Exercise 26 (The Laplace Transform). Find the Laplace Transform of the following functions:

(a) $f(t) = e^{-2t}$

(b) $f(t) = 3t^2$

(c) $f(t) = \cos^2 2t$

(d) $f(t) = t\cos t + te^t$

(e) $f(t) = \frac{\sinh t}{t}$

(f) $f(t) = t^2 \cos 2t$

(g) $f(t) = \frac{e^{3t}-1}{t}$

(h) $f(t) = te^{-t}\sin^2 t$

(i) $f(t) = \begin{cases} 2 & 0 < t \le 3 \\ 0 & t > 3 \end{cases}$

(j) $f(t) = \begin{cases} \sin 2t & \pi \le t \le 2\pi \\ 0 & t < \pi \text{ or } t > 2\pi \end{cases}$

Solution 26.

(a) Note that

$$\mathcal{L}\left[e^{-2t}\right](s) = \int_0^\infty e^{-st}e^{-2t}\,dt = \int_0^\infty e^{-t(s+2)}\,dt = \frac{1}{s+2}$$

(b) Note that

$$\mathcal{L}\left[t^2\right] = (-1)^2\frac{d^2}{ds^2}\left(\mathcal{L}[1]\right) = (-1)^2\frac{d^2}{ds^2}\left(\frac{1}{s}\right) = (-1)^2\frac{d}{ds}\left(-\frac{1}{s^2}\right) = \frac{2}{s^3}.$$

Consequently, we get

$$\mathcal{L}\left[3t^2\right] = 3\mathcal{L}\left[t^3\right] = \frac{6}{s^3}.$$

(c) Note that $\cos^2 2t = \frac{1}{2}(1+\cos 4t)$ which implies that

$$\mathcal{L}\left[\cos^2 2t\right] = \frac{1}{2}\mathcal{L}\left[1+\cos 4t\right] = \frac{1}{2}\left(\frac{1}{s} + \frac{s}{s^2+16}\right).$$

(d) Since multiplication by t is equivalent to taking derivative with respect to s and multiplying by -1, it follows that

$$\begin{aligned}\mathcal{L}\left[t\cos t + te^t\right] &= (-1)\frac{d}{ds}\mathcal{L}\left[\cos t + e^t\right] = (-1)\frac{d}{ds}\left(\frac{s}{s^2+1} + \frac{1}{s-1}\right) \\ &= (-1)\left(\frac{1-s^2}{s^2+1} - \frac{1}{(s-1)^2}\right) = \frac{s^2-1}{s^2+1} + \frac{1}{(s-1)^2}.\end{aligned}$$

(e) Note first that since

$$(-1)\frac{d}{ds}\mathcal{L}\left[\frac{\sinh t}{t}\right] = \mathcal{L}\left[t\frac{\sinh t}{t}\right] = \mathcal{L}\left[\sinh t\right] = \frac{1}{s^2-1}$$

we have that

$$\mathcal{L}\left[\frac{\sinh t}{t}\right] = -\int \frac{ds}{s^2-1} = -\frac{1}{2}\int\left(\frac{1}{s-1} - \frac{1}{s+1}\right)ds = \frac{1}{2}\ln\left(\frac{s+1}{s-1}\right).$$

(f) We calculate that

$$\begin{aligned}\mathcal{L}\left[t^2\cos 2t\right] &= (-1)^2\frac{d^2}{ds^2}\mathcal{L}\left[\cos 2t\right] = (-1)^2\frac{d^2}{ds^2}\left(\frac{s}{s^2+4}\right) = \frac{d}{ds}\left(\frac{s^2+4-2s^2}{(s^2+4)^2}\right) = \frac{d}{ds}\left(\frac{4-s^2}{(s^2+4)^2}\right) \\ &= \frac{-2s\left(s^2+4\right)^2 - \left(4-s^2\right)4s\left(s^2+4\right)}{\left(s^2+4\right)^4} = \frac{-2s\left(s^2+4\right) - \left(4-s^2\right)4s}{\left(s^2+4\right)^3} \\ &= \frac{2s^3-24s}{\left(s^2+4\right)^3}.\end{aligned}$$

(g) Note first that

$$(-1)\frac{d}{ds}\mathcal{L}\left[\frac{e^{3t}-1}{t}\right] = \mathcal{L}\left(t\frac{e^{3t}-1}{t}\right) = \mathcal{L}\left[e^{3t}-1\right] = \frac{1}{s-3} - \frac{1}{s}.$$

