Exercise 21 (The Method of Undetermined Coefficients). Find the general solutions of the following ODEs:

(a) $y'' - 2y' - 3y = 3e^{2t}$

(b) $y'' + 2y' + 5y = 3\cos 2t$

(c) $y'' - 2y' - 3y = 2 - 3te^{-t}$

(d) $y'' + 2y' = 3 + 4\sin 2t$

(e) $y'' + 9y = t^2e^{3t} + 6$

(f) $y'' + 2y' + y = 2e^{-t}$

(g) $2y'' + 3y' + y = t^3 + 3\sin t$

(h) $y'' + y = 3\sin 2t + t\cos 2t$

(i) $y'' + y' + 4y = 2\sinh t$

Solution 21.

(a) First we must consider the homogeneous equation
$$y'' - 2y' - 3y = 0.$$
The characteristic equation is
$$0 = r^2 - 2r - 3 = (r-3)(r+1)$$
which implies that $r_1 = 3$ and $r_2 = -1$. Hence the general solution of the homogeneous equation is
$$y = c_1e^{3t} + c_2e^{-t}.$$
Next we must find a particular solution to our ODE. Since e^{2t} does not solve the homogeneous equation, our ODE does not have resonance. Thus we try the ansatz $Y(t) = Ae^{2t}$ for some constant A. Then we calculate that $Y' = 2Ae^{2t}$, that $Y'' = 4Ae^{2t}$ and that
$$\begin{aligned} 3e^{2t} &= Y'' - 2Y' - 3Y \\ &= 4Ae^{2t} - 2(2Ae^{2t}) - 3(Ae^{2t}) = -3Ae^{2t}. \end{aligned}$$
Thus we must have $A = -1$. Therefore the general solution to our ODE is
$$y = c_1e^{3t} + c_2e^{-t} - e^{2t}.$$

(b) $y = c_1e^{-t}\cos 2t + c_2e^{-t}\sin 2t + \frac{12}{17}\sin 2t + \frac{3}{17}\cos 2t$

(c) $y = c_1e^{3t} + c_2e^{-t} + \frac{1}{192}(72t^2 + 36t + 9 - 128e^t)e^{-t}$

(d) $y = c_1 + c_2e^{-2t} + \frac{3}{2}t - \frac{1}{2}\sin 2t - \frac{1}{2}\cos 2t$

(e) $y = c_1\cos 3t + c_2\sin 3t + \frac{1}{162}(9t^2 - 6t + 1)e^{3t} + \frac{2}{3}$

(f) The homogeneous equation $y'' + 2y' + y = 2e^{-t}$ has characteristic equation
$$0 = r^2 + 2r + r = (r+1)^2$$
and general solution $y = c_1e^{-t} + c_2te^{-t}$.

Next we need to find a particular solution to our ODE. Our equation has resonance since both e^{-t} and te^{-t} solve the homogeneous equation. Hence we must multiply by t again and consider the ansatz $Y(t) = At^2e^{-t}$ for some constant A. Then we calculate that $Y' = 2Ate^{-t} - At^2e^{-t}$, that $Y'' = 2Ae^{-t} - 4Ate^{-t} + At^2e^{-t}$ and that
$$\begin{aligned} 2e^{-t} &= Y'' + 2Y' + Y \\ &= e^{-t}\big((2A - 4At + At^2) + 2(2At - At^2) + (At^2)\big) \\ &= 2Ae^{-t}. \end{aligned}$$
Therefore the general solution to our ODE is
$$y = c_1e^{-t} + c_2te^{-t} + t^2e^{-t}.$$

(g) $y = c_1e^{-t} + c_2e^{-\frac{t}{2}} + t^3 - 9t^2 + 47t - 90 - \frac{3}{10}\sin t - \frac{9}{10}\cos t$

