MATH 216 MATHEMATICS IV
Homework 3

A. Find the Laplace transformation of the following functions:

1. $f(t) = e^{-2t}$.

Solutions : Note that

$$\begin{aligned} \mathcal{L}\left(e^{-2t}\right) &= \int_0^\infty e^{-st}e^{-2t}\,dt = \int_0^\infty e^{-t(s+2)}\,dt = \frac{-1}{s+2}\int_0^\infty e^{-t(s+2)}\,\left(-\left(s+2\right)\right)dt \\ \frac{-1}{s+2}\left[e^{-t(s+2)}\right]_0^\infty &= \frac{1}{(s+2)}. \end{aligned}$$

2. $f(t) = \sqrt{t} + 3t^2$.

Solutions : Recall that multiplying $x(t)$ by t^n in $t - domain$ is equivalent to taking the n^{th} derivative in $s - domain$ and multtiplying with $(-1)^n$. Consequently, we get

$$\mathcal{L}\left(t^2\right) = (-1)^2\frac{d^2}{ds^2}\left(\mathcal{L}\left(1\right)\right) = (-1)^2\frac{d^2}{ds^2}\left(\frac{1}{s}\right) = (-1)^2\frac{d}{ds}\left(-\frac{1}{s^2}\right) = \frac{2}{s^3}.$$

For the first term, we have

$$\mathcal{L}\left(t^\alpha\right) = \frac{\Gamma\left(\alpha+1\right)}{s^{\alpha+1}} \Longrightarrow \mathcal{L}\left(t^{1/2}\right) = \frac{\Gamma\left(\frac{1}{2}+1\right)}{s^{3/2}}.$$

Since $\Gamma\left(z+1\right) = z\Gamma\left(z\right)$, it follows that $\Gamma\left(\frac{1}{2}+1\right) = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{1}{2}\sqrt{\pi}$. This implies that

$$\mathcal{L}\left(t^{1/2}\right) = \frac{\sqrt{\pi}}{2s^{3/2}} \Longrightarrow \mathcal{L}\left(\sqrt{t}+3t^2\right) = \frac{\sqrt{\pi}}{2s^{3/2}} + \frac{2}{s^3}.$$

3. $f(t) = \cos^2 2t$.

Solution : Note that $\cos^2 2t = \frac{1}{2}\left(1+\cos 4t\right)$ which implies that

$$\mathcal{L}\left(\cos^2 2t\right) = \frac{1}{2}\mathcal{L}\left(1+\cos 4t\right) = \frac{1}{2}\left[\frac{1}{s} + \frac{s}{s^2+16}\right].$$

4. $f(t) = t\cos t + te^t$.

Solution : Since multiplication by t is equivalent to taking derivative with respect to s and multiplying by -1, it follows that

$$\begin{aligned} \mathcal{L}\left(t\cos t + te^t\right) &= (-1)\frac{d}{ds}\left(\frac{s}{s^2+1} + \frac{1}{s-1}\right) = (-1)\left(\frac{1-s^2}{s^2+1} - \frac{1}{(s-1)^2}\right), \\ &= \frac{s^2-1}{s^2+1} + \frac{1}{(s-1)^2}. \end{aligned}$$

5. $f(t) = \frac{\sinh t}{t}$.

Solution : Since $\mathcal{L}\left(\sinh t\right) = \mathcal{L}\left(t\frac{\sinh t}{t}\right)$, it follows that $\mathcal{L}\left(\sinh t\right)$ is the derivative with respect to s of $\mathcal{L}\left(\frac{\sinh t}{t}\right)$ multiplied by -1. Therefore, we have

$$\begin{aligned} \mathcal{L}\left(\sinh t\right) &= \mathcal{L}\left(t\frac{\sinh t}{t}\right) = \frac{1}{s^2-1} \Longrightarrow \mathcal{L}\left(\frac{\sinh t}{t}\right) = -\int\frac{ds}{s^2-1} \\ &= -\frac{1}{2}\int\left(\frac{1}{s-1} - \frac{1}{s+1}\right)ds = \frac{1}{2}\ln\left(\frac{s+1}{s-1}\right). \end{aligned}$$

6. $f(t) = t^2 \cos 2t$.

