Exercise 35 (Non-Homogeneous Systems of Equations)**.**
Use the Method of Undetermined Coefficients to solve the following systems of ODEs:

(a) $\mathbf{x}' = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix} \mathbf{x} + \begin{bmatrix} e^t \\ t \end{bmatrix}$ (b) $\mathbf{x}' = \begin{bmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} e^t \\ \sqrt{3}e^{-t} \end{bmatrix}$

Use the Method of Diagonalisation (use the substitution $\mathbf{x} = T\mathbf{y}$) to solve the following systems of ODEs:

(c) $\mathbf{x}' = \begin{bmatrix} 1 & 1 \\ 4 & -2 \end{bmatrix} \mathbf{x} + \begin{bmatrix} e^{-2t} \\ -2e^t \end{bmatrix}$ (d) $\mathbf{x}' = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \mathbf{x} + \begin{bmatrix} -\cos t \\ \sin t \end{bmatrix}$

Use the Method of Variation of Parameters ($\mathbf{x}(t) = \Psi(t) \int \Psi^{-1}(s)\mathbf{g}(s)\,ds$) to solve the following systems of ODEs:

(e) $\mathbf{x}' = \begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} t^{-1} \\ 2t^{-1} + 4 \end{bmatrix}, \quad t > 0$ (f) $\mathbf{x}' = \begin{bmatrix} 4 & -2 \\ 8 & -4 \end{bmatrix} \mathbf{x} + \begin{bmatrix} t^{-3} \\ -t^{-2} \end{bmatrix}, \quad t > 0$

Solution 35.

(a) Note that our ODE can be written as $\mathbf{x}' = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^t + \begin{bmatrix} 0 \\ 1 \end{bmatrix} t$.

The eigenvalues of $A = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix}$ are $r_1 = 1$ and $r_2 = -1$. The corresponding eigenvectors are $\xi^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\xi^{(2)} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$. Therefore the general solution of the homogeneous equation $\mathbf{x}' = A\mathbf{x}$ is

$$\mathbf{x}(t) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^t + c_2 \begin{bmatrix} 1 \\ 3 \end{bmatrix} e^{-t}.$$

Next we need to find a particular solution of $\mathbf{x}' = A\mathbf{x} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^t$. Since $r = 1$ is an eigenvalue of A, we try the ansatz $\mathbf{x} = \mathbf{a}te^t + \mathbf{b}e^t$ where $\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \in \mathbb{R}^2$. Then we calculate that

$$\mathbf{a}e^t + \mathbf{a}te^t + \mathbf{b}e^t = \mathbf{x}' = A\mathbf{x} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^t = A\mathbf{a}te^t + A\mathbf{b}e^t + \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^t$$

$$\mathbf{a} + \mathbf{a}t + \mathbf{b} = A\mathbf{a}t + A\mathbf{b} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Since this must be true for all t, we must have

$$\begin{cases} \mathbf{a} = A\mathbf{a} \\ A\mathbf{b} - \mathbf{b} = \mathbf{a} - \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \end{cases}$$

The former equation tells us that $\mathbf{a}$ must be an eigenvector of A corresponding to $r = 1$. So $\mathbf{a} = \begin{bmatrix} \alpha \\ \alpha \end{bmatrix}$ for some $\alpha \in \mathbb{R}$ that we need to find. Then the latter equation becomes

$$\begin{bmatrix} b_1 - b_2 \\ 3b_1 - 3b_2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 3 & -3 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = (A - I)\mathbf{b} = A\mathbf{b} - \mathbf{b} = \mathbf{a} - \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \alpha - 1 \\ \alpha \end{bmatrix}.$$

Thus

$$3(\alpha - 1) = 3(b_1 - b_2) = 3b_1 - 3b_2 = \alpha \implies 2\alpha = 3 \implies \alpha = \frac{3}{2}.$$

Then we have $b_1 - b_2 = \frac{1}{2}$ which implies that $\mathbf{b} = \begin{bmatrix} k \\ k - \frac{1}{2} \end{bmatrix}$ for any k. I choose $k = 0$. Therefore

$$\mathbf{x}(t) = \begin{bmatrix} \frac{3}{2} \\ \frac{3}{2} \end{bmatrix} te^t - \begin{bmatrix} 0 \\ \frac{1}{2} \end{bmatrix} e^t.$$

