FORENAME:

SURNAME:

STUDENT NO:

DEPARTMENT:

TEACHER: ☐ Neil Course ☐ Vasfi Eldem ☐ M.Tuba Gülpınar ☐ Hasan Özekes

SIGNATURE:

Question	Points	Score
1	25	
2	20	
3	25	
4	30	
Total:	100	

- The time limit is 80 minutes.
- Any attempts at cheating or plagiarizing and assisting of such actions in any form would result in getting an automatic zero (0) from the exam. Disciplinary action will also be taken in accordance with the regulations of the Council of Higher Education.
- Give your answers in exact form (for example $\frac{\pi}{3}$ or $5\sqrt{3}$), except as noted in particular problems.
- Calculators, mobile phones, smart watches, etc. are not allowed.
- In order to receive credit, you must **show all of your work**. If you do not indicate the way in which you solved a problem, you may get little or no credit for it, even if your answer is correct.
- Place a box around your answer to each question.
- Use a **BLUE ball-point pen** to fill the cover sheet. Please make sure that your exam is complete.
- Do not write in the table above.

Elementary Laplace Transforms: Suppose that $a, b \in \mathbb{R}$, $n \in \mathbb{N}$, and $\mathcal{L}\{f(t)\}$ exists and $F(s) = \mathcal{L}\{f(t)\}$

- $\mathcal{L}\{1\} = \frac{1}{s}, s > 0$
- $\mathcal{L}\{e^{at}\} = \frac{1}{s-a}, \ s > a,$
- $\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}, \ s > 0,$
- $\mathcal{L}\{t^n e^{at}\} = \frac{n!}{(s-a)^{n+1}}$
- $\mathcal{L}\{e^{at}\sin bt\} = \frac{b}{(s-a)^2+b^2}$
- $f * g = \int_0^t f(t-\tau)g(\tau)d\tau$
- $\mathcal{L}\{\cos at\} = \frac{s}{s^2+a^2}, s > 0$
- $\mathcal{L}\{\sin at\} = \frac{a}{s^2+a^2}, \ s > 0$
- $\mathcal{L}\{\cosh at\} = \frac{s}{s^2-a^2}, \ s > |a|$
- $\mathcal{L}\{\sinh at\} = \frac{a}{s^2-a^2}, \ s > |a|$
- $\mathcal{L}\{e^{at}\cos bt\} = \frac{s-a}{(s-a)^2+b^2}$
- $f * g = g * f = \int_0^t g(t-\tau)f(\tau)d\tau$
- $\mathcal{L}\{f(ct)\} = \frac{1}{c}F(\frac{s}{c}), \ c > 0$
- $\mathcal{L}\{u_c(t)f(t-c)\} = e^{-cs}\mathcal{L}\{f(t)\}$
- $\mathcal{L}\{u_c(t)\} = \frac{e^{-cs}}{s}, \ s > 0$
- $\mathcal{L}\{e^{at}f(t)\} = F(s-a)$
- $\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n F(s)}{ds^n}$
- $\mathcal{L}\{f * g\} = F(s)G(s)$

1. (a) [15 points] Find the solution of the following initial value problem. $y''' - 4y'' - 12y' = 0, y(0) = 1, \ y'(0) = -6, \ y''(0) = 12$.

Solution: The characteristic equation of the given differential equatin is

$$r^3 - 4r^2 - 12r = 0 \Rightarrow r(r^2 - 4r - 12) = 0 \Rightarrow r(r-6)(r+2) = 0 \Rightarrow r_1 = 0, r_2 = 6, r_3 = -2$$

The general solution of the differential equation is

$$\begin{aligned} y(t) &= c_1e^{0t} + c_2e^{6t} + c_3e^{-2t} \\ y(t) &= c_1 + c_2e^{6t} + c_3e^{-2t} \\ y(0) = 1 &\Rightarrow c_1 + c_2 + c_3 = 1 \\ y'(0) = -6 \Rightarrow y'(t) = 6c_2e^{6t} - 2c_3e^{-2t} &\Rightarrow 6c_2 - 2c_3 = -6 \\ y''(0) = 12 \Rightarrow y'(t) = 36c_2e^{6t} + 4c_3e^{-2t} &\Rightarrow 36c_2 + 4c_3 = 12 \\ c_1 &= -2, c_2 = 0, c_3 = 3 \\ y(t) &= -2 + 3e^{-2t} \end{aligned}$$

(b) 10 points Find the general solution of $\dfrac{dy}{dx} = -\dfrac{2x(y^2+1)}{y}$.

