is the oblique asymptote.

   (e) 4 Points Sketch the graph of f using your results in (a), (b), (c) and (d).

Solution:



2. [15 Points] Find the volume of the largest right circular cone that can be inscribed in a sphere ofradius 3.



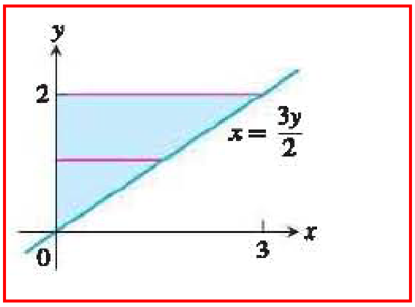
Solution: p.220, pr.12

The volume of the cone is $V = \frac{1}{3}\pi r^2 h$ where $r = x = \sqrt{9-y^2}$ and $h = y+3$ (from the figure). Thus, $V(y) = \frac{\pi}{3}(9-y^2)(y+3) = \frac{\pi}{3}(27+9y-3y^2-y^3) \Rightarrow V'(y) = \frac{\pi}{3}(9-6y-3y^2) = \pi(1-y)(3+y)$. The critical points are -3 and 1, but -3 is not in the domain. Thus $V''(1) = \frac{\pi}{3}(-6-(6)(1)) < 0 \Rightarrow$ at $y=1$ we have a maximum volume of

$$V(1) = \frac{\pi}{3}(8)(4) = \frac{32\pi}{3}$$

cubic units.

3. [15 Points] Find the volume of the solid generated by revolving the shaded region about y-axis.



Solution: p.414, pr.108

$$\begin{aligned} R(y) = x = \frac{3y}{2} \Rightarrow V &= \int_0^2 \pi[R(y)]^2\,dy \\ &= \pi\int_0^2 \left(\frac{3y}{2}\right)^2 dy \\ &= \pi\int_0^2 \left(\frac{9}{4}\right) y^2\,dy \\ &= \pi\left[\frac{3}{4}y^3\right]_0^2 \\ &= \pi\cdot\frac{3}{4}\cdot 8 = 6\pi. \end{aligned}$$

4. [15 Points] Using definite integral for $y = \sqrt{r^2-x^2}$, $0 \le x \le \frac{r}{\sqrt{2}}$ verify that the circumference of the circle of radius r is $2\pi r$.

Solution:

$$\begin{aligned} y &= \sqrt{r^2-x^2} \Rightarrow \frac{dy}{dx} = \frac{-x}{\sqrt{r^2-x^2}} \Rightarrow \left(\frac{dx}{dy}\right)^2 = \frac{1}{2y-1} \Rightarrow L = \int_0^{r/\sqrt{2}} \sqrt{1+\left(\frac{-x}{\sqrt{r^2-x^2}}\right)^2}\,dx \\ &= \int_0^{r/\sqrt{2}} \sqrt{1+\frac{x^2}{r^2-x^2}}\,dx = \int_0^{r/\sqrt{2}} \frac{r}{\sqrt{r^2-x^2}}\,dx = \left[r\sin^{-1}\left(\frac{x}{r}\right)\right]_0^{r/\sqrt{2}} \\ &\longrightarrow L = r\sin^1\left(\frac{r\sqrt{2}}{r}\right) - r\sin^{-1}(0) = r(\frac{\pi}{4}) = \frac{\pi r}{4} \end{aligned}$$

The total circumference of the circle is $C = 8L = 8(\frac{\pi r}{4}) = 2\pi r$.

5. (a) [6 Points] $e^{2x} = \sin(x+3y) \Rightarrow \frac{dy}{dx} = ?$

Solution: We differentiate the equality implicity.

$$\begin{aligned} \frac{d}{dx}e^{2x} &= \frac{d}{dx}\sin(x+3y) \Rightarrow 2e^{2x} = \cos(x+3y)(1+3\frac{dy}{dx}) = \ln\frac{x^2}{9} \\ &\Rightarrow 2e^{2x} = \cos(x+3y) + 3\cos(x+3y)\frac{dy}{dx} \\ \frac{dy}{dx} &= \frac{2e^{2x} - \cos(x+3y)}{3\cos(x+3y)} \end{aligned}$$

(b) 9 Points $\int_0^{x^2} f(t)\,dt = x\cos(\pi x) \Rightarrow f(4) = ?$

Solution: By the Fundamental Theorem of Calculus Part 1, we have

$$\frac{d}{dx}\int_0^{x^2} f(t)\,dt = \frac{d}{dx}\left(x\cos(\pi x)\right).$$

Hence we have

$$f(x^2)2x = \cos(\pi x) - x\pi\sin(\pi x).$$

Now plug $x = 2$. Hence

$$f(2^2)(2)(2) = \cos(2\pi) - 2\pi\sin(2\pi) = 1 - 2\pi(0) \Rightarrow f(4) = \frac{1}{4}$$

6. (a) 10 Points $\int_1^e \frac{\sqrt{\ln x}}{x}\,dx = ?$

Solution: Let $u = \ln x$ and so $du = \frac{1}{x}dx$. When $x = 1$ we have $u = \ln 1 = 0$ and also when $x = e$, we have $u = \ln e = 1$. Hence

$$\int_1^e \frac{\sqrt{\ln x}}{x}\,dx = \int_0^1 \sqrt{u}\,du = \left[\frac{u^{3/2}}{3/2}\right]_0^1 = \frac{2}{3}$$

(b) 10 Points $\int x3^{x^2}\,dx = ?$

Solution: Let $u = x^2$ and so $du = 2x\,dx$. Then we have

$$\int x3^{x^2}\,dx = \frac{1}{2}\int 3^{x^2}2x\,dx = \frac{1}{2}\int 3^u\,du = \frac{1}{2}\frac{3^u}{\ln 3} + C = \frac{1}{2}\frac{3^{x^2}}{\ln 3} + C$$