Your Name / Ad - Soyad ( **75 min.** ) Signature / İmza

Student ID # / Öğrenci No ( **use a blue pen!** )

| Problem | 1 | 2 | 3 | 4 | Total |
|---|---|---|---|---|---|
| Points: | 32 | 23 | 22 | 23 | 100 |
| Score: | | | | | |

Time limit is **75 minutes**. If you need more room on your exam paper, you may use the empty spaces on the paper. You have to show your. Answers without reasonable-even if your results are true- work will either get zero or very little credit.

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1. (a) (8 Points) Find the derivative of $y = \left(\dfrac{3t-4}{5t+2}\right)^{-5}$.

**Solution:** We have

$$\ln y = \ln\left(\frac{3t-4}{5t+2}\right)^{-5} = -5\ln\frac{3t-4}{5t+2} = -5\left(\ln(3t-4) - \ln(5t+2)\right)$$

$$\frac{1}{y}\frac{dy}{dt} = -5\frac{3}{3t-4} + 5\frac{5}{5t+2} \Rightarrow \frac{dy}{dt} = y\left(\frac{-15}{3t-4} + \frac{25}{5t+2}\right)$$

$$\boxed{\frac{dy}{dt} = \left(\frac{3t-4}{5t+2}\right)^{-5}\left(\frac{-15}{3t-4} + \frac{25}{5t+2}\right)}$$

(b) (8 Points) Evaluate the integral $\displaystyle\int_1^4 \frac{\ln 2 \log_2 x}{x}\,\mathrm{d}x$.

**Solution:** We have

$$\int_1^4 \frac{\ln 2\log_2 x}{x}\,\mathrm{d}x = \int_1^4 \frac{\ln 2}{x}\frac{\ln x}{\ln 2}\,\mathrm{d}x = \int_1^4 \frac{\ln x}{x}\,\mathrm{d}x = \left[\frac{1}{2}(\ln x)^2\right]_1^4$$

$$= \frac{1}{2}\left[(\ln 4)^2 - (\ln 1)^2\right] = \boxed{\frac{1}{2}(\ln 4)^2}$$

(c) (8 Points) For $y = x^{\sin x}$, use logarithmic differentiation to find the derivative $\dfrac{dy}{dx}$.

**Solution:** We have

$$y = x^{\sin x} \Rightarrow \ln y = \ln x^{\sin x} = (\sin x)\ln x$$

$$\frac{y'}{y} = (\cos x)\ln x + (\sin x)\frac{1}{x} = \frac{\sin x + x(\ln x)(\cos x)}{x}$$

$$\boxed{y' = x^{\sin x}\left[\frac{\sin x + x(\ln x)(\cos x)}{x}\right]}$$

p.73, pr.27

(d) (8 Points) Use L'Hôpital's Rule to evaluate the limit $\displaystyle\lim_{\theta\to 0}\frac{\theta - \sin\theta\cos\theta}{\tan\theta - \theta}$.

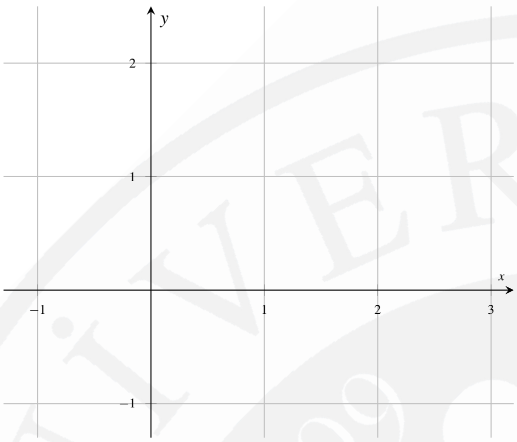
**Solution:** From L'Hôpital's Rule, we have

$$\lim_{\theta\to 0}\frac{\theta-\sin\theta\cos\theta}{\tan\theta-\theta} = \lim_{\theta\to 0}\frac{1+\sin^2\theta-\cos^2\theta}{\sec^2\theta - 1} = \lim_{\theta\to 0}\frac{2\sin^2\theta}{\tan^2\theta} = \lim_{\theta\to 0} 2\cos^2\theta = \boxed{2}$$

p.73, pr.27

2. (a) i. (5 Points) Sketch the graph of

$$f(x) = \begin{cases} x & 0 \le x < 1 \\ 2-x & 1 \le x \le 2 \end{cases}$$




ii. (5 Points) Is the function $f$ continuous at $x = 1$? Give reasons.

**Solution:** Since

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (2-x) = 1$$

and

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (x) = 1$$

exist and are both equal to $f(1) = 1$, the function is continuous at $x = 1$.

iii. (5 Points) Is the function $f$ differentiable at $x = 1$? Give reasons.

**Solution:** Since

$$\begin{aligned} f'_+(1) &= \lim_{h\to 0^+} \frac{f(1+h)-f(1)}{h} \\ &= \lim_{h\to 0^+} \frac{2-(1+h)-(2-1)}{h} \\ &= \lim_{h\to 0^+} \frac{-h}{h} = -1 \end{aligned}$$

and

$$\begin{aligned} f'_-(1) &= \lim_{h\to 0^-} \frac{f(1+h)-f(1)}{h} \\ &= \lim_{h\to 0^-} \frac{(1+h)-(2-1)}{h} \\ &= \lim_{h\to 0^-} \frac{h}{h} = 1 \end{aligned}$$

do not agree, $f$ is not differentiable at $x = 1$.