Solution : Using the derivative in s-domain, we get

$$\begin{aligned}
\mathcal{L}\left(t^2 \cos 2t\right) &= (-1)^2 \frac{d^2}{ds^2}\left(\frac{s}{s^2+4}\right) = \frac{d}{ds}\left(\frac{s^2+4-2s^2}{\left(s^2+4\right)^2}\right) = \frac{d}{ds}\left(\frac{4-s^2}{\left(s^2+4\right)^2}\right), \\
&= \frac{-2s\left(s^2+4\right)^2 - \left(4-s^2\right)4s\left(s^2+4\right)}{\left(s^2+4\right)^4} = \frac{-2s\left(s^2+4\right) - \left(4-s^2\right)4s}{\left(s^2+4\right)^3} \\
&= \frac{2s^3-24s}{\left(s^2+4\right)^3}.
\end{aligned}$$

7. $f(t) = \frac{e^{3t}-1}{t}$.

Solution : Note that

$$\begin{aligned}
\mathcal{L}\left(e^{3t}-1\right) &= \mathcal{L}\left(t\frac{e^{3t}-1}{t}\right) = \frac{1}{s-3} - \frac{1}{s} \Longrightarrow \mathcal{L}\left(\frac{e^{3t}-1}{t}\right) = -\int\left(\frac{1}{s-3}-\frac{1}{s}\right)ds \Longrightarrow \\
\mathcal{L}\left(\frac{e^{3t}-1}{t}\right) &= \ln\left(\frac{s}{s-3}\right).
\end{aligned}$$

8. $f(t) = te^{-t}\sin^2 t$.

Solution : Recall that $\sin^2 t = \frac{1}{2}(1-\cos 2t)$. This implies that

$$\begin{aligned}
\mathcal{L}\left(te^{-t}\sin^2 t\right) &= \mathcal{L}\left(te^{-t}\frac{1}{2}(1-\cos 2t)\right) = \frac{1}{2}\mathcal{L}\left(te^{-t} - te^{-t}\cos 2t\right) = \frac{1}{2}(-1)\frac{d}{ds}\left(\frac{1}{s+1} - \frac{(s+1)}{4+(s+1)^2}\right), \\
&= \frac{1}{2}(-1)\left(-\frac{1}{(s+1)^2} - \frac{(s+1)^2+4-(s+1)2(s+1)}{\left((s+1)^2+4\right)^2}\right) = \frac{1}{2}\left(\frac{1}{(s+1)^2} + \frac{4-(s+1)^2}{\left((s+1)^2+4\right)^2}\right).
\end{aligned}$$

9. $f(t) = \left\{\begin{array}{cc} 2 & 0 < t \le 3 \\ 0 & t > 3 \end{array}\right\}$.

Solution : Let $\tau = t-3$. Then, we have

$$\begin{aligned}
\mathcal{L}(f(t)) &= \int_0^\infty e^{-st}2dt - \int_3^\infty 2e^{-st}\,dt = \frac{2}{s} - \int_0^\infty 2e^{-s(\tau+3)}\,d\tau \\
&= \frac{2}{s} - \frac{2e^{-3s}}{s} = \frac{2\left(1-e^{-3s}\right)}{s}.
\end{aligned}$$

10. $f(t) = \left\{\begin{array}{cc} \sin 2t & \pi \le t \le 2\pi \\ 0 & t < \pi \text{ or } t > 2\pi \end{array}\right\}$.

