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DEPARTMENT:

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SIGNATURE:

Question	Points	Score
1	25	
2	25	
3	25	
4	25	
Total:	100	

- The time limit is 90 minutes.
- Give your answers in exact form (for example $\frac{\pi}{3}$ or $5\sqrt{3}$), except as noted in particular problems.
- All communication between students, either verbally or non-verbally, is strictly forbidden.
- Calculators, mobile phones, smart watches, and any digital means of communication are forbidden. The sharing of pens, erasers or any other item between students is forbidden.
- In order to receive credit, you must **show all of your work**. If you do not indicate the way in which you solved a problem, you may get little or no credit for it, even if your answer is correct.
- Place a box around your answer to each question.
- Please do not write in the table above.

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1. 25 points Solve the initial value problem $\begin{cases} y' = \dfrac{1+3x^2}{3y^2-12y} \\ y(0) = 1. \end{cases}$

2. (a) 15 points **Use the Laplace Transform** to solve $\begin{cases} y'' - 2y' + 2y = e^{-t} \\ y(0) = 1 \\ y'(0) = 1. \end{cases}$

Solution: Taking the Laplace Transform of the ODE gives

$$\begin{aligned}
\mathcal{L}[y''] - 2\mathcal{L}[y'] + 2\mathcal{L}[y] &= \mathcal{L}[e^{-t}] \\
\big(s^2Y - sy(0) - y'(0)\big) - 2\big(sY - y(0)\big) + 2Y &= \frac{1}{s+1} \\
(s^2 - 2s + 2)Y &= \frac{1}{s+1} \\
Y &= \frac{s}{(s^2+1)(s^2-3s+2)} \\
&= \frac{s}{(s^2+1)(s-2)(s-1)} \\
&= \frac{As+B}{s^2+1} + \frac{C}{s-2} + \frac{D}{s-1} \\
&= \frac{(As+B)(s-2)(s-1) + C(s^2+1)(s-1) + D(s^2+1)(s-2)}{(s^2+1)(s-2)(s-1)} \\
&\quad \left(A = \tfrac{1}{10},\ B = -\tfrac{3}{10},\ C = \tfrac{2}{5},\ D = -\tfrac{1}{2}\right) \\
&= \frac{\frac{1}{10}s - \frac{3}{10}}{s^2+1} + \frac{\frac{2}{5}}{s-2} - \frac{\frac{1}{2}}{s-1} \\
&= \frac{1}{10}\left(\frac{s}{s^2+1}\right) - \frac{3}{10}\left(\frac{1}{s^2+1}\right) + \frac{2}{5}\left(\frac{1}{s-2}\right) - \frac{1}{2}\left(\frac{1}{s-1}\right) \\
&= \frac{1}{10}\mathcal{L}\left[\cos t\right] - \frac{3}{10}\mathcal{L}\left[\sin t\right] + \frac{2}{5}\mathcal{L}\left[e^{2t}\right] - \frac{1}{2}\mathcal{L}\left[e^{t}\right].
\end{aligned}$$

Therefore the solution to the IVP is

$$y(t) = \frac{1}{10}\cos t - \frac{3}{10}\sin t + \frac{2}{5}e^{2t} - \frac{1}{2}e^{t}.$$

(b) 10 points Find the inverse Laplace Transform of $F(s) = \dfrac{3}{s^2+3s-4}$.

Solution:
First we calculate that

$$\begin{aligned}
F(s) = \frac{2s-5}{s^2+2s+10} &= \frac{2s-5}{(s+1)^2+3^2} = \frac{2s+2}{(s+1)^2+3^2} + \frac{-7}{(s+1)^2+3^2} \\
&= 2\left(\frac{s+1}{(s+1)^2+3^2}\right) - \frac{7}{3}\left(\frac{3}{(s+1)^2+3^2}\right) \\
&= 2\mathcal{L}\left[e^{-t}\cos 3t\right] - \frac{7}{3}\mathcal{L}\left[e^{-t}\sin 3t\right].
\end{aligned}$$

Therefore

$$f(t) = \mathcal{L}^{-1}[F](t) = 2e^{-t}\cos 3t - \frac{7}{3}e^{-t}\sin 3t.$$

(a table of Laplace Transforms will be printed here when the answers are hidden)

3. For the given system $\begin{cases} \frac{dx}{dt} &= x + y + z \\ \frac{dy}{dt} &= 2y \\ \frac{dz}{dt} &= y - z, \end{cases}$

(a) 5 points Write the system in matrix form.

Solution: $\mathbf{x}' = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 0 \\ 0 & 1 & -1 \end{bmatrix} \mathbf{x}.$

(b) 20 points Find the general solution for this system.

