Your Name / Adınız - Soyadınız

Your Signature / İmza

Student ID # / Öğrenci No

Professor's Name / Öğretim Üyesi

Your Department / Bölüm

- In order to receive credit, you must **show all of your work**. If you do not indicate the way in which you solved a problem, you may get little or no credit for it, even if your answer is correct. **Show your work in evaluating any limits, derivatives**.
- Place a box around your answer to each question.
- Use a **BLUE ball-point pen** to fill the cover sheet. Please make sure that your exam is complete.
- Time limit is 70 min.

Do not write in the table to the right.

Problem	Points	Score
1	30	
2	25	
3	20	
4	25	
Total:	100	

1. (a) 15 Points Solve $y' - 4y = t^2e^{4t}$.

Solution: This is a linear differential equation of the form $y' + P(t)y = Q(t)$ where $P(t) = -4$ and $Q(t) = t^2e^{4t}$. Multiply by the integrating factor $\mu(t) = \exp\int(-4)\,\mathrm{d}t = e^{-4t}$. Hence

$$e^{-4t}y' - 4e^{-4t}y = e^{-4t}t^2e^{4t}$$

$$(e^{-4t}y)' = t^2 \Rightarrow e^{-4t}y = \int(t^2)\,\mathrm{d}t = \frac{1}{3}t^3 + C$$

$$\boxed{y = \frac{1}{3}t^3e^{4t} + Ce^{4t}}$$

p.491, pr.86

(b) 15 Points Solve the initial value problem:

$$y' = \frac{e^{-x} - e^x}{3+4y}, \quad y(0) = 2.$$

Solution: Separating the variables, we have

$$\frac{dy}{dx} = \frac{e^{-x} - e^x}{3+4y} \Rightarrow (3+4y)\,\mathrm{d}y = (e^{-x} - e^x)\,\mathrm{d}x \Rightarrow \int(3+4y)\,\mathrm{d}y = \int(e^{-x} - e^x)\,\mathrm{d}x$$

Hence we have

$$3y + 2y^2 = -e^{-x} - e^x + c$$

Impose the initial condition:

$$(3)(2) + (2)(2)^2 = -e^{-0} - e^0 + c \Rightarrow 6 + 8 = -1 - 1 + c \Rightarrow c = 16$$

Then the solution to IVP is

$$\boxed{3y + 2y^2 = -e^{-x} - e^x + 16}$$

p.491, pr.86

2. Given the equation $(3x^2y+2xy+y^3)+(x^2+y^2)y'=0$.

   (a) 10 Points Find a if there is an integrating factor of the form $\mu(x)=e^{ax}$.

Solution: Multiplying through by e^{ax} gives

$$e^{ax}(3x^2y+2xy+y^3)\,\mathrm{d}x+e^{ax}(x^2+y^2)\,\mathrm{d}y=0$$

an equation of the form $M\,\mathrm{d}x+N\,\mathrm{d}y=0$ where $M=e^{ax}(3x^2y+2xy+y^3)$ and $N=e^{ax}(x^2+y^2)$. Then $M_y=e^{ax}(3x^2+2x+3y^2)$ and $N_x=ae^{ax}(x^2+y^2)+2xe^{ax}$. Since e^{ax} is an integrating factor we must have $M_y=N_x$, that is we have

$$\cancel{e^{ax}}(3x^2+2x+3y^2)=a\cancel{e^{ax}}(x^2+y^2)+2x\cancel{e^{ax}}\Rightarrow 3x^2+2x+3=ax^2+ay^2\Rightarrow \boxed{a=3}$$

p.491, pr.86

   (b) 15 Points Solve the equation.

Solution: Re-write the equation as

$$e^{3x}(3x^2y+2xy+y^3)\,\mathrm{d}x+e^{3x}(x^2+y^2)\,\mathrm{d}y=0.$$

The general solution is of the form $F(x,y)=C$, where

$$\frac{\partial F}{\partial x}=e^{3x}(3x^2y+2xy+y^3)\text{ and }\frac{\partial F}{\partial y}=e^{3x}(x^2+y^2)$$

The second equation implies that F is of the form

$$F=x^2e^{3x}y+\frac{1}{3}y^3e^{3x}+\varphi(x)$$

Substituting this formula into the first equation gives

$$2xe^{3x}y+3x^2e^{3x}y+y^3e^{3x}+\varphi'(x)=e^{3x}(3x^2y+2xy+y^3)\Rightarrow\varphi'(x)=0\Rightarrow\varphi(x)=c_1$$

The general solution is given implicitly by

$$F=x^2e^{3x}y+\frac{1}{3}y^3e^{3x}=C$$

p.491, pr.86

3. [20 Points] Solve the initial values problem

$$y''' + y' = 0, \quad y(0) = 1, \quad y'(0) = 1, \quad y''(0) = 2.$$

Solution: The characteristic equations

$$0 = r^3 + r = r(r^2 + 1)$$

has roots $r_1 = 0$, $r_2 = i$ and $r_3 = -i$. Therefore the general solution to the homogeneous equation $y''' + y' = 0$ is

$$y(t) = c_1 + c_2 \cos t + c_3 \sin t$$

We next use the initial conditions:

$$\begin{cases} y(t) = c_1 + c_2 \cos t + c_3 \sin t \\ y'(t) = -c_2 \sin t + c_3 \cos t \\ y''(t) = -c_2 \cos t - c_3 \sin t \end{cases} \Rightarrow \begin{cases} y(0) = c_1 + c_2 = 1 \\ y'(0) = c_3 = 1 \\ y''(0) = -c_2 = 2 \end{cases} \Rightarrow \begin{cases} c_1 = 3 \\ c_3 = 1 \\ c_2 = -2 \end{cases}$$

The general solution to the initial value problem is thus

$$\boxed{y(t) = 3 - 2\cos t + \sin t}$$

p.583, pr.17

4. 25 Points Find the general solution of

$$y''' - y'' - y' + y = 4e^{-t} + 3.$$

Solution: The characteristic equations

$$0 = r^3 - r^2 - r + 1 = r^2(r-1) - (r-1) = (r-1)^2(r+1)$$

has roots $r_1 = 1$, $r_2 = 1$ and $r_3 = -1$. Therefore the general solution to the homogeneous equation $y''' - y'' - y' + y = 0$ is

$$y(t) = c_1 e^t + c_2 t e^t + c_3 e^{-t}.$$

Next we must find a particular solution to the non-homogeneous equation. Since 3 is a polynomial, we try the ansatz $y_p(t) = Ate^{-t} + B$. Since $y' = Ae^{-t} - Ate^{-t}$, $y'' = -Ae^{-t} - Ae^{-t} + Ate^{-t} = -2Ae^{-t} + Ate^{-t}$ and $y''' = 3Ae^{-t} - Ate^{-t}$, we get

$$4e^{-t} + 3 = y''' - y'' - y' + y = Ae^{-t} + B.$$

Clearly we require $4A = 4$ and $B = 3$. Therefore

$$y_p(t) = te^{-t} + 3$$

is a particular solution.
The general solution to the non-homogeneous equation is thus

$$y(t) = c_1 e^t + c_2 t e^t + c_3 e^{-t} + te^{-t} + 3.$$

p.573, pr.38