It follows that

$$\mathcal{L}\left[\frac{e^{3t}-1}{t}\right] = -\int\left(\frac{1}{s-3} - \frac{1}{s}\right)ds = \ln\left(\frac{s}{s-3}\right).$$

(h) Recall that $\sin^2 t = \frac{1}{2}(1-\cos 2t)$. This implies that

$$\begin{aligned}
\mathcal{L}\left[te^{-t}\sin^2 t\right] &= \mathcal{L}\left[te^{-t}\frac{1}{2}(1-\cos 2t)\right] = \frac{1}{2}\mathcal{L}\left(te^{-t}-te^{-t}\cos 2t\right) = \frac{1}{2}(-1)\frac{d}{ds}\left(\frac{1}{s+1}-\frac{(s+1)}{4+(s+1)^2}\right) \\
&= \frac{1}{2}(-1)\left(-\frac{1}{(s+1)^2}-\frac{(s+1)^2+4-(s+1)2(s+1)}{\left((s+1)^2+4\right)^2}\right) = \frac{1}{2}\left(\frac{1}{(s+1)^2}+\frac{4-(s+1)^2}{\left((s+1)^2+4\right)^2}\right) \\
&= \frac{1}{2}\left(\frac{(s+1)^4+12(s+1)^2+16-(s+1)^4}{(s+1)^2\left((s+1)^2+4\right)^2}\right) = \left(\frac{6(s+1)^2+8}{(s+1)^2\left((s+1)^2+4\right)^2}\right) \\
&= \frac{\left(6s^2+12s+14\right)}{(s+1)^2\left(s^2+2s+5\right)^2}.
\end{aligned}$$

(i) We calculate that

$$\mathcal{L}[f(t)] = \int_0^\infty e^{-st}f(t)\ dt = \int_0^3 2e^{-st}\ dt = \left[\frac{-2e^{-st}}{s}\right]_0^3 = \frac{2-2e^{-3s}}{s}.$$

(j) Let $\tau = t-\pi$ and $\varphi = t-2\pi$. Then, we get

$$\begin{aligned}
\mathcal{L}(f(t)) &= \int_0^\infty e^{-st}f(t)\,dt \\
&= \int_\pi^{2\pi} e^{-st}f(t)\,dt \\
&= \int_\pi^\infty e^{-st}\sin 2t\ dt - \int_{2\pi}^\infty \sin 2te^{-st}\,dt \\
&= \int_0^\infty e^{-s(\tau+\pi)}\sin(2\tau+2\pi)\ d\tau - \int_0^\infty \sin(2\varphi+4\pi)\,e^{-s(\varphi+2\pi)}\,d\varphi \\
&= \frac{2e^{-\pi s}}{s^2+4}-\frac{2e^{-2\pi s}}{s^2+4} \\
&= \frac{2\left(e^{-\pi s}-e^{-2\pi s}\right)}{s^2+4}.
\end{aligned}$$

Exercise 27 (The Inverse Laplace Transform). Find the inverse Laplace Transform of the following functions:

(a) $F(s) = \frac{1}{s-2}$

(b) $F(s) = \frac{1}{s} - \frac{2}{s^{5/2}}$

(c) $F(s) = \frac{3s+1}{s^2+4}$

(d) $F(s) = \frac{2e^{-3s}}{s}$

(e) $F(s) = \frac{1}{s(s-3)}$

(f) $F(s) = \frac{2s+1}{s(s^2+9)}$

(g) $F(s) = \frac{s^3}{(s-4)^4}$

(h) $F(s) = \frac{s^2-2s}{s^4+5s^2+4}$

(i) $F(s) = \frac{2s^3-s^2}{(4s^2-4s+5)^2}$

(j) $F(s) = \ln\left(1+\frac{1}{s^2}\right)$

(k) $F(s) = \arctan\left(\frac{3}{s+2}\right)$

(l) $F(s) = \frac{s}{(s^2+1)^3}$

(m) $F(s) = \frac{e^{-s}}{s+2}$

Solution 27.