(h) $y = c_1\cos t + c_2\sin t - \frac{1}{3}t\cos 2t - \frac{5}{9}\sin 2t$

(i) First we consider the homogeneous equation
$$y'' + y' + 4y = 0.$$
Its characteristic equation, $r^2 + r + 4 = 0$, has roots
$$r_{1,2} = \frac{-1 \pm \sqrt{1^2 - 16}}{2} = -\frac{1}{2} \pm \frac{\sqrt{15}}{2}i.$$
Hence $\lambda = -\frac{1}{2}$ and $\mu = \frac{\sqrt{15}}{2}$. Therefore this homogeneous ODE has general solution
$$y = c_1e^{-\frac{t}{2}}\cos\frac{\sqrt{15}t}{2} + c_2e^{-\frac{t}{2}}\sin\frac{\sqrt{15}t}{2}.$$
Now recall that $\sinh t = \frac{1}{2}(e^t - e^{-t})$. Thus we try the ansatz $Y(t) = Ae^t + Be^{-t}$ for constants A and B. We calculate that $Y' = Ae^t - Be^{-t}$ and $Y'' = Y$. Therefore
$$\begin{aligned} e^t - e^{-t} = 2\sinh t &= Y'' + Y' + 4Y \\ &= (Ae^t + Be^{-t}) + (Ae^t - Be^{-t}) + 4(Ae^t + Be^{-t}) \\ &= 6Ae^t + 4Be^{-t} \end{aligned}$$
which implies that $A = \frac{1}{6}$ and $B = -\frac{1}{4}$. Therefore the general solution to the ODE is
$$y = c_1e^{-\frac{t}{2}}\cos\frac{\sqrt{15}t}{2} + c_2e^{-\frac{t}{2}}\sin\frac{\sqrt{15}t}{2} + \frac{1}{6}e^t - \frac{1}{4}e^{-t}.$$

Exercise 22 (The Method of Undetermined Coefficients). Solve the following IVPs:

(a) $\begin{cases} y'' + y' - 2y = 2t \\ y(0) = 0 \\ y'(0) = 1 \end{cases}$

(b) $\begin{cases} y'' - 2y' + y = te^t + 4 \\ y(0) = 1 \\ y'(0) = 1 \end{cases}$

(c) $\begin{cases} y'' + 4y = t^2 + 3e^t \\ y(0) = 0 \\ y'(0) = 2 \end{cases}$

(d) $\begin{cases} -y'' + 6y' - 16y = 1 + 6e^{3t}\sin(2t) \\ y(0) = \frac{15}{16} \\ y'(0) = -1 \end{cases}$

Solution 22.

(a) $y = e^t - \frac{1}{2}e^{-2t} - t - \frac{1}{2}$

(b) $y = 4te^t - 3e^t + \frac{1}{6}t^3e^t + 4$

(c) $y = \frac{7}{10}\sin 2t - \frac{19}{40}\cos 2t + \frac{1}{4}t^2 - \frac{1}{8} + \frac{3}{5}e^t$

(d) First consider the homogeneous equation
$$-y'' + 6y' - 16y = 0.$$
The characteristic equation is
$$-r^2 + 6r - 16 = 0$$
which has roots
$$r = 3 \pm i\sqrt{7}.$$
Therefore the general solution of
$$-y'' + 6y' - 16y = 0$$
is
$$y(t) = c_1e^{3t}\sin(\sqrt{7}t) + c_2e^{3t}\cos(\sqrt{7}t).$$

Next consider
$$-y'' + 6y' - 16y = 1.$$
Trying the ansatz $Y(t) = C$, we see that
$$1 = -Y'' + 6Y' - 16Y = -16C.$$
We must choose $C = -\frac{1}{16}$. Hence $Y(t) = -\frac{1}{16}$.

