Your Name / Adınız - Soyadınız

Your Signature / İmza

Student ID # / Öğrenci No

Professor's Name / Öğretim Üyesi

Your Department / Bölüm

- This exam is closed book.
- Give your answers in exact form (for example $\frac{\pi}{3}$ or $5\sqrt{3}$), except as noted in particular problems.
- Calculators, cell phones are not allowed.
- In order to receive credit, you must **show all of your work**. If you do not indicate the way in which you solved a problem, you may get little or no credit for it, even if your answer is correct. **Show your work in evaluating any limits, derivatives.**
- Place a box around your answer to each question.
- If you need more room, use the backs of the pages and indicate that you have done so.
- Do not ask invigilator anything.
- This exam has 3 pages plus this cover sheet and 6 problems. Please make sure that your exam is complete.
- Time limit is 90 min.

Problem	Points	Score
1	20	
2	15	
3	15	
4	15	
5	15	
6	20	
Total:	100	

Do not write in the table to the right.

1. Suppose $f(x) = \dfrac{x^2}{x^2-1}$, $\quad f'(x) = \dfrac{-2x}{(x^2-1)^2}$, $\quad f''(x) = \dfrac{6x^2+2}{(x^2-1)^3}$, p.212, pr.80

(a) (4 Points) All critical points of f, and the intervals where f is increasing and decreasing;

Solution: The critical points are only when $f'(x) = \dfrac{-2x}{(x^2-1)^2} = 0$, that is when $2x = 0$ and so the point is at $x = 0$. This splits the real line into 4 open subintervals, namely, $(-\infty,-1)$, $(-1,0)$, $(0,1)$, and $(1,\infty)$. By considering test values on each of these intervals, we see that f is increasing on $(-\infty,-1)\cup(-1,0)$ and decreasing on $(0,1)\cup(1,\infty)$.

(b) (4 Points) All inflection points of f, and the open intervals where f is concave up resp. concave down.

Solution: Since $f''(x) = \dfrac{6x^2+2}{(x^2-1)^3} \neq 0$ and there is no domain point for f where the second derivative is undefined, there are no points of inflection for the graph of f. By looking at the sign for f'', we see that the graph is concave up on $(\infty,-1)\cup(1,+\infty)$ and concave down on $(-1,1)$.

(c) (4 Points) Classify the critical points of f as either local maxima, local minima or neither.

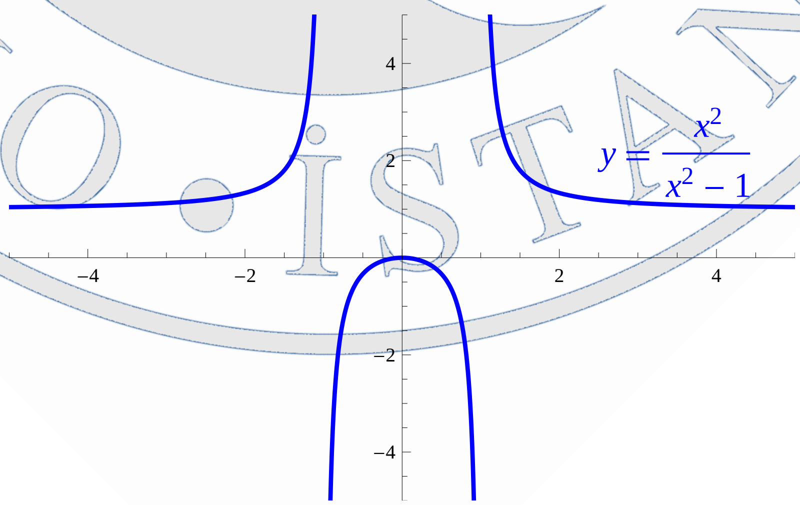
Solution: At $x = 0$, the graph has a local maximum, namely, the point $(0,0)$ is a point of local maximum and there is no point of local minimum.

(d) (4 Points) All asymptotes of f.

