MATH 216 MATHEMATICS IV
Homework 3 (first eight problems)

A. Find the Laplace transformation of the following functions:

1. $f(t) = e^{-2t}$.

   Solution : Note that

$$\begin{aligned}
\mathcal{L}\left(e^{-2t}\right) &= \int_0^\infty e^{-st}e^{-2t}\,dt = \int_0^\infty e^{-t(s+2)}\,dt \\
&= \frac{-1}{s+2}\int_0^\infty e^{-t(s+2)}\,\left(-\left(s+2\right)\right)dt \Longrightarrow \\
\frac{-1}{s+2}\left[e^{-t(s+2)}\right]_0^\infty &= \frac{1}{(s+2)}.
\end{aligned}$$

2. $f(t) = \sqrt{t} + 3t^2$.

   Solution : Recall that multiplying $x(t)$ by t^n in $t - domain$ is equivalent to taking the n^{th} derivative in $s - domain$ and multtiplying with $(-1)^n$. Consequently, we get

$$\mathcal{L}\left(t^2\right) = (-1)^2\frac{d^2}{ds^2}\left(\mathcal{L}\left(1\right)\right) = (-1)^2\frac{d^2}{ds^2}\left(\frac{1}{s}\right) = (-1)^2\frac{d}{ds}\left(-\frac{1}{s^2}\right) = \frac{2}{s^3}.$$

   For the first term, we have

$$\mathcal{L}\left(t^\alpha\right) = \frac{\Gamma\left(\alpha+1\right)}{s^{\alpha+1}} \Longrightarrow \mathcal{L}\left(t^{1/2}\right) = \frac{\Gamma\left(\frac{1}{2}+1\right)}{s^{3/2}}.$$

   Since $\Gamma\left(z+1\right) = z\Gamma\left(z\right)$, it follows that $\Gamma\left(\frac{1}{2}+1\right) = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{1}{2}\sqrt{\pi}$. This implies that

$$\mathcal{L}\left(t^{1/2}\right) = \frac{\sqrt{\pi}}{2s^{3/2}} \Longrightarrow \mathcal{L}\left(\sqrt{t}+3t^2\right) = \frac{\sqrt{\pi}}{2s^{3/2}} + \frac{2}{s^3}.$$

3. $f(t) = \cos^2 2t$.

   Solution : Note that $\cos^2 2t = \frac{1}{2}\left(1+\cos 4t\right)$ which implies that

$$\mathcal{L}\left(\cos^2 2t\right) = \frac{1}{2}\mathcal{L}\left(1+\cos 4t\right) = \frac{1}{2}\left[\frac{1}{s} + \frac{s}{s^2+16}\right].$$

4. $f(t) = t\cos t + te^t$.

   Solution : Since multiplication by t is equivalent to taking derivative with respect to s and multiplying by -1, it follows that

$$\begin{aligned}
\mathcal{L}\left(t\cos t + te^t\right) &= (-1)\frac{d}{ds}\left(\frac{s}{s^2+1} + \frac{1}{s-1}\right) = (-1)\left(\frac{1-s^2}{s^2+1} - \frac{1}{\left(s-1\right)^2}\right), \\
&= \frac{s^2-1}{s^2+1} + \frac{1}{\left(s-1\right)^2}.
\end{aligned}$$

5. $f(t) = \frac{\sinh t}{t}$.

Solution : Since $\mathcal{L}(\sinh t) = \mathcal{L}\left(t \frac{\sinh t}{t}\right)$, it follows that $\mathcal{L}(\sinh t)$ is the derivative with respect to s of $\mathcal{L}\left(\frac{\sinh t}{t}\right)$ multiplied by -1. Therefore, we have

$$\begin{aligned}\mathcal{L}(\sinh t) &= \mathcal{L}\left(t\frac{\sinh t}{t}\right) = \frac{1}{s^2-1} \Longrightarrow \mathcal{L}\left(\frac{\sinh t}{t}\right) = -\int \frac{ds}{s^2-1} \\ &= -\frac{1}{2}\int\left(\frac{1}{s-1} - \frac{1}{s+1}\right) ds = \frac{1}{2}\ln\left(\frac{s+1}{s-1}\right).\end{aligned}$$

6. $f(t) = t^2 \cos 2t$.

Solution : Using the derivative in s-domain, we get

$$\begin{aligned}\mathcal{L}\left(t^2\cos 2t\right) &= (-1)^2 \frac{d^2}{ds^2}\left(\frac{s}{s^2+4}\right) = \frac{d}{ds}\left(\frac{s^2+4-2s^2}{\left(s^2+4\right)^2}\right) = \frac{d}{ds}\left(\frac{4-s^2}{\left(s^2+4\right)^2}\right), \\ &= \frac{-2s\left(s^2+4\right)^2 - \left(4-s^2\right)4s\left(s^2+4\right)}{\left(s^2+4\right)^4} = \frac{-2s\left(s^2+4\right) - \left(4-s^2\right)4s}{\left(s^2+4\right)^3} \\ &= \frac{2s^3-24s}{\left(s^2+4\right)^3}.\end{aligned}$$

7. $f(t) = \frac{e^{3t}-1}{t}$.

Solution : Note that

$$\begin{aligned}\mathcal{L}\left(e^{3t}-1\right) &= \mathcal{L}\left(t\frac{e^{3t}-1}{t}\right) = \frac{1}{s-3} - \frac{1}{s} \\ &\Longrightarrow \mathcal{L}\left(\frac{e^{3t}-1}{t}\right) = -\int\left(\frac{1}{s-3} - \frac{1}{s}\right) ds \Longrightarrow \\ \mathcal{L}\left(\frac{e^{3t}-1}{t}\right) &= \ln\left(\frac{s}{s-3}\right).\end{aligned}$$

8. $f(t) = te^{-t}\sin^2 t$.

Solution : Recall that $\sin^2 t = \frac{1}{2}(1-\cos 2t)$. This implies that

$$\begin{aligned}
\mathcal{L}\left(te^{-t}\sin^2 t\right) &= \mathcal{L}\left(te^{-t}\frac{1}{2}(1-\cos 2t)\right) = \frac{1}{2}\mathcal{L}\left(te^{-t} - te^{-t}\cos 2t\right) \\
&= \frac{1}{2}(-1)\frac{d}{ds}\left(\frac{1}{s+1} - \frac{(s+1)}{4+(s+1)^2}\right) \\
&= \frac{1}{2}(-1)\left(-\frac{1}{(s+1)^2} - \frac{(s+1)^2+4-(s+1)\,2\,(s+1)}{\left((s+1)^2+4\right)^2}\right) \\
&= \frac{1}{2}\left(\frac{1}{(s+1)^2} + \frac{4-(s+1)^2}{\left((s+1)^2+4\right)^2}\right).
\end{aligned}$$