Finally we must find a particular solution of $\mathbf{x}' = A\mathbf{x} + \begin{bmatrix}0\\1\end{bmatrix}t$. Here we try the ansatz $\mathbf{x} = \mathbf{c}t + \mathbf{d}$ where $\mathbf{c} = \begin{bmatrix}c_1\\c_2\end{bmatrix}, \mathbf{d} = \begin{bmatrix}d_1\\d_2\end{bmatrix} \in \mathbb{R}^2$. Then we calculate that

$$\mathbf{c} = \mathbf{x}' = A\mathbf{x} + \begin{bmatrix}0\\1\end{bmatrix}t = A\mathbf{c}t + A\mathbf{d} + \begin{bmatrix}0\\1\end{bmatrix}t.$$

Since this must be true for all t, we must have $A\mathbf{c} + \begin{bmatrix}0\\1\end{bmatrix} = \mathbf{0}$ and $A\mathbf{d} = \mathbf{c}$. Using the former equation, we calculate that

$$\begin{bmatrix}0\\-1\end{bmatrix} = A\mathbf{c} = \begin{bmatrix}2 & -1\\3 & -2\end{bmatrix}\begin{bmatrix}c_1\\c_2\end{bmatrix} = \begin{bmatrix}2c_1 - c_2\\3c_1 - 2c_2\end{bmatrix} \implies c_1 = 1,\ c_2 = 2 \implies \mathbf{c} = \begin{bmatrix}1\\2\end{bmatrix}.$$

(Or equivalently, we could calculate that $\mathbf{c} = A^{-1}\begin{bmatrix}0\\-1\end{bmatrix} = \begin{bmatrix}1\\2\end{bmatrix}$.) Then the latter equation gives us

$$\begin{bmatrix}1\\2\end{bmatrix} = A\mathbf{d} = \begin{bmatrix}2 & -1\\3 & -2\end{bmatrix}\begin{bmatrix}d_1\\d_2\end{bmatrix} = \begin{bmatrix}2d_1 - d_2\\3d_1 - 2d_2\end{bmatrix} \implies d_1 = 0,\ d_2 = -1 \implies \mathbf{d} = \begin{bmatrix}0\\-1\end{bmatrix}.$$

Hence

$$\mathbf{x} = \begin{bmatrix}1\\2\end{bmatrix}t + \begin{bmatrix}0\\-1\end{bmatrix}.$$

Adding these three solutions together, we obtain the general solution of the problem:

$$\mathbf{x}(t) = c_1\begin{bmatrix}1\\1\end{bmatrix}e^t + c_2\begin{bmatrix}1\\3\end{bmatrix}e^{-t} + \frac{3}{2}\begin{bmatrix}1\\1\end{bmatrix}te^t - \frac{1}{2}\begin{bmatrix}0\\1\end{bmatrix}e^t + \begin{bmatrix}1\\2\end{bmatrix}t - \begin{bmatrix}0\\1\end{bmatrix}$$

(b) Using the same method as in (a), we find that $\mathbf{x}(t) = c_1\begin{bmatrix}\sqrt{3}\\1\end{bmatrix}e^{2t} + c_2\begin{bmatrix}1\\-\sqrt{3}\end{bmatrix}e^{-2t} - \begin{bmatrix}\frac{2}{3}\\\frac{1}{\sqrt{3}}\end{bmatrix}e^t + \begin{bmatrix}-1\\\frac{2}{\sqrt{3}}\end{bmatrix}e^{-t}$.

(c) The eigenvalues of $A = \begin{bmatrix}1 & 1\\4 & -2\end{bmatrix}$ are $r_1 = -3$ and $r_2 = 2$; and the eigenvectors are $\xi^{(1)} = \begin{bmatrix}1\\-4\end{bmatrix}$ and $\xi^{(2)} = \begin{bmatrix}1\\1\end{bmatrix}$. Let $T = \begin{bmatrix}1 & 1\\-4 & 1\end{bmatrix}$. Then $T^{-1} = \frac{1}{5}\begin{bmatrix}1 & -1\\4 & 1\end{bmatrix}$.