Solution : Similar to the previous case, let $\tau = t-\pi$ and $\varphi = t-2\pi$. Then, we get

$$\begin{aligned}
\mathcal{L}(f(t)) &= \int_0^\infty e^{-st}f(t)\,dt = \int_\pi^\infty e^{-st}\sin 2t\;dt - \int_{2\pi}^\infty e^{-st}\sin 2t\,dt \\
&= \int_0^\infty e^{-s(\tau+\pi)}\sin(2\tau+2\pi)\;d\tau - \int_0^\infty \sin(2\varphi+4\pi)\,e^{-s(\varphi+2\pi)}\,d\varphi \\
&= \frac{2e^{-\pi s}}{s^2+4} - \frac{2e^{-2\pi s}}{s^2+4} = \frac{2\left(e^{-\pi s}-e^{-2\pi s}\right)}{s^2+4}.
\end{aligned}$$

B. Find the inverse Laplace Transform of the following functions:

1. $F(s)=\frac{1}{s-2}$
   A: $f(t)=e^{2t}$
2. $F(s)=\frac{1}{s}-\frac{2}{s^{5/2}}$
   A: $f(t)=1-\frac{8}{3\sqrt{\pi}}t^{\frac{3}{2}}$
3. $F(s)=\frac{3s+1}{s^2+4}$
   A: $f(t)=3\cos 2t+\frac{1}{2}\sin 2t$
4. $F(s)=\frac{2e^{-3s}}{s}$
   A: $f(t)=2u(t-3)$
5. $F(s)=\frac{1}{s(s-3)}$
   A: $f(t)=\frac{1}{3}\left(e^{3t}-1\right)$
6. $F(s)=\frac{2s+1}{s(s^2+9)}$
   A: $f(t)=\frac{1}{9}\left(6\sin 3t-\cos 3t+1\right)$
7. $F(s)=\frac{s^3}{(s-4)^4}$
   A: $f(t)=e^{4t}\left(1+12t+24t^2+\frac{32}{3}t^3\right)$
8. $F(s)=\frac{s^2-2s}{s^4+5s^2+4}$
   A: $f(t)=\frac{1}{3}\left(2\cos 2t-2\sin 2t-2\cos t+\sin t\right)$
9. $F(s)=\frac{2s^3-s^2}{(4s^2-4s+5)^2}$
   A: $f(t)=\frac{1}{64}e^{\frac{t}{2}}\left[(4t+8)\cos t+(4-3t)\sin t\right]$
10. $F(s)=\ln\left(1+\frac{1}{s^2}\right)$
    Here we use $\mathcal{L}\left(tf(t)\right)=(-1)\frac{dF}{ds}\Rightarrow\mathcal{L}^{-1}\left(\frac{dF}{ds}\right)=(-1)tf(t)$. Therefore,

$$\begin{aligned}
\frac{dF}{ds} &= \frac{\frac{-2}{s^3}}{\left(1+\frac{1}{s^2}\right)}=-\frac{2}{s\left(s^2+1\right)}\\
\mathcal{L}^{-1}\left(\frac{dF}{ds}\right) &= -\mathcal{L}^{-1}\left(\frac{2}{s\left(s^2+1\right)}\right)=(-1)tf(t)\\
\mathcal{L}^{-1}\left(\frac{2}{s\left(s^2+1\right)}\right) &= \mathcal{L}^{-1}\left(\frac{2}{s}-\frac{2s}{s^2+1}\right)=tf(t)\\
2-2\cos t &= tf(t)\\
f(t) &= \frac{2\left(1-\cos t\right)}{t}
\end{aligned}$$

11. $F(s)=\arctan\left(\frac{3}{s+2}\right)$
    Solution: Here, we again use the $\mathcal{L}\left(tf(t)\right)=(-1)\frac{dF}{ds}\Rightarrow\mathcal{L}^{-1}\left(\frac{dF}{ds}\right)=(-1)tf(t)$. Therefore

$$\begin{aligned}
\frac{dF}{ds} &= \frac{\frac{-3}{(s+2)^2}}{1+\left(\frac{3}{s+2}\right)^2}=-\frac{3}{s^2+4s+13}=-\frac{3}{(s+2)^2+9}\\
\mathcal{L}^{-1}\left(\frac{dF}{ds}\right) &= -\mathcal{L}^{-1}\left(\frac{3}{(s+2)^2+9}\right)=(-1)tf(t)\\
e^{-2t}\sin 3t &= tf(t)\\
f(t) &= \frac{e^{-2t}\sin 3t}{t}.
\end{aligned}$$