Now consider
$$-y'' + 6y' - 16y = 6e^{3t}\sin(2t).$$
We try the ansatz
$$Y(t) = Ae^{3t}\cos 2t + Be^{3t}\sin 2t$$
and find that
$$\begin{aligned} 6e^{3t}\sin 2t = -Y'' + 6Y' - 16Y \\ &= -e^{3t}\Big((5A+12B)\cos 2t + (5B-12A)\sin 2t\Big) \\ &\quad + 6e^{2t}\big((3A+2B)\cos 2t + (3B-2A)\sin 2t\big) \\ &\quad - 16e^{3t}\big(A\cos 2t + B\sin 2t\big) \\ &= e^{3t}\cos 2t\big(-5A - 12B + 16A + 12B - 16A\big) \\ &\quad + e^{3t}\sin 2t\big(-5B + 12A + 18B - 12A - 16B\big) \\ &= e^{3t}\cos 2t\big(-5A\big) + e^{3t}\sin 2t\big(-3B\big). \end{aligned}$$
Thus, we need $A = 0$ and $B = -2$. Hence
$$Y(t) = -2e^{3t}\sin 2t.$$

Next we add these 3 solutions together. Therefore, the general solution of the ODE is
$$y(t) = c_1e^{3t}\sin(\sqrt{7}t) + c_2e^{3t}\cos(\sqrt{7}t) - 2e^{3t}\sin(2t) - \frac{1}{16}.$$
The final step is to satisfy the initial conditions. We calculate that
$$\frac{15}{16} = y(0) = 0 + c_2 - 0 - \frac{1}{16} \qquad \implies \qquad c_2 = 1.$$
and
$$\begin{aligned} -1 &= y'(0) \\ &= 3c_1e^{3t}\sin(\sqrt{7}t) + \sqrt{7}c_1e^{3t}\cos(\sqrt{7}t) + 3e^{3t}\cos(\sqrt{7}t) \\ &\quad - \sqrt{7}e^{3t}\sin(\sqrt{7}t) - 6e^{3t}\sin(2t) - 4e^{3t}\cos(2t)\big|_{t=0} \\ &= 0 + \sqrt{7}c_1 + 3 - 0 - 0 - 4 \qquad \implies \qquad c_1 = 0. \end{aligned}$$
Therefore, the solution of the IVP is
$$y(t) = e^{3t}\cos(\sqrt{7}t) - 2e^{3t}\sin(2t) - \frac{1}{16}.$$

Exercise 23 (The Method of Variation of Parameters). Find the general solutions of the following ODEs:

(a) $y'' + y = \tan t, \quad 0 < t < \frac{\pi}{2}$

(b) $y'' + 4y = 3 \operatorname{cosec} 2t, \quad 0 < t < \frac{\pi}{2}$

(c) $y'' + 4y' + 4y = t^{-2}e^{-2t}, \quad t > 0$

(d) $y'' - 2y' + y = \dfrac{e^t}{1+t^2}$

Solution 23.

(a) Note first that $y_1(t) = \cos t$ and $y_2(t) = \sin t$ form a fundamental set of solutions of the homogeneous equation $y'' + y = 0$. The Wronskian of y_1 and y_2 is
$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{vmatrix} = \cos^2 t + \sin^2 t = 1.$

Using the theorem from class, we calculate that

$$\begin{aligned}
Y(t) &= -y_1 \int \frac{y_2 g}{W} + y_2 \int \frac{y_1 g}{W} \\
&= -\cos t \int \sin t \tan t \, dt + \sin t \int \cos t \tan t \, dt \\
&= -\cos t \int \frac{\sin^2 t}{\cos t} dt + \sin t \int \sin t \, dt \\
&= -\cos t \int \frac{1 - \cos^2 t}{\cos t} dt + \sin t \int \sin t \, dt \\
&= \cos t \int \cos t - \sec t \, dt + \sin t \int \sin t \, dt \\
&= \cos t(\sin t - \ln(\sec t + \tan t)) + \sin t(-\cos t) \\
&= -(\cos t) \ln(\sec t + \tan t)
\end{aligned}$$

is a particular solution to the non-homogeneous ODE.