Using the substitution $\mathbf{x} = T\mathbf{y}$ we convert $\mathbf{x}' = A\mathbf{x} + \mathbf{g}$ into two first order linear ODEs as follows:

$$\begin{aligned}\mathbf{x}' &= A\mathbf{x} + \mathbf{g}\\ T\mathbf{y}' &= AT\mathbf{y} + \mathbf{g}\\ \mathbf{y}' &= T^{-1}AT\mathbf{y} + T^{-1}\mathbf{g}\\ \mathbf{y}' &= \begin{bmatrix}-3 & 0\\0 & 2\end{bmatrix}\mathbf{y} + \frac{1}{5}\begin{bmatrix}1 & -1\\4 & 1\end{bmatrix}\begin{bmatrix}e^{-2t}\\-2e^t\end{bmatrix} = \begin{bmatrix}-3 & 0\\0 & 2\end{bmatrix}\mathbf{y} + \frac{1}{5}\begin{bmatrix}e^{-2t} + 2e^t\\4e^{-2t} - 2e^t\end{bmatrix}\end{aligned}$$

$$\begin{cases}y_1' = -3y_1 + \frac{1}{5}\left(e^{-2t} + 2e^t\right)\\ y_2' = 2y_2 + \frac{1}{5}\left(4e^{-2t} - 2e^t\right)\end{cases}$$

Using the integrating factors $\mu_1(t) = e^{3t}$ and $\mu_2(t) = e^{-2t}$ respectively, we can solve these two first order linear ODEs to obtain

$$\mathbf{y}(t) = \begin{bmatrix}y_1(t)\\y_2(t)\end{bmatrix} = \begin{bmatrix}\frac{1}{5}e^{-2t} + \frac{1}{10}e^t + c_1e^{-3t}\\-\frac{1}{5}e^{-2t} + \frac{2}{5}e^t + c_2e^{2t}\end{bmatrix}.$$

Finally multiplying by T gives

$$\begin{aligned}\mathbf{x}(t) &= T\mathbf{y}\\ &= \begin{bmatrix}1 & 1\\-4 & 1\end{bmatrix}\begin{bmatrix}\frac{1}{5}e^{-2t} + \frac{1}{10}e^t + c_1e^{-3t}\\-\frac{1}{5}e^{-2t} + \frac{2}{5}e^t + c_2e^{2t}\end{bmatrix}\\ &= c_1\begin{bmatrix}1\\-4\end{bmatrix}e^{-3t} + c_2\begin{bmatrix}1\\1\end{bmatrix}e^{2t} - \begin{bmatrix}0\\1\end{bmatrix}e^{-2t} + \frac{1}{2}\begin{bmatrix}1\\0\end{bmatrix}e^t.\end{aligned}$$

(d) $\mathbf{x}(t) = c_1\begin{bmatrix}1\\0\end{bmatrix}e^t + c_2\begin{bmatrix}1\\1\end{bmatrix}e^{2t} + \frac{1}{5}\begin{bmatrix}4\cos t - 2\sin t\\-\cos t - 2\sin t\end{bmatrix}$.

(e) The eigenvalues of $A = \begin{bmatrix}-4 & 2\\2 & -1\end{bmatrix}$ are $r_1 = 0$ and $r_2 = -5$; and the eigenvectors are $\xi^{(1)} = \begin{bmatrix}1\\2\end{bmatrix}$ and $\xi^{(2)} = \begin{bmatrix}-2\\1\end{bmatrix}$. Thus

$$\Psi(t) = \begin{bmatrix}1 & -2e^{-5t}\\2 & e^{-5t}\end{bmatrix}$$

is a fundamental matrix for $\mathbf{x}' = A\mathbf{x}$. Using the formula $\begin{bmatrix}a & b\\c & d\end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix}d & -b\\-c & a\end{bmatrix}$ we calculate that

$$\Psi^{-1}(t) = \frac{1}{e^{-5t} + 4e^{-5t}}\begin{bmatrix}e^{-5t} & 2e^{-5t}\\-2 & 1\end{bmatrix} = \frac{1}{5}\begin{bmatrix}1 & 2\\-2e^{5t} & e^{5t}\end{bmatrix}.$$