Solution: There are three asymptotes. First $x = -1$ and $x = 1$ are vertical asymptotes since $\lim_{x\to\pm1^{\pm}} f(x) = +\infty$. Next $y = 1$ is the horizontal asymptote, since $\dfrac{x^2}{x^2-1} \to 1$ as $x \to \pm\infty$

(e) (4 Points) Sketch the graph of f using your results in (a), (b), (c) and (d).

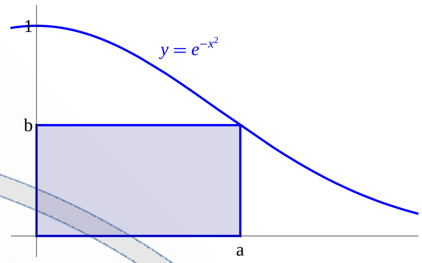
Solution:



2. The rectangle shown here has one side on the positive y-axis, one side on the positive x-axis, and its upper right-hand vertex on the curve $y = e^{-x^2}$. p.432, pr.121

(a) (5 Points) Write the area of the rectangle in terms of a.

Solution: The area is $A = ab = ae^{-a^2}$.



(b) (10 Points) What dimensions give the rectangle its largest area, and what is that area?

Solution: From part (a), we have $A = ae^{-a^2}$. Differentiating with respect to a gives

$$\frac{dA}{da} = e^{-a^2} + (a)(-2a)e^{-a^2} = e^{-a^2}(1-2a^2).$$

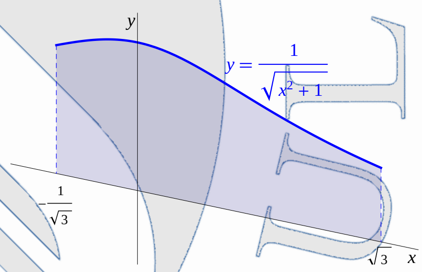
Hence $\frac{dA}{da} < 0$ for $a > \frac{1}{\sqrt{2}}$ and $\frac{dA}{da} > 0$ for $0 < a < \frac{1}{\sqrt{2}}$.

Solving $\dfrac{dA}{da} = 0$ for a, we have absolute maximum of $\frac{1}{\sqrt{2}}e^{-1/2} = \frac{1}{\sqrt{2e}}$ at $a = \frac{1}{\sqrt{2}}$ units long by $b = e^{-1/2} = \frac{1}{\sqrt{e}}$ units high.

3. (15 Points) Use the method of disks to find the volume of the solid generated by revolving, about the line x-axis, the region bounded by : $y = \dfrac{1}{\sqrt{1+x^2}}$, $y = 0$, $\dfrac{-1}{\sqrt{3}} \le x \le \sqrt{3}$. p.415, pr.121

Solution:

$$\begin{aligned}
V &= \pi \int_{-\sqrt{3}/3}^{\sqrt{3}} \left(\frac{1}{\sqrt{1+x^2}}\right)^2 dx \\
&= \pi \int_{-\sqrt{3}/3}^{\sqrt{3}} \frac{1}{1+x^2}\, dx = \pi \left[\tan^{-1} x\right]_{-\sqrt{3}/3}^{\sqrt{3}} \\
&= \pi \left[\tan^{-1}\sqrt{3} - \tan^{-1}\left(-\frac{\sqrt{3}}{3}\right)\right] \\
&= \pi \left[\frac{\pi}{3} - \left(-\frac{\pi}{6}\right)\right] = \frac{\pi^2}{2}
\end{aligned}$$



4. (15 Points) Find the length of the curve given by $x = y^{2/3}$, $1 \le y \le 8$ p.298 pr.39