Therefore the general solution of the ODE is
$y(t) = c_1 \cos t + c_2 \sin t - (\cos t) \ln(\tan t + \sec t).$

(b) $y = c_1 \cos 2t + c_2 \sin 2t + \frac{3}{4}(\sin 2t) \ln \sin 2t - \frac{3}{2} t \cos 2t$

(c) $y = c_1 e^{-2t} + c_2 t e^{-2t} - e^{-2t} \ln t$

(d) $y = c_1 e^t + c_2 t e^t - \frac{1}{2} e^t \ln(1 + t^2) + t e^t \tan^{-1} t$

Exercise 24 (Going Backwards). Find linear, homogeneous ODEs with constant coefficients, which have general solutions equal to the functions given below. The first one is done for you.

(ω) $y(t) = c_1 e^t + c_2 e^{2t} + c_3 e^{3t}$.

Clearly $r_1 = 1$, $r_2 = 2$ and $r_3 = 3$. We need to give an ODE which has characteristic equation $0 = (r - r_1)(r - r_2)(r - r_3) = (r-1)(r-2)(r-3) = r^3 - 6r^2 + 11r - 6$. One possible answer is $y''' - 6y'' + 11y' - 6y = 0$.

(a) $y(t) = c_1 + c_2 t + c_3 e^{3t} \sin t + c_4 e^{3t} \cos t + c_5 e^{3t} \sin 2t + c_6 e^{3t} \cos 2t$

(b) $y(t) = c_1 e^t + c_2 t e^t + c_3 e^{2t} \sin t + c_4 e^{2t} \cos t + c_5 e^{2t} t \sin t + c_6 e^{2t} t \cos t$

(c) $y(t) = c_1 e^{2t} + c_2 t e^{2t} + c_3 t^2 e^{2t} + c_4 e^{-t} \sin 3t + c_5 e^{-t} \cos 3t$

Solution 24.

(a) $\dfrac{d^6y}{dt^6} - 12\dfrac{d^5y}{dt^5} + 59\dfrac{dy^4}{dt^4} - 138\dfrac{d^3y}{dt^3} + 130\dfrac{d^2y}{dt^2} = 0$

(b) The first two terms correspond to a double root $r = 1$. The last four terms correspond to a double complex root $r = 2 \pm i$. Consequently, the characteristic equation is

$$0 = (r-1)^2(r^2 - 4r + 5)^2 = r^6 - 10r^5 + 43r^4 - 100r^3 + 131r^2 - 90r + 25$$

Then, the differential equation is

$$\frac{d^6y}{dt^6} - 10\frac{d^5y}{dt^5} + 43\frac{d^4y}{dt^4} - 100\frac{d^3y}{dt^3} + 131\frac{d^2y}{dt^2} - 90\frac{dy}{dt} + 25y = 0.$$

(c) $\dfrac{d^5y}{dt} - 6\dfrac{d^4y}{dt^4} + 10\dfrac{d^3y}{dt^3} - 44\dfrac{d^2y}{dt^2} + 104\dfrac{dy}{dt} - 80y = 0.$

Exercise 25 (Higher Order Linear ODEs)**.**

(a) Given that $\sin t$ is a solution of $y^{(4)} + 2y''' + 6y'' + 2y' + 5y = 0$, find the general solution of this ODE.

(b) Find the general solution of $y^{(4)} + y'' = 3x^2 + 4\sin x - 2\cos x$.

(c) Solve $\begin{cases} \frac{d^3y}{dx^3} - 2\frac{d^2y}{dx^2} + 4\frac{dy}{dx} - 8y = 0 \\ y(0) = 2 \\ y'(0) = 0 \\ y''(0) = 0. \end{cases}$

Solution 25. Thank you to Prof. Eldem for the following solutions.

