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FORENAME:

SURNAME:

STUDENT NO:

DEPARTMENT:

TEACHER: ☐ M.Tuba Gülpınar ☐ Sezgin Sezer

SIGNATURE:

Question	Points	Score
1	20	
2	22	
3	23	
4	10	
5	25	
Total:	100	

- The time limit is 75 minutes.
- Give your answers in exact form (for example $\frac{\pi}{3}$ or $5\sqrt{3}$), except as noted in particular problems.
- Calculators, mobile phones, smart watches, etc. are not allowed.
- In order to receive credit, you must **show all of your work**. If you do not indicate the way in which you solved a problem, you may get little or no credit for it, even if your answer is correct.
- Place a box around your answer to each question.
- Use a **BLUE ball-point pen** to fill the cover sheet. Please make sure that your exam is complete.
- Do not write in the table above.

1. 20 points Find the general solution of $\dfrac{d^2y}{dt^2} + \dfrac{4}{t}\dfrac{dy}{dt} = \dfrac{5}{t^4}$ where $t > 0$.

Solution: It is a reducable (dependent variable missing) differential equation. Therefore

$$p = y' \Rightarrow \frac{dp}{dt} = \frac{d^2y}{dt^2}$$

$$\frac{d^2y}{dt^2} + \frac{4}{t}\frac{dy}{dt} = \frac{5}{t^4} \Rightarrow \frac{dp}{dt} + \frac{4}{t}p = \frac{5}{t^4}$$

It is a linear differential equation. Let us find the integrating factor to find the general solution.

$$\lambda(x) = e^{\int \frac{4}{t}dt} = e^{4\ln t} = t^4$$

$$t^4 p = \int t^4 \frac{5}{t^4} = \int 5dt = 5t + C_1$$

$$p = \frac{5t}{t^4} + \frac{C_1}{t^4}$$

$$y' = \frac{5}{t^3} + \frac{C_1}{t^4}$$

$$\int dy = \int \left(\frac{5}{t^3} + \frac{C_1}{t^4}\right) dt$$

$$y(t) = -\frac{5}{2t^2} - \frac{C_1}{3t^3} + C_2$$

2. 22 points Solve the initial value problem $(xy - xe^x)dx + (x^2 + x\cos y)dy = 0$, $y(0) = 0$.

Solution:

$$\lambda = e^{\int \frac{M_y - N_x}{N} dx} = e^{\int \frac{x-2x-\cos y}{x^2+x\cos y} dx} = e^{\int \frac{-x-\cos y}{x(x+\cos y)} dx} = e^{int - \frac{1}{x} dx} = e^{-\ln x} = \frac{1}{x}$$

$$\frac{1}{x}\big((xy - xe^x)dx + (x^2 + x\cos y)dy = 0\big)$$

$$(y - e^x)dx + (x + \cos y)dy = 0$$

Let us take $P(y,x) = y - e^x$ and $Q(x,y) = x + \cos y$. $P_y = 1 = Q_x$ Therefore, it is an exact differential equation. There exists a function $F(x,y) = 0$ such that$F_x dx + F_y dy = 0$.

$$F_x = y - e^x \Rightarrow F(x,y) = \int (y - e^x)dx = yx - e^x + h(y)$$

$$F_y = x + \cos y \Rightarrow F_y = x + h'(y) = x + \cos y$$

$$h'(y) = \cos y \Rightarrow h(y) = \int \cos y dy = \sin y + C$$

$$F(x,y) = yx - e^x + \sin y + C = 0$$

$$y(0) = 0 \Rightarrow 0 - e^0 + \sin 0 + C = 0 \Rightarrow C = 1$$

$$yx - e^x + \sin y + 1 = 0$$

3. 23 points Find the solution of the initial value problem $y' = \dfrac{2y}{3x+y}$, $y(1) = 2$ by using the substitution $y = vx$.