Solution:

$$\begin{aligned}
x &= y^{2/3} \Rightarrow \frac{dx}{dy} = \frac{2}{3}y^{-1/3} \Rightarrow \left(\frac{dx}{dy}\right)^2 = \frac{4y^{-2/3}}{9} \Rightarrow L = \int_1^8 \sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy \\
L &= \int_1^8 \sqrt{1+\frac{4}{9y^{2/3}}}\, dy = \int_1^8 \frac{\sqrt{9y^{2/3}+4}}{3y^{1/3}}\, dy = \frac{1}{3}\int_1^8 \sqrt{9y^{2/3}+4}\,(y^{-1/3})\, dy \\
;[u &= 9y^{2/3}+4 \Rightarrow du = 6y^{-1/3}\, dy; y = 1 \Rightarrow u = 13, y = 8 \Rightarrow u = 40] \\
&\longrightarrow L = \frac{1}{18}\int_{13}^{40} u^{1/2}\, du = \frac{1}{18}\left[\frac{2}{3}u^{3/2}\right]_{13}^{40} = \frac{1}{27}\left[40^{3/2} - 13^{3/2}\right] \approx 7.634
\end{aligned}$$

5. (a) (6 Points) If $9e^{2y} = x^2$, then solve for y in terms of x. p.431, pr.81

Solution: We isolate y.

$$\begin{aligned} 9e^{2y} &= x^2 \Rightarrow e^{2y} = \frac{x^2}{9} \Rightarrow \ln e^{2y} = \ln\frac{x^2}{9} \Rightarrow 2y\ln e = \ln\frac{x^2}{9} \\ &\Rightarrow \quad y = \frac{1}{2}\ln\frac{x^2}{9} = \ln\sqrt{\frac{x^2}{9}} = \ln\left|\frac{x}{3}\right| \\ y &= \ln|x| - \ln 3 \end{aligned}$$

(b) (9 Points) $\lim_{x\to 0}\frac{5-5\cos x}{e^x - x - 1} = ?$ p.431, pr.101

Solution: This has the indeterminate form $\frac{0}{0}$. Hence the L'Hôpital's Rule applies.

$$\begin{aligned} \lim_{x\to 0}\frac{5-5\cos x}{e^x-x-1} &\overset{\text{L'H}}{=} \lim_{x\to 0}\frac{5\sin x}{e^x-1} \overset{\text{L'H}}{=} \lim_{x\to 0}\frac{5\cos x}{e^x} \\ &= \frac{5\cos(0)}{e^0} = 5 \end{aligned}$$

6. Evaluate the following integrals.

(a) (10 Points) $\int_{-\ln 4}^{-\ln 2} 2e^{\theta}\cosh\theta\, d\theta$ p.431, pr.53

Solution: Since $\cosh\theta = \frac{e^{\theta}+e^{-\theta}}{2}$, we have

$$\begin{aligned} \int_{-\ln 4}^{-\ln 2} 2e^{\theta}\cosh\theta\, d\theta &= \int_{-\ln 4}^{-\ln 2} 2e^{\theta}\left(\frac{e^{\theta}+e^{-\theta}}{2}\right) d\theta = \int_{-\ln 4}^{-\ln 2}\left(e^{2\theta}+1\right) d\theta = \left[\frac{e^{2\theta}}{2}+\theta\right]_{-\ln 4}^{-\ln 2} \\ &= \left(\frac{e^{-2\ln 2}}{2} - \ln 2\right) - \left(\frac{e^{-2\ln 4}}{2} - \ln 4\right) = \left(\frac{1}{8} - \ln 2\right) - \left(\frac{1}{32} - \ln 4\right) \\ &= \frac{3}{32} - \ln 2 + 2\ln 2 = \frac{3}{32} + \ln 2 \end{aligned}$$

(b) (10 Points) $\int \frac{dx}{\sqrt{9-4x^2}}$ p.431, pr.65

Solution: We use the substitution $u = \frac{2x}{3}$ and so $du = \frac{2}{3}dx$.

$$\begin{aligned} \int \frac{dx}{\sqrt{9-4x^2}} &= \frac{1}{3}\int \frac{dx}{\sqrt{1-\left(\frac{2x}{3}\right)^2}} \\ &= \frac{1}{2}\int \frac{du}{\sqrt{1-u^2}}, \text{ where } u = \frac{2x}{3}, du = \frac{2}{3}dx; \\ &= \frac{1}{2}\sin^{-1}\left(\frac{2x}{3}\right) + C \end{aligned}$$