Solution:

$$\frac{dy}{dx} = -\frac{2x(y^2+1)}{y} \Rightarrow \frac{y}{y^2+1}dy = -2xdx \Rightarrow \int \frac{y}{y^2+1}dy = \int -2xdx \Rightarrow \frac{1}{2}\ln(y^2+1) = -x^2 + C$$

2. (a) 10 points Find the Laplace Transform of $f(t) = \int_0^t (t-\tau)^2 \cos 3\tau d\tau$.

Solution:

$$\mathcal{L}\{f * g\} = \mathcal{L}\{f(t)\}.\mathcal{L}\{g(t)\}$$

$$t^2 * \cos 3t = \int_0^t (t-\tau)^2 \cos 3\tau d\tau$$

$$\Rightarrow \mathcal{L}\left\{\int_0^t (t-\tau)^2 \cos 3\tau d\tau\right\} = \mathcal{L}\left\{t^2 * \cos 3t\right\} = \mathcal{L}\left\{t^2\right\}\mathcal{L}\{\cos 3t\} = \frac{2}{s^3}.\frac{s}{s^2+9}$$

(b) 10 points Write the convolution integral that represents the inverse Laplace Transform of $F(s) = \dfrac{s}{s^2(s^2+4)}$.

Solution:

$$\mathcal{L}^{-1}\left\{\frac{s}{s^2(s^2+4)}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2}\frac{s}{s^2+4}\right\} = t * \cos 2t = \int_0^t (t-\tau)\cos 2\tau d\tau$$

3. (a) [3 points] Is the differential equation $(3x^2 - y)dx + xdy = 0$ exact?

Solution: First of all, let us take $M(x,y) = 3x^2 - y$ and $N(x,y) = x$. $M_y = -1$, $N_x = 1$ It is not an exact equation.

(b) [5 points] If the answer is no, then find the integrating factor $\mu(x)$ which depends on x such that $\mu(x)(3x^2 - y)dx + \mu(x)xdy = 0$ is an exact differential equation.

Solution:

$$\frac{M_y - N_x}{N} = \frac{-1-1}{x} = -\frac{2}{x} \Rightarrow \mu(x) = e^{\int -\frac{2}{x}dx} = e^{-2\ln x} = \frac{1}{x^2}$$

(c) [17 points] Find the general solution of $(3x^2 - y)dx + xdy = 0$.

Solution:

$$\mu(x)\left[M(x,y)dx + N(x,y)dy = 0\right] \Rightarrow \frac{1}{x^2}\left[(3x^2 - y)dx + xdy = 0\right] \Rightarrow \left(3 - \frac{y}{x^2}\right)dx + \frac{1}{x}dy = 0$$

$$P(x,y) = 3 - \frac{y}{x^2} \text{ and } Q(x,y) = \frac{1}{x} \Rightarrow P_y = -\frac{1}{x^2} = Q_x \text{ Exact Differential Equation}$$

Therefore, there exists a function $F(x,y) = 0$ such that $F_x dx + F_y dy = 0$.

$$F_x = 3 - \frac{y}{x^2} \text{ and } F_y = \frac{1}{x}$$

$$F(x,y) = \int \frac{1}{x}dy \Rightarrow F(x,y) = \frac{y}{x} + g(x)$$

$$F_x = -\frac{y}{x^2} + g'(x) = 3 - \frac{y}{x^2} \Rightarrow g'(x) = 3 \Rightarrow g(x) = 3x + C$$

$$F(x,y) = \frac{y}{x} + 3x + C = 0$$

4. (a) 10 points Find the general solution of the system $\mathbf{x}' = A\mathbf{x}$ where $A = \begin{bmatrix} 3 & -1 \\ 4 & -1 \end{bmatrix}$ with eigenvalues $\lambda_1 = \lambda_2 = 1$ and the corresponding eigenvector to λ is $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

Solution:

$$(A - \lambda I)\mathbf{w} = \mathbf{v} \Rightarrow \left[\begin{array}{cc|c} 2 & -1 & 1 \\ 4 & -2 & 2 \end{array}\right] \sim \left[\begin{array}{cc|c} 2 & -1 & 1 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow \mathbf{w} = \begin{bmatrix} w_1 \\ 2w_1 - 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} w_1 + \begin{bmatrix} 0 \\ -1 \end{bmatrix} \Rightarrow \mathbf{w} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

$$x(t) = c_1 \mathbf{v} e^t + c_2 [\mathbf{v}t + \mathbf{w}] e^t = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^t + c_2 \left[\begin{bmatrix} 1 \\ 2 \end{bmatrix} t + \begin{bmatrix} 0 \\ -1 \end{bmatrix} \right] e^t$$

$$x(t) = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^t + c_2 \begin{bmatrix} t \\ 2t - 1 \end{bmatrix} e^t$$