Then

$$\Psi^{-1}(t)\mathbf{g}(t) = \frac{1}{5}\begin{bmatrix} 1 & 2 \\ -2e^{5t} & e^{5t} \end{bmatrix}\begin{bmatrix} t^{-1} \\ 2t^{-1}+4 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} t^{-1}+4t^{-1}+8 \\ -2t^{-1}e^{5t}+2t^{-1}e^{5t}+4e^{5t} \end{bmatrix} = \begin{bmatrix} t^{-1}+\frac{8}{5} \\ \frac{4}{5}e^{5t} \end{bmatrix}$$

and

$$\int \Psi^{-1}(t)\mathbf{g}(t)\,dt = \int \begin{bmatrix} t^{-1}+\frac{8}{5} \\ \frac{4}{5}e^{5t} \end{bmatrix} dt = \begin{bmatrix} \ln t + \frac{8}{5}t + c_1 \\ \frac{4}{25}e^{5t} + c_2 \end{bmatrix}.$$

It follows that

$$\mathbf{x}(t) = \Psi(t)\int \Psi^{-1}(s)\mathbf{g}(s)\,ds = \begin{bmatrix} 1 & -2e^{-5t} \\ 2 & e^{-5t} \end{bmatrix}\begin{bmatrix} \ln t + \frac{8}{5}t + c_1 \\ \frac{4}{25}e^{5t} + c_2 \end{bmatrix} = \begin{bmatrix} \ln t + \frac{8}{5}t - \frac{8}{25} + c_1 - 2c_2e^{-5t} \\ 2\ln t + \frac{16}{5}t + \frac{4}{25} + 2c_1 + c_2e^{-5t} \end{bmatrix}$$

$$= c_1\begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2\begin{bmatrix} -2 \\ 1 \end{bmatrix}e^{-5t} + \begin{bmatrix} 1 \\ 2 \end{bmatrix}\ln t + \frac{8}{5}\begin{bmatrix} 1 \\ 2 \end{bmatrix}t + \frac{4}{25}\begin{bmatrix} -2 \\ 1 \end{bmatrix}.$$

(f) Using the Method of Variation of Parameters, we can find that $\mathbf{x}(t) = c_1\begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}t - \frac{1}{2}\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) - 2\begin{bmatrix} 1 \\ 2 \end{bmatrix}\ln t + \begin{bmatrix} 2 \\ 5 \end{bmatrix}t^{-1} - \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix}t^{-2}$.

Exercise 36 (The Laplace Transform). Use the Laplace Transform to solve the following IVPs:

(a) $\begin{cases} x' = x - 2y \\ y' = 5x - y \\ x(0) = -1 \\ y(0) = 2 \end{cases}$ (b) $\begin{cases} x' = -x + y \\ y' = 2x \\ x(0) = 0 \\ y(0) = 1 \end{cases}$ (c) $\begin{cases} 2x' + y' - 2x = 1 \\ x' + y' - 3x - 3y = 2 \\ x(0) = 0 \\ y(0) = 0 \end{cases}$ (d) $\begin{cases} 2x' + y' - y - t = 0 \\ x' + y' - t^2 = 0 \\ x(0) = 1 \\ y(0) = 0 \end{cases}$

[Hint: For (c) and (d), you must first rearrange the ODEs to the form $\begin{cases} x' = f_1(x,y) \\ y' = f_2(x,y) \end{cases}$.]

Solution 36.

(a) The equations above can be written as

$$\mathbf{x}' = \mathbf{A}\mathbf{x} = \begin{bmatrix} 1 & -2 \\ 5 & -1 \end{bmatrix}\mathbf{x}, \text{ where } \mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}.$$

If we take the Laplace transform of the both sides of the above equation, we get

$$\begin{aligned} (s\mathbf{I} - \mathbf{A})\mathbf{X}(s) &= \mathbf{x}(0) \Longrightarrow \begin{bmatrix} s-1 & 2 \\ -5 & s+1 \end{bmatrix}\mathbf{X}(s) = \begin{bmatrix} -1 \\ 2 \end{bmatrix} \Longrightarrow \\ \mathbf{X}(s) &= \frac{1}{s^2+9}\begin{bmatrix} s+1 & -2 \\ 5 & s-1 \end{bmatrix}\begin{bmatrix} -1 \\ 2 \end{bmatrix} = \frac{1}{s^2+9}\begin{bmatrix} -(s+5) \\ 2s-7 \end{bmatrix}. \end{aligned}$$