12. $F(s) = \frac{s}{(s^2+1)^3}$

Solution:

$$f(t) = \mathcal{L}^{-1}\left(\frac{s}{(s^2+1)^3}\right)$$

then we have

$$f(t) = \frac{1}{8}\left(t\sin t - t^2\cos t\right)$$

13. $F(s) = \frac{e^{-s}}{s+2}$

Solution:

$$f(t) = \mathcal{L}^{-1}\left(\frac{e^{-s}}{s+2}\right) = \left\{\begin{array}{cc} e^{-2(t-1)} & t \geq 1 \\ 0 & t < 1 \end{array}\right\}$$

A: $f(t) = u(t-1)e^{-2(t-1)}$ where $u(.)$ is the unit step function.

C. Solve the following initial value problems by using Laplace Transform:

1. $x'' + 4x = 0;\ \ x(0) = 5, x'(0) = 0$

Solution:

$$\begin{aligned}
\mathcal{L}\left(x^{''} + 4x\right) &= \mathcal{L}(0) \\
\left[s^2F(s) - sx(0) - x'(0)\right] + 4F(s) &= 0 \\
\left(s^2+4\right)F(s) - 5s &= 0 \\
F(s) &= \frac{5s}{(s^2+4)} \\
x(t) &= \mathcal{L}^{-1}\left(\frac{5s}{(s^2+4)}\right) \\
x(t) &= 5\cos 2t
\end{aligned}$$

A: $x(t) = 5\cos 2t$

2. $x'' - x' - 2x = 0;\ \ x(0) = 0, x'(0) = 2$

Solution:

$$\begin{aligned}
\mathcal{L}\left(x^{''} - x' - 2x\right) &= \mathcal{L}(0) \\
\left[s^2F(s) - sx(0) - x'(0)\right] - [sF(s) - x(0)] - 2F(s) &= 0 \\
\left(s^2 - s - 2\right)F(s) - 2 &= 0 \\
F(s) &= \frac{2}{(s^2-s-2)} \\
x(t) &= \mathcal{L}^{-1}\left(\frac{2}{3(s-2)} - \frac{2}{3\,(s+1)}\right) \\
x(t) &= \frac{2}{3}e^{2t} - \frac{2}{3}e^{-t}
\end{aligned}$$

3. $x'' + 9x = 1; x(0) = 0, x'(0) = 0$

Solution:

$$\begin{aligned}
\mathcal{L}\left(x^{''}+9x\right) &= \mathcal{L}(1)\\
\left[s^2F(s)-sx(0)-x'(0)\right]+9F(s) &= \frac{1}{s}\\
\left(s^2+9\right)F(s) &= \frac{1}{s}\\
F(s) &= \frac{1}{s\left(s^2+9\right)}\\
x(t) &= \mathcal{L}^{-1}\left(\frac{1}{9s}-\frac{s}{9\left(s^2+9\right)}\right)\\
x(t) &= \frac{1}{9}-\frac{1}{9}\cos 3t
\end{aligned}$$

4. $x''+6x'+25x=0; x(0)=2, x'(0)=3$

   Solution:

$$\begin{aligned}
\mathcal{L}\left(x^{''}+6x'+25x\right) &= \mathcal{L}\left(0\right)\\
\left[s^2F(s)-sx(0)-x'(0)\right]+6\left[sF(s)-x(0)\right]+25F(s) &= 0\\
\left(s^2+6s+25\right)F(s)-2s-3-12 &= 0\\
F(s) &= \frac{2s+15}{s^2+6s+25}\\
x(t) &= \mathcal{L}^{-1}\left(\frac{2s+15}{(s+3)^2+16}\right)=\\
x(t) &= \mathcal{L}^{-1}\left(\frac{2(s+3)}{(s+3)^2+16}+\frac{9}{(s+3)^2+16}\right)\\
x(t) &= 2e^{-3t}\cos 4t+\frac{9}{4}e^{-3t}\sin 4t
\end{aligned}$$