(b) 20 points Find the solution of the initial value problem $\mathbf{x}' = \begin{bmatrix} 3 & -1 \\ 4 & -1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 5 \\ 4e^{-t} \end{bmatrix}$, $\mathbf{x}(0) = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$.

Solution:

$$\mathbf{x}_p = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} e^{-t} \Rightarrow \mathbf{x}_p' = -\begin{bmatrix} B_1 \\ B_2 \end{bmatrix} e^{-t}$$

$$\mathbf{x}_p' = \begin{bmatrix} 3 & -1 \\ 4 & -1 \end{bmatrix} \mathbf{x}_p + \begin{bmatrix} 5 \\ 2e^{-t} \end{bmatrix}$$

$$-\begin{bmatrix} B_1 \\ B_2 \end{bmatrix} e^{-t} = \begin{bmatrix} 3 & -1 \\ 4 & -1 \end{bmatrix} \left(\begin{bmatrix} A_1 \\ A_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} e^{-t} \right) + \begin{bmatrix} 5 \\ 4e^{-t} \end{bmatrix}$$

$$\begin{bmatrix} -B_1 \\ -B_2 \end{bmatrix} e^{-t} = \begin{bmatrix} 3A_1 - A_2 \\ 4A_1 - A_2 \end{bmatrix} + \begin{bmatrix} 3B_1 - B_2 \\ 4B_1 - B_2 \end{bmatrix} e^{-t} + \begin{bmatrix} 5 \\ 4e^{-t} \end{bmatrix} = \begin{bmatrix} 3A_1 - A_2 + 5 \\ 4A_1 - A_2 \end{bmatrix} + \begin{bmatrix} 3B_1 - B_2 \\ 4B_1 - B_2 + 4 \end{bmatrix} e^{-t}$$

$$\begin{bmatrix} 3A_1 - A_2 + 5 \\ 4A_1 - A_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} 3B_1 - B_2 \\ 4B_1 - B_2 + 4 \end{bmatrix} = \begin{bmatrix} -B_1 \\ -B_2 \end{bmatrix} \Rightarrow \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 20 \end{bmatrix} \text{ and } \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = \begin{bmatrix} -1 \\ -4 \end{bmatrix}$$

$$\mathbf{x}_p = \begin{bmatrix} 5 \\ 20 \end{bmatrix} + \begin{bmatrix} -1 \\ -4 \end{bmatrix} e^{-t}$$

The general solution of the system is

$$\mathbf{x}(t) = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^t + c_2 \begin{bmatrix} t \\ 2t - 1 \end{bmatrix} e^t + \begin{bmatrix} 5 \\ 20 \end{bmatrix} + \begin{bmatrix} -1 \\ -4 \end{bmatrix} e^{-t}$$

$$\mathbf{x}(0) = \begin{bmatrix} 3 \\ -1 \end{bmatrix} \Rightarrow \mathbf{x}(0) = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^0 + c_2 \begin{bmatrix} 0 \\ -1 \end{bmatrix} e^0 + \begin{bmatrix} 5 \\ 20 \end{bmatrix} + \begin{bmatrix} -1 \\ -4 \end{bmatrix} e^0 = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} c_1 + 5 - 1 \\ 2c_1 - c_2 + 20 - 4 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix} \Rightarrow \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 15 \end{bmatrix}$$

$$\mathbf{x}(t) = -\begin{bmatrix} 1 \\ 2 \end{bmatrix} e^t + 15 \begin{bmatrix} t \\ 2t - 1 \end{bmatrix} e^t + \begin{bmatrix} 5 \\ 20 \end{bmatrix} + \begin{bmatrix} -1 \\ -4 \end{bmatrix} e^{-t}$$

$$\mathbf{x}(t) = \begin{bmatrix} 15t - 1 \\ 30t - 17 \end{bmatrix} e^t + \begin{bmatrix} 5 \\ 20 \end{bmatrix} + \begin{bmatrix} -1 \\ -4 \end{bmatrix} e^{-t}$$