Solution:

Learning Objectives:

LO1	first order ODEs	25 points	Q1
LO2	higher order ODEs	0 points	
LO3	Laplace T.	25 points	Q2
LO4	systems	50 points	Q3 & Q4

4. (a) 20 points Solve $\begin{cases} \mathbf{x}' = \begin{bmatrix} 4 & -1 \\ 1 & 2 \end{bmatrix} \mathbf{x} \\ \mathbf{x}(0) = \begin{bmatrix} 3 \\ 5 \end{bmatrix} \end{cases}$

Solution: Solving

$$0 = \det(A - rI) = \begin{vmatrix} 2-r & 1 \\ -1 & 2-r \end{vmatrix} = (2-r)^2 + 1 = 4 - 4r + r^2 + 1 = r^2 - 4r + 5$$

gives

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{4 \pm \sqrt{16 - 20}}{2} = 2 \pm i.$$

So $\lambda = 2$ and $\mu = 1$.
Next we must find an eigenvector for $r = 2 + i$. We calculate that

$$\mathbf{0} = (A - rI)\xi = \begin{bmatrix} 2-(2+i) & 1 \\ -1 & 2-(2+i) \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} -i\xi_1 + \xi_2 \\ -\xi_1 - i\xi_2 \end{bmatrix}$$

which implies that $0 = -i\xi_1 + \xi_2$. If we choose $\xi_1 = 1$, then we must have $\xi_2 = i\xi_1 = i$. Hence we let $\xi = \begin{bmatrix} 1 \\ i \end{bmatrix}$. Thus

$$\mathbf{x}^{(1)}(t) = \xi^{(1)} e^{r_1 t} = \begin{bmatrix} 1 \\ i \end{bmatrix} e^{2t}(\cos t + i \sin t) = \begin{bmatrix} e^{2t}\cos t \\ -e^{2t}\sin t \end{bmatrix} + i \begin{bmatrix} e^{2t}\sin t \\ e^{2t}\cos t \end{bmatrix}.$$

It follows that the general solution to the linear system is

$$\mathbf{x}(t) = c_1 \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} \sin t \\ \cos t \end{bmatrix} e^{2t}$$

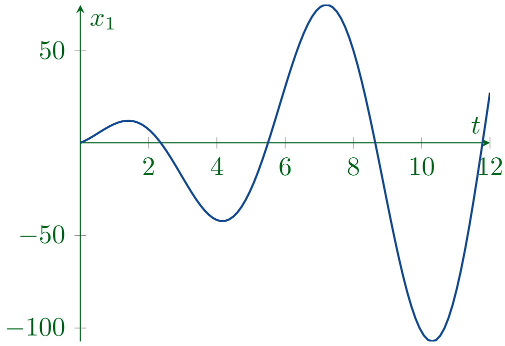
for constants c_1 and c_2.
Finally we use the initial condition $\mathbf{x}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ to find the constants. We calculate that

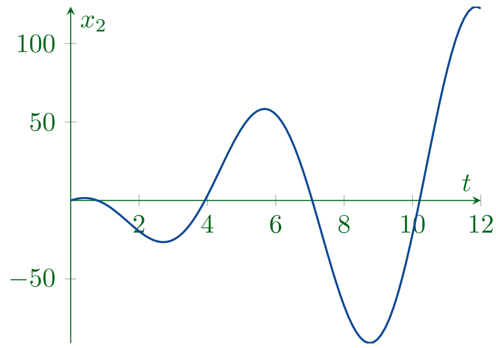
$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \mathbf{x}(0) = c_1 \begin{bmatrix} \cos 0 \\ -\sin 0 \end{bmatrix} e^0 + c_2 \begin{bmatrix} \sin 0 \\ \cos 0 \end{bmatrix} e^0 = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

which implies that $c_1 = 1 = c_2$. Therefore the solution is

$$\mathbf{x}(t) = \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix} e^{2t} + \begin{bmatrix} \sin t \\ \cos t \end{bmatrix} e^{2t} = \begin{bmatrix} \cos t + \sin t \\ \cos t - \sin t \end{bmatrix} e^{2t}$$

which looks like this:





(b) 5 points Give a fundamental matrix for the above system.

Solution: Since the general solution to this linear system is

$$\mathbf{x}(t) = c_1\mathbf{u}(t) + c_2\mathbf{v}(t) = c_1 \begin{bmatrix} e^{2t}\cos t \\ -e^{2t}\sin t \end{bmatrix} + c_2 \begin{bmatrix} e^{2t}\sin t \\ e^{2t}\cos t \end{bmatrix}$$

it follows that a fundamental matrix for this linear system is

$$\Psi(t) = \begin{bmatrix} \mathbf{u}(t) & \mathbf{v}(t) \end{bmatrix} \begin{bmatrix} e^{2t}\cos t & e^{2t}\sin t \\ -e^{2t}\sin t & e^{2t}\cos t \end{bmatrix}.$$