5. $x''-6x'+8x=2; x(0)=0, x'(0)=0$

   Solution:

$$\begin{aligned}
\mathcal{L}\left(x^{''}-6x'+8x\right) &= \mathcal{L}(2)\\
\left[s^2F(s)-sx(0)-x'(0)\right]-6\left[sF(s)-x(0)\right]+8F(s) &= \frac{2}{s}\\
\left(s^2-6s+8\right)F(s) &= \frac{2}{s}\\
F(s) &= \frac{2}{s\left(s^2-6s+8\right)}\\
x(t) &= \mathcal{L}^{-1}\left(\frac{1}{4(s-4)}-\frac{1}{2(s-2)}+\frac{1}{4s}\right)\\
x(t) &= \frac{1}{4}e^{4t}-\frac{1}{2}e^{2t}+\frac{1}{4}
\end{aligned}$$

6. $x''-4x=3t; x(0)=0, x'(0)=0$

Solution:

$$\begin{aligned}
\mathcal{L}\left(x'' - 4x\right) &= \mathcal{L}(3t) \\
\left[s^2F(s) - sx(0) - x'(0)\right] - 4F(s) &= \frac{3}{s^2} \\
\left(s^2-4\right)F(s) &= \frac{3}{s^2} \\
F(s) &= \frac{3}{s^2\left(s^2-4\right)} \\
x(t) &= \mathcal{L}^{-1}\left(\frac{3}{16(s-2)} - \frac{3}{4s^2} - \frac{3}{16(s+2)}\right) \\
x(t) &= \frac{3}{16}e^{2t} - \frac{3}{4}t - \frac{3}{16}e^{-2t} \\
x(t) &= \frac{1}{8}\left(-6t + 3\sinh 2t\right)
\end{aligned}$$

7. $x'' + 4x' + 8x = e^{-t}; x(0) = 0, x'(0) = 0$

   Solution:

$$\begin{aligned}
\mathcal{L}\left(x'' + 4x' + 8x\right) &= \mathcal{L}\left(e^{-t}\right) \\
\left[s^2F(s) - sx(0) - x'(0)\right] + 4\left[sF(s) - x(0)\right] + 8F(s) &= \frac{1}{s+1} \\
\left(s^2+4s+8\right)F(s) &= \frac{1}{s+1} \\
F(s) &= \frac{1}{(s+1)\left(s^2+4s+8\right)} \\
x(t) &= \mathcal{L}^{-1}\left(\frac{1}{5}\frac{1}{s+1} - \frac{1}{5}\frac{s+3}{\left(s^2+4s+8\right)}\right) \\
x(t) &= \mathcal{L}^{-1}\left(\frac{1}{5}\frac{1}{s+1} - \frac{1}{5}\frac{s+2}{(s+2)^2+4} - \frac{1}{10}\frac{2}{(s+2)^2+4}\right) \\
x(t) &= \frac{1}{5}e^{-t} - \frac{1}{5}e^{-2t}\cos 2t - \frac{1}{10}e^{-2t}\sin 2t
\end{aligned}$$

8. $x^{(4)} + 8x'' + 16x = 0; x(0) = x'(0) = x''(0) = 0, x^{(3)}(0) = 1$

   Solution:

$$\begin{aligned}
\mathcal{L}\left(x^{(4)} + 8x'' + 16x\right) &= \mathcal{L}(0) \\
\left[s^4F(s) - s^3x(0) - s^2x'(0) - sx''(0) - x^{(3)}(0)\right] + 8\left[s^2F(s) - sx(0) - x'(0)\right] + 16F(s) &= 0 \\
\left(s^4+8s^2+16\right)F(s) - 1 &= 0 \\
\left(s^4+8s^2+16\right)F(s) &= 1
\end{aligned}$$