(a) Note that the characteristic equation is $r^4 + 2r^3 + 6r^2 + 2r + 5 = 0$. Since $\sin t$ is a solution, two roots are $\pm i$. Thus the characteristic equation has $(r^2 + 1)$ as a factor. Dividing the characteristic equation by $(r^2 + 1)$, we get

$$r^2 + 2r + 5 = 0.$$

Thus the other roots are $-1 \pm 2i$. Consequently, the general solution is

$$y(t) = c_1 \cos t + c_2 \sin t + c_3 e^{-t} \cos 2t + c_4 e^{-t} \sin 2t.$$

(b) The characteristic equation is $0 = r^4 + r^2 = r^2(r^2 + 1)$ and its roots are 0, 0, $\pm i$. Consequently, the general solution of the homogeneous equation is

$$y(x) = c_1 + c_2 x + c_3 \cos x + c_4 \sin x.$$

There is resonance for all the terms on the right hand side of the equation. For the first term on the right, we try the ansatz $y_{p1} = x^2(a + bx + cx^2)$ because the degree of the zero root is two. For the second term, we try the ansatz $y_{p2} = x(d\cos x + f\sin x)$ because the multiplicity of the imaginary root is one. Thus, we have

$$\begin{aligned} y'_{p1} &= 2ax + 3bx^2 + 4cx^3 \\ y''_{p1} &= 2a + 6bx + 12cx^2 \\ y'''_{p1} &= 6b + 24cx \\ y^{(4)}_{p1} &= 24c. \end{aligned}$$

Using these expressions in the equation, we get

$$24c + 2a + 6bx + 12cx^2 = 3x^2.$$

This implies that $24c + 2a = 0$, $b = 0$ and $c = \frac{1}{4}$. Thus $a = -3$. Consequently, we have $y_{p1} = \frac{1}{4}x^4 - 3x^2$. For the second term, we get

$$\begin{aligned} y'_{p2} &= d(\cos x - x\sin x) + f(\sin x + x\cos x) \\ y''_{p2} &= d(-2\sin x - x\cos x) + f(2\cos x - x\sin x) \\ y'''_{p2} &= d(-3\cos x + x\sin x) + f(-3\sin x - x\cos x) \\ y^{(4)}_{p2} &= d(4\sin x + x\cos x) + f(-4\cos x + x\sin x) \end{aligned}$$

Using these expressions in the equation, we get

$$d(4\sin x + x\cos x) + f(-4\cos x + x\sin x) + d(-2\sin x - x\cos x) + f(2\cos x - x\sin x) = 4\sin x - 2\cos x.$$

This implies that $d = 2$ and $f = 1$. Hence

$$y_{p2} = 2x\cos x + x\sin x.$$

Therefore, the general solution is

$$\begin{aligned} y(x) &= c_1 + c_2 x + c_3 \cos x + c_4 \sin x + y_{p1} + y_{p2} \\ &= c_1 + c_2 x + c_3 \cos x + c_4 \sin x + \frac{1}{4}x^4 - 3x^2 + 2x\cos x + x\sin x \\ &= c_1 + c_2 x - 3x^2 + \frac{1}{4}x^4 + (c_3 + 2x)\cos x + (c_4 + x)\sin x. \end{aligned}$$

(c) The characteristic equation is

$$0 = r^3 - 2r^2 + 4r - 8 = (r^2 + 4)(r - 2)$$

and its roots are 2 and $\pm 2i$. The general solution of the ODE is

$$y(x) = c_1 e^{2x} + c_2 \cos 2x + c_3 \sin 2x.$$

Since $y(0) = 2$, we get $c_1 + c_2 = 2$. Since $y'(x) = 2c_1 e^{2x} - 2c_2 \sin 2x + 2c_3 \cos 2x$ and $y'(0) = 0$, it follows that $2c_1 + 2c_3 = 0$. Furthermore, since $y''(x) = 4c_1 e^{2x} - 4c_2 \cos 2x - 4c_3 \sin 2x$ and $y''(0) = 0$, we also have $4c_1 - 4c_2 = 0$. Thus $c_1 = c_2 = 1$ and $c_3 = -1$. Therefore the solution of the initial value problem is

$$y(x) = e^{2x} + \cos 2x - \sin 2x.$$