Note that

$$\begin{aligned} \frac{-s-5}{s^2+9} &= -\frac{s}{s^2+9} - \frac{5}{3}\frac{3}{s^2+9} \Longrightarrow \mathcal{L}^{-1}\left(\frac{-s-5}{s^2+9}\right) = -\cos 3t - \frac{5}{3}\sin 3t. \\ \frac{2s-7}{s^2+9} &= 2\frac{s}{s^2+9} - \frac{7}{3}\frac{3}{s^2+9} \Longrightarrow \mathcal{L}^{-1}\left(\frac{2s-7}{s^2+9}\right) = 2\cos 3t - \frac{7}{3}\sin 3t. \end{aligned}$$

Then, the solution of the initial value problem is

$$\mathbf{x}(t) = \begin{bmatrix} -\cos 3t - \frac{5}{3}\sin 3t \\ 2\cos 3t - \frac{7}{3}\sin 3t \end{bmatrix}.$$

(b) The equations above can be written as

$$\mathbf{x}' = \mathbf{A}\mathbf{x} = \begin{bmatrix} -1 & 1 \\ 2 & 0 \end{bmatrix}\mathbf{x}, \text{ where } \mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}.$$

If we take the Laplace transform of the both sides of the above equation, we get

$$\begin{aligned} (s\mathbf{I} - \mathbf{A})\mathbf{X}(s) &= \mathbf{x}(0) \Longrightarrow \begin{bmatrix} s+1 & -1 \\ -2 & s \end{bmatrix}\mathbf{X}(s) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Longrightarrow \\ \mathbf{X}(s) &= \frac{1}{(s^2+s-2)}\begin{bmatrix} s & 1 \\ 2 & s+1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{(s^2+s-2)}\begin{bmatrix} 1 \\ s+1 \end{bmatrix}. \end{aligned}$$

Note that

$$\begin{aligned} \frac{1}{(s^2+s-2)} &= \frac{1}{3}\left(\frac{1}{s-1} - \frac{1}{s+2}\right) \Longrightarrow \mathcal{L}^{-1}\left(\frac{1}{(s^2+s-2)}\right) = \frac{1}{3}\left(e^t - e^{-2t}\right). \\ \frac{s+1}{(s^2+s-2)} &= \frac{2}{3}\frac{1}{s-1} + \frac{1}{3}\frac{1}{s+2} \Longrightarrow \mathcal{L}^{-1}\left(\frac{s+1}{(s^2+s-2)}\right) = \frac{1}{3}\left(2e^t + e^{-2t}\right). \end{aligned}$$

Then, the solution of the initial value problem is

$$\mathbf{x}(t) = \frac{1}{3}\begin{bmatrix} e^t - e^{-2t} \\ 2e^t + e^{-2t} \end{bmatrix}.$$

(c) The equations above can be written as

$$\begin{aligned} x' + x + 3y &= -1, \\ y' - 4x - 6y &= 3. \end{aligned}$$

This implies that

$$\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{h} = \begin{bmatrix} -1 & -3 \\ 4 & 6 \end{bmatrix} \mathbf{x} + \begin{bmatrix} -1 \\ 3 \end{bmatrix}, \text{ where } \mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}.$$

If we take the Laplace transfo rm of the both sides of the above equation, we get

$$\begin{aligned} (s\mathbf{I} - \mathbf{A}) \mathbf{X}(s) &= \mathbf{x}(0) + \mathbf{H}(\mathbf{s}) \Longrightarrow \begin{bmatrix} s+1 & 3 \\ -4 & s-6 \end{bmatrix} \mathbf{X}(s) = \frac{1}{s} \begin{bmatrix} -1 \\ 3 \end{bmatrix} \Longrightarrow \\ \mathbf{X}(s) &= \frac{1}{(s^2 - 5s + 6)} \begin{bmatrix} s-6 & -3 \\ 4 & s+1 \end{bmatrix} \frac{1}{s} \begin{bmatrix} -1 \\ 3 \end{bmatrix} = \frac{1}{s(s^2 - 5s + 6)} \begin{bmatrix} -s-3 \\ 3s-1 \end{bmatrix}. \end{aligned}$$