This implies that

$$\begin{aligned}
F(s) &= \frac{1}{s^4+8s^2+16} \\
x(t) &= \mathcal{L}^{-1}\left(\frac{1}{\left(s^2+4\right)^2}\right) \\
x(t) &= \frac{1}{8}\left(\sin 2t - t\cos 2t\right).
\end{aligned}$$

9. $x^{(4)}+2x''+x=e^{2t}; x(0)=x'(0)=x''(0)=x^{(3)}(0)=1$

Solution:

$$\begin{aligned}
\mathcal{L}\left(x^{(4)}+2x''+x\right) &= \mathcal{L}\left(e^{2t}\right)\\
\left[s^{(4)}F(s)-s^3x(0)-s^2x'(0)-sx''(0)-x^{(3)}(0)\right]+2\left[s^2F(s)-sx(0)-x'(0)\right]+F(s) &= \frac{1}{s-2}\\
\left(s^4+2s^2+1\right)F(s)-s^3-s^2-s-1-2s-2 &= \frac{1}{s-2}
\end{aligned}$$

$$\begin{aligned}
\left(s^4+2s^2+1\right)F(s) &= \frac{1}{s-2}+s^3+s^2+3s+3=\frac{-3s+s^2-s^3+s^4-5}{s-2}\\
F(s) &= \frac{-3s+s^2-s^3+s^4-5}{(s-2)\left(s^4+2s^2+1\right)}\\
x(t) &= \mathcal{L}^{-1}\left(\frac{-3s+s^2-s^3+s^4-5}{(s-2)\left(s^2+1\right)^2}\right)\\
x(t) &= \mathcal{L}^{-1}\left(\frac{1}{25}\frac{1}{s-2}+\frac{1}{25}\frac{24s+23}{s^2+1}+\frac{1}{5}\frac{9s+8}{\left(s^2+1\right)^2}\right)\\
x(t) &= \frac{1}{25}\left(e^{2t}+24\cos t+23\sin t\right)+\frac{9}{10}t\sin t+\frac{4}{5}(\sin t-t\cos t)
\end{aligned}$$

A:$x(t)=\frac{1}{50}\left[2e^{2t}+(48-40t)\cos t+(45t+86)\sin t\right]$

10. $x^{(3)}+4x''+5x'+2x=10\cos t;\quad x(0)=x'(0)=0, x''(0)=3$

Solution:

$$\begin{aligned}
\mathcal{L}\left(x^{(3)}+4x''+5x'+2x\right) &= \mathcal{L}\left(10\cos t\right)\\
\left[s^3F(s)-s^2x(0)-sx'(0)-x''(0)\right]+4\left[s^2F(s)-sx(0)-x'(0)\right]+5\left[sF(s)-x(0)\right]+2F(s) &= \frac{10s}{s^2+1}\\
\left(s^3+4s^2+5s+2\right)F(s)-3 &= \frac{10s}{s^2+1}
\end{aligned}$$

$$\begin{aligned}
\left(s^3+4s^2+5s+2\right)F(s) &= \frac{10s}{s^2+1}+3=\frac{3s^2+10s+3}{s^2+1}\\
F(s) &= \frac{3s^2+10s+3}{\left(s^2+1\right)\left(s^3+4s^2+5s+2\right)}\\
x(t) &= \mathcal{L}^{-1}\left(\frac{3s^2+10s+3}{\left(s^2+1\right)\left(s^3+4s^2+5s+2\right)}\right)\\
x(t) &= \mathcal{L}^{-1}\left(\frac{2}{s+1}-\frac{2}{(s+1)^2}-\frac{1}{s+2}-\frac{s}{s^2+1}+\frac{2}{s^2+1}\right)\\
x(t) &= 2e^{-t}-2te^{-t}-e^{-2t}-\cos t+2\sin t
\end{aligned}$$