(a) $f(t) = e^{2t}$

(b) $f(t) = 1 - \frac{8}{3\sqrt{\pi}}t^{\frac{3}{2}}$

(c) $f(t) = 3\cos 2t + \frac{1}{2}\sin 2t$

(d) $f(t) = 2u_3(t)$

(e) $f(t) = \frac{1}{3}\left(e^{3t}-1\right)$

(f) $f(t) = \frac{1}{9}\left(6\sin 3t - \cos 3t + 1\right)$

(g) $f(t) = e^{4t}\left(1 + 12t + 24t^2 + \frac{32}{3}t^3\right)$

(h) $f(t) = \frac{1}{3}\left(2\cos 2t + 2\sin 2t - 2\cos t - \sin t\right)$

(i) $f(t) = \frac{1}{64}e^{\frac{t}{2}}\left[(4t+8)\cos t + (4-3t)\sin t\right]$

(j) Note that since $\mathcal{L}\left[tf(t)\right] = (-1)\frac{dF}{ds}$, we have that $\mathcal{L}^{-1}\left[\frac{dF}{ds}\right] = (-1)tf(t)$. Therefore

$$\begin{aligned}
\frac{dF}{ds} &= \frac{\frac{-2}{s^3}}{\left(1+\frac{1}{s^2}\right)} = -\frac{2}{s\left(s^2+1\right)} \\
\mathcal{L}^{-1}\left[\frac{dF}{ds}\right] &= -\mathcal{L}^{-1}\left[\frac{2}{s\left(s^2+1\right)}\right] = (-1)tf(t) \\
\mathcal{L}^{-1}\left[\frac{2}{s\left(s^2+1\right)}\right] &= \mathcal{L}^{-1}\left[\frac{2}{s} - \frac{2s}{s^2+1}\right] = tf(t) \\
2 - 2\cos t &= tf(t) \\
f(t) &= \frac{2\left(1-\cos t\right)}{t}
\end{aligned}$$

(k) Here, we again use the formula $\mathcal{L}^{-1}\left[\frac{dF}{ds}\right] = (-1)tf(t)$ to calculate that

$$\begin{aligned}
\frac{dF}{ds} &= \frac{\frac{-3}{(s+2)^2}}{1+\left(\frac{3}{s+2}\right)^2} \\
&= -\frac{3}{s^2+4s+13} = -\frac{3}{(s+2)^2+9} \\
\mathcal{L}^{-1}\left[\frac{dF}{ds}\right] &= -\mathcal{L}^{-1}\left(\frac{3}{(s+2)^2+9}\right) = (-1)tf(t) \\
e^{-2t}\sin 3t &= tf(t) \\
f(t) &= \frac{e^{-2t}\sin 3t}{t}.
\end{aligned}$$

(l) $f(t) = \frac{1}{8}\left(t\sin t - t^2\cos t\right)$

(m)

$$\begin{aligned}
f(t) = \mathcal{L}^{-1}\left(\frac{e^{-s}}{s+2}\right) &= \begin{cases} e^{-2(t-1)} & t \geq 1 \\ 0 & t < 1 \end{cases} \\
&= u_1(t)e^{-2(t-1)}
\end{aligned}$$

where u is the unit step function.

$f(t)$	$F(s) = \mathcal{L}[f](s)$	
1	$\frac{1}{s}$	$s > 0$
e^{at}	$\frac{1}{s-a}$	$s > a$
$t^n \quad (n \in \mathbb{N})$	$\frac{n!}{s^{n+1}}$	$s > 0$
$\sin at$	$\frac{a}{s^2+a^2}$	$s > 0$
$\cos at$	$\frac{s}{s^2+a^2}$	$s > 0$
$\sinh at$	$\frac{a}{s^2-a^2}$	$s > \lvert a \rvert$
$\cosh at$	$\frac{s}{s^2-a^2}$	$s > \lvert a \rvert$
$e^{at} \sin bt$	$\frac{b}{(s-a)^2+b^2}$	$s > a$
$e^{at} \cos bt$	$\frac{s-a}{(s-a)^2+b^2}$	$s > a$
$t^n e^{at} \quad (n \in \mathbb{N})$	$\frac{n!}{(s-a)^{n+1}}$	$s > a$
$u_c(t)$	$\frac{e^{-cs}}{s}$	$s > 0$
$u_c(t) f(t-c)$	$e^{-cs} F(s)$	
$e^{ct} f(t)$	$F(s-c)$	
$f(ct) \quad (c > 0)$	$\frac{1}{c} F\left(\frac{s}{c}\right)$	
$\int_0^t f(t-\tau) g(\tau) d\tau$	$F(s) G(s)$	
$t^n f(t)$	$(-1)^n F^{(n)}(s)$	