Note that

$$\begin{aligned} \frac{-s-3}{s(s^2-5s+6)} &= \frac{-1}{2}\frac{1}{s} + \frac{5}{2}\frac{1}{s-2} - 2\frac{1}{s-3} \Longrightarrow \mathcal{L}^{-1}\left(\frac{-s-3}{s(s^2-5s+6)}\right) = \frac{-1}{2} + \frac{5}{2}e^{2t} - 2e^{3t}. \\ \frac{3s-1}{s(s^2-5s+6)} &= \frac{-1}{6}\frac{1}{s} - \frac{5}{2}\frac{1}{s-2} + \frac{8}{3}\frac{1}{s-3} \Longrightarrow \mathcal{L}^{-1}\left(\frac{3s-1}{s(s^2-5s+6)}\right) = \frac{-1}{6} - \frac{5}{2}e^{2t} + \frac{8}{3}e^{3t}. \end{aligned}$$

Then, the solution of the initial value problem is

$$\mathbf{x}(t) = \frac{1}{6} \begin{bmatrix} 15e^{2t} - 12e^{3t} - 3 \\ -15e^{2t} + 16e^{3t} - 1 \end{bmatrix}.$$

(d) The equations above can be written as

$$\begin{aligned} x' - y + t^2 - t &= 0, \\ y' + y + t - 2t^2 &= 0. \end{aligned}$$

This implies that

$$\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{h} = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} t - t^2 \\ 2t^2 - t \end{bmatrix}, \text{ where } \mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}.$$

If we take the Laplace transform of the both sides of the above equation, we get

$$\begin{aligned} (s\mathbf{I} - \mathbf{A}) \mathbf{X}(s) &= \mathbf{x}(0) + \mathbf{H}(\mathbf{s}) \Longrightarrow \begin{bmatrix} s & -1 \\ 0 & s+1 \end{bmatrix} \mathbf{X}(s) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{1}{s^2} - \frac{2}{s^3} \\ \frac{4}{s^3} - \frac{1}{s^2} \end{bmatrix} \Longrightarrow \\ \mathbf{X}(s) &= \frac{1}{s(s+1)} \begin{bmatrix} s+1 & 1 \\ 0 & s \end{bmatrix} \frac{1}{s^3} \begin{bmatrix} s^3 + s - 2 \\ 4 - s \end{bmatrix} \\ &= \frac{1}{s^4(s+1)} \begin{bmatrix} s^4 + s^3 + s^2 - 2s + 2 \\ 4s - s^2 \end{bmatrix}. \end{aligned}$$

Note that

$$\begin{aligned} \frac{s^4 + s^3 + s^2 - 2s + 2}{s^4(s+1)} &= \frac{5}{s+1} - 4\frac{1}{s} + 5\frac{1}{s^2} - 4\frac{1}{s^3} + 2\frac{1}{s^4} \Longrightarrow \\ \mathcal{L}^{-1}\left(\frac{s^4 + s^3 + s^2 - 2s + 2}{s^4(s+1)}\right) &= 5e^{-t} - 4 + 5t - 2t^2 + \frac{1}{3}t^3. \\ \frac{4s - s^2}{s^4(s+1)} &= -5\frac{1}{s+1} + 5\frac{1}{s} - 5\frac{1}{s^2} + 4\frac{1}{s^3} \Longrightarrow \\ \mathcal{L}^{-1}\left(\frac{4s - s^2}{s^4(s+1)}\right) &= -5e^{-t} + 5 - 5t + 2t^2. \end{aligned}$$

Then, the solution of the initial value problem is

$$\mathbf{x}(t) = \begin{bmatrix} 5e^{-t} - 4 + 5t - 2t^2 + \frac{1}{3}t^3 \\ -5e^{-t} + 5 - 5t + 2t^2 \end{bmatrix}.$$