11. $x''+4x'+13x=te^{-t}; x(0)=0, x'(0)=2$

Solution:

$$\begin{aligned}
\mathcal{L}\left(x'' + 4x' + 13x\right) &= \mathcal{L}\left(te^{-t}\right) \\
\left[s^2F(s) - sx(0) - x'(0)\right] + 4\left[sF(s) - x(0)\right] + 13F(s) &= \frac{1}{(s+1)^2} \\
\left(s^2 + 4s + 13\right)F(s) - 2 &= \frac{1}{(s+1)^2} \\
\left(s^2 + 4s + 13\right)F(s) &= \frac{1}{(s+1)^2} + 2 = \frac{4s + 2s^2 + 3}{(s+1)^2} \\
F(s) &= \frac{4s + 2s^2 + 3}{(s+1)^2\left(s^2 + 4s + 13\right)}
\end{aligned}$$

and

$$\begin{aligned}
\frac{4s + 2s^2 + 3}{(s+1)^2\left(s^2 + 4s + 13\right)} &= \frac{A}{s+1} + \frac{B}{(s+1)^2} + \frac{Cs + D}{s^2 + 4s + 13} \\
&\Rightarrow A = \frac{-1}{50}, B = \frac{1}{10}, C = \frac{1}{50}, D = \frac{98}{50} \\
x(t) &= \mathcal{L}^{-1}\left(-\frac{1}{50}\frac{1}{s+1} + \frac{1}{10}\frac{1}{(s+1)^2} + \frac{1}{50}\frac{s+2}{(s+2)^2 + 9} + \frac{32}{50}\frac{3}{(s+2)^2 + 9}\right) \\
x(t) &= -\frac{1}{50}e^{-t} + \frac{1}{10}te^{-t} + \frac{1}{50}e^{-2t}\cos 3t + \frac{32}{50}e^{-2t}\sin 3t
\end{aligned}$$

12. $x'' + x = \sin 2t; x(\frac{\pi}{2}) = 2, x'(\frac{\pi}{2}) = 0$

Solution: Let us use the substitution $k = t - \frac{\pi}{2}$ then we have

$$\begin{aligned}
x'' + x &= \sin\left(2k + \pi\right) \\
x'' + x &= -\sin 2k \\
\mathcal{L}\left(x'' + x\right) &= -\mathcal{L}\left(\sin 2k\right) \\
s^2F(s) - sx(0) - x'(0) + F(s) &= -\frac{2}{s^2 + 4} \\
\left(s^2 + 1\right)F(s) - 2s &= -\frac{2}{s^2 + 4} \\
\left(s^2 + 1\right)F(s) &= -\frac{2}{s^2 + 4} + 2s = \frac{2s^3 + 8s - 2}{s^2 + 4} \\
F(s) &= \frac{2s^3 + 8s - 2}{\left(s^2 + 1\right)\left(s^2 + 4\right)}
\end{aligned}$$

By using the partial fraction we obtain

$$
\begin{aligned}
F(s) &= \frac{2s^3+8s-2}{(s^2+1)(s^2+4)} = \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+4} \\
2s^3+8s-2 &= 4B+D+As^3+Bs^2+Cs^3+s^2D+4As+Cs \\
A &= 2, B=-\frac{2}{3}, C=0, D=\frac{2}{3} \\
F(s) &= \frac{2s^3+8s-2}{(s^2+1)(s^2+4)} = \frac{2s}{s^2+1} - \frac{2}{3}\frac{1}{s^2+1} + \frac{1}{3}\frac{2}{s^2+4} \\
x(k) &= 2\cos k - \frac{2}{3}\sin k + \frac{1}{3}\sin 2k \\
x(t) &= 2\cos\left(t-\frac{\pi}{2}\right) - \frac{2}{3}\sin\left(t-\frac{\pi}{2}\right) + \frac{1}{3}\sin(2t-\pi) \\
x(t) &= -\frac{2}{3}\cos t + 2\sin t - \frac{1}{3}\sin 2t
\end{aligned}
$$