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(b) (8 Points) Find the length of the curve $y = x^{3/2}$ from $x = 0$ to $x = 4$.

**Solution:**

$$\frac{dy}{dx} = \frac{3}{2}\sqrt{x} \Rightarrow L = \int_0^4 \sqrt{1+\frac{9}{4}x\,\mathrm{d}x};$$

$$\left[u = 1+\frac{9}{4}x \Rightarrow \mathrm{d}u = \frac{9}{4}\mathrm{d}x \Rightarrow \frac{4}{9}\mathrm{d}u = \mathrm{d}x;\;\; x=0 \Rightarrow u=1 \;\; x=4 \Rightarrow u=10\right]$$

$$\to L = \int_1^{10} u^{1/2}\frac{4}{9}\mathrm{d}u = \frac{4}{9}\left[\frac{2}{3}u^{3/2}\right]_1^{10} = \boxed{\frac{8}{27}(10\sqrt{10}-1)}$$

3. (a) (10 Points) Suppose the derivative of the the function $y = f(x)$ is

$$y' = (x-1)^2(x-2)(x-4).$$

At what points, if any, does the graph of $f$ have a local minimum, local maximum, or point of inflection?

**Solution:** When $y' = (x-1)^2(x-2)(x-4)$, then $y'' = 2(x-1)(x-2)(x-4) + (x-1)^2(x-4) + (x-1)^2(x-2)$ $y'' = (x-1)[2(x^2-6x+8) + (x^2-5x+4) + (x^2-3x+2)] = 2(x-1)(2x^2-10x+11)$. The curve rises on $(-\infty, 2)$ and $(4, \infty)$ and falls on $(2,4)$. At $x = 2$ there is a local maximum and at $x = 4$ a local minimum. The curve is concave downward on $(-\infty, 1)$ and $\left(\frac{5-\sqrt{3}}{2}, \frac{5+\sqrt{3}}{2}\right)$ and concave upward on $\left(1, \frac{5-\sqrt{3}}{2}\right)$ and $\left(\frac{5+\sqrt{3}}{2}, \infty\right)$. At $x = 1$, $\frac{5-\sqrt{3}}{2}$ and $\frac{5-\sqrt{3}}{2}$, there are inflection points.

(b) (12 Points) Find the inverse of $f(x) = \dfrac{x+3}{x-2}$. Find the domain and the range for $f^{-1}$. Then verify the equality $f(f^{-1}(x)) = x$.

**Solution:** Let $y = \dfrac{x+3}{x-2}$. Then

$y(x-2) = x+3 \Rightarrow xy - 2y = x+3 \Rightarrow xy - x = 2y+3 \Rightarrow \boxed{x = \dfrac{2y+3}{y-1}}$

Now $y = \dfrac{2x+3}{x-1} = f^{-1}(x)$;

Domain of $f^{-1}: (-\infty, 1) \cup (1, \infty)$  Range of $f^{-1}: (-\infty, 2) \cup (2, \infty)$

$$f(f^{-1}(x)) = \frac{\left(\frac{2x+3}{x-1}\right) + 3}{\left(\frac{2x+3}{x-1}\right) - 2} = \frac{2(x+3) + 3(x-2)}{(x+3) - (x-2)} = \frac{5x}{5} = x$$

4. (a) (10 Points) Find the area bounded by the curves $y = x^2 - 2$ and $y = 2$.

**Solution:** We shall find this area by integrating with respect to $x$ since this is easier than $y$. Therefore we have

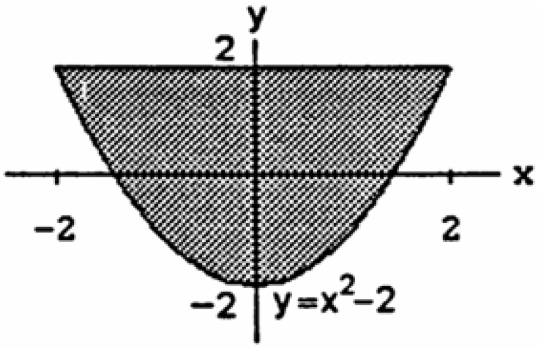
$$A = \int_{-2}^{2} \Bigg( \underbrace{2}_{\text{upper curve}} - \underbrace{(x^2-2)}_{\text{lower curve}} \Bigg) \, dx = \int_{-2}^{2} (4 - x^2)\, dx$$

$$= \left[ 4x - \frac{x^3}{3} \right]_{-2}^{2}$$

$$= 8 - \frac{8}{3} - \left( -8 + \frac{8}{3} \right) = 16 - \frac{16}{3}$$

$$= \boxed{\frac{32}{3}}$$

p.212, pr.85




(b) (13 Points) Use the shell method to find the volume of the solid generated by revolving the region bounded by $x = 3 - y^2$, $y = \sqrt{3}$ and $x = 3$ about the $x$-axis.

**Solution:** For the sketch given, $c = 0$, $d = \sqrt{3}$,

$$V = \int_c^d 2\pi \begin{pmatrix} \text{shell} \\ \text{radius} \end{pmatrix} \begin{pmatrix} \text{shell} \\ \text{height} \end{pmatrix} dy = \int_0^{\sqrt{3}} 2\pi y \cdot \left[ 3 - (3 - y^2) \right] dy = \int_0^{\sqrt{3}} 2\pi y^3 \, dy$$

$$= 2\pi \left[ \frac{y^4}{4} \right]_0^{\sqrt{3}} = \boxed{\frac{9\pi}{2}}$$

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