OKAN UNIVERSITY

Department of Mathematics

Math 112
Calculus for Engineering II
Final Exam
SOLUTIONS

May 14, 2012
13:00-15:00

Surname : ________________
Name : ________________
ID # : ________________
Department : ________________
Section : ________________
Instructor : ________________
Signature : ________________

- The exam consists of 5 questions
- Please read the questions carefully and write your answers under the corresponding questions. Be neat.
- Show all your work. Correct answers without sufficient explanation might not get full credit.
- Calculators are not allowed.

GOOD LUCK!

Please do not write below this line.

Q1	Q2	Q3	Q4	Q5	TOTAL
16	16	26	18	24	100

1. If $-1 \leq x \leq 1$, then find the *total area* of the region bounded by $y = x^3$ and the x-axis. **(16pts)**
SOLUTION:
Since

$$x^3 \geq 0 \text{for all} x \in [-1,0] \, \text{and} 0 \geq x^3 \text{for all } x \in [-1,0] \, ,$$

the area we want is

$$\begin{aligned} AREA &= \int_{-1}^{0} -x^3 \; dx + \int_{0}^{1} x^3 \; dx \\ &= \left[-\frac{1}{4}x^4 \right]_{-1}^{0} + \left[\frac{1}{4}x^4 \right]_{0}^{1} \\ &= \frac{1}{4}(0)^4 - \left(-\frac{1}{4}(-1)^4 \right) + \frac{1}{4}(1)^4 - \frac{1}{4}(0)^4 \\ &= \frac{1}{2} units^2 \end{aligned}$$

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2. Evaluate.

$$\int \frac{x+2}{x^3+x}\,dx \quad \textbf{(16pts)}$$

SOLUTION:
From the Partial Fraction Decomposition, we have

$$\begin{aligned} \frac{x+2}{x^3+x} &= \frac{x+2}{x\left(x^2+1\right)} \\ &= \frac{A}{x} + \frac{Bx+C}{x^2+1} \end{aligned}$$

$$\begin{aligned} x+2 &= A\left(x^2+1\right) + (Bx+C)\,x \\ &= Ax^2 + A + Bx^2 + Cx \\ &= (A+B)\,x^2 + Cx + A \\ A &= ;C = 1; B = -2 \end{aligned}$$

Now we can integrate

$$\begin{aligned} \frac{x+2}{x^3+x} &= \frac{2}{x} + \frac{-2x+1}{x^2+1} \\ \int \frac{x+2}{x^3+x}\,dx &= \int \left(\frac{2}{x} + \frac{-2x+1}{x^2+1}\right)\,dx \\ &= \int \left(\frac{2}{x} - \frac{2x}{x^2+1} + \frac{1}{x^2+1}\right)\,dx \\ &= 2\ln|x| - \ln\left(x^2+1\right) + \tan^{-1} x + C. \end{aligned}$$

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3. (a) If it exists, compute the integral $\int_0^2 \frac{x\ dx}{\sqrt{4-x^2}}$.**(13pts)**

(b) Investigate the convergence or divergence of $\int_1^{\infty} \frac{dx}{1+x^3}$.**(13pts)**

SOLUTION:

(a)

$$\begin{aligned}
\int_0^2 \frac{x\ dx}{\sqrt{4-x^2}} &= \lim_{b\to 2^+} \int_0^b \frac{x\ dx}{\sqrt{4-x^2}} \\
&= -\lim_{b\to 2^+} \int_4^{4-b^2} \frac{du}{2\sqrt{u}} \quad (\text{where} u = 4-x^2 \text{and} du = -2x\ dx) \\
&= -\lim_{b\to 2^+} \left[\sqrt{4-b^2} - \sqrt{4}\right] = 2
\end{aligned}$$

Therefore, the improper integral **converges** and has value 2.

(b)

First, notice that

$$1+x^3 > x^3, \quad \text{for all } x \geq 1.$$

We then have that

$$0 < \frac{1}{1+x^3} < \frac{1}{x^3}, \quad \text{for all } x \geq 1.$$

Further, since

$$\begin{aligned}
\int_1^{\infty} \frac{1}{x^3}\ dx &= \lim_{b\to\infty} \int_1^b \frac{1}{x^3}\ dx \\
&= \lim_{b\to\infty} \left[\frac{x^{-3+1}}{-3+1}\right]_1^b = -\frac{1}{2}\lim_{b\to\infty}\left(\frac{1}{b^2}-1\right) = \frac{1}{2},
\end{aligned}$$

the integral $\int_1^{\infty} \frac{1}{x^3}\ dx$ **converges**.

Hence by the **Direct Comparison Test**, the integral $\int_1^{\infty} \frac{dx}{1+x^3}$ **converges**.

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4. Suppose $a_n = (-1)^n \dfrac{n+2}{2n-1}$, for $n \geq 1$.(p.541, pr.28)

(a) Find the values of a_1, a_2, a_3, and a_4.**(8pts)**

(b) Find $\lim_{n\to\infty} a_n$, if it exists. **(10pts)**.

SOLUTION:

(a)

We have

$$\begin{aligned} a_1 &= (-1)^1 \frac{1+2}{1} = -3 \\ a_2 &= (-1)^2 \frac{2+2}{2(2)-1} = \frac{4}{3} \\ a_3 &= (-1)^3 \frac{3+2}{2(3)-1} = -1 \\ a_4 &= (-1)^4 \frac{4+2}{2(4)-1} = \frac{6}{7}. \end{aligned}$$

(b)

First, notice that

$$\begin{aligned} \lim_{n\to\infty} \frac{n+2}{2n-1} &= \lim_{n\to\infty} \frac{1+\frac{2}{n}}{2-\frac{1}{n}} \\ &= \frac{1+0}{2-0} \\ &= \frac{1}{2} \neq 0, \end{aligned}$$

the even-numbered terms of a_n approach $\frac{1}{2}$ whereas the odd-numbered terms approach $-\frac{1}{2}$. Hence the limit of the sequence does not exist.

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5. (a) Investigate the convergence or divergence of

$$\textbf{(a)} \sum_{n=1}^{\infty} \frac{n+1}{n!}, \ \textbf{(12pts)}$$

Be sure to state the *test* you are using.

(b) Find the sum of the series

$$\sum_{n=0}^{\infty} \left(\frac{5}{2^n} - \frac{2}{3^{n-1}} \right), \ \textbf{(12pts)}$$

SOLUTION:
(a)
The series **converges by the Ratio Test**, since

$$\begin{aligned} \rho &= \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)+1}{(n+1)!} \Big/ \frac{n+1}{n!} \\ &= \lim_{n\to\infty} \frac{(n+2)}{(n+1)\,n!} \frac{n!}{(n+1)} = \lim_{n\to\infty} \frac{(n+2)}{(n+1)^2} = 0 < 1. \end{aligned}$$

(b)
The series is the difference of two **convergent geometric series**, so its sum is

$$\begin{aligned} \sum_{n=0}^{\infty} \left(\frac{5}{2^n} - \frac{2}{3^{n-2}} \right) &= \sum_{n=0}^{\infty} \frac{5}{2^n} - \sum_{n=0}^{\infty} \frac{18}{3^n} \\ &= \frac{5}{1-\frac{1}{2}} - \frac{18}{1-\frac{1}{3}} \\ &= 10 - 27 = -17 \end{aligned}$$