Solution: It is a homogeneous differential equation.

$$y = vx \Rightarrow y' = xv' + v$$
$$y' = \frac{2y}{3x+y} \Rightarrow xv' + v = \frac{2vx}{3x+vx}$$
$$xv' + v = \frac{2v}{3+v}$$
$$x\frac{dv}{dx} = \frac{2v}{3+v} - v = -\frac{v+v^2}{3+v}$$
$$\frac{3+v}{v(v+1)}dv = -\frac{1}{x}dx$$
$$\int \frac{3+v}{v(v+1)}dv = \int -\frac{1}{x}dx$$
$$\int \left(\frac{3}{v} - \frac{3}{v+1}\right) = \int -\frac{1}{x}dx$$
$$3\ln|v| - 3\ln|v+1| = -\ln|x| + C$$
$$3\ln\left|\frac{v}{v+1}\right| = -\ln|x| + C$$
$$3\ln\left|\frac{\frac{y}{x}}{\frac{y}{x}+1}\right| = -\ln|x| + C \Rightarrow 3\ln\left|\frac{y}{y+x}\right| = -\ln|x| + C$$
$$\left(\frac{y}{y+x}\right)^3 = \frac{A}{x}\text{where } (A = e^C)$$
$$y(1) = 2 \Rightarrow \left(\frac{2}{2+1}\right)^3 = \frac{A}{1} \Rightarrow A = \frac{8}{27}$$
$$\left(\frac{y}{y+x}\right)^3 = \frac{8}{27x}$$

4. 10 points Find the general solution of $y''' - 2y'' + 17y' = 0$.

Solution: Let us find the characteristic equation and its roots.

$$r^3 - 2r^2 + 17r = 0 \Rightarrow r(r^2 - 2r + 17) = 0 \Rightarrow r[(r-1)^2 + 16] = 0 \Rightarrow r_1 = 0, r_2 = 1 + 4i, r_3 = 1 - 4i$$
$$y(x) = c_1 e^{0x} + c_2 e^x \cos 4x + c_3 e^x \sin 4x$$
$$y(x) = c_1 + c_2 e^x \cos 4x + c_3 e^x \sin 4x$$

5. 25 points Find the general solution of $y'' - 4y' + 4y = xe^{2x} + \cos x$.

Solution:

$$r^2 - 4r + 4 = 0 \Rightarrow (r-2)^2 = 0 \Rightarrow r_1 = r_2 = 2 \Rightarrow y_h(x) = c_1e^{2x} + c_2xe^{2x}$$

$$y_p(x) = (Ax+B)x^2e^{2x} + C\cos x + D\sin x = (Ax^3 + Bx^2)e^{2x} + C\cos x + D\sin x$$

$$y_p'(x) = (3Ax^2 + 2Bx)e^{2x} + 2(Ax^3 + Bx^2)e^{2x} - C\sin x + D\cos x$$

$$y_p' = 2Ax^3e^{2x} + (3A+2B)x^2e^{2x} + 2Bxe^{2x} - C\sin x + D\cos x$$

$$y_p''(x) = (6Ax + 2B)e^{2x} + 4(3Ax^2 + 2Bx)e^{2x} + 4(Ax^3 + Bx^2)e^{2x} - C\cos x - D\sin x$$

$$y_p''(x) = 4Ax^3e^{2x} + (12A + 4B)x^2e^{2x} + (6A + 8B)xe^{2x} + 2Be^{x} - C\cos x - D\sin x$$

$$y'' - 4y' + 4y = xe^{2x} + \cos x$$

$$\left[4Ax^3e^{2x} + (12A+4B)x^2e^{2x} + (6A+8B)xe^{2x} + 2Be^{x} - C\cos x - D\sin x\right]$$

$$-4[2Ax^3e^{2x} + (3A+2B)x^2e^{2x} + 2Bxe^{2x} - C\sin x + D\cos x]$$

$$+4[(Ax^3 + Bx^2)e^{2x} + C\cos x + D\sin x] = xe^{2x} + \cos x$$

$$(4A - 8A + 4A)x^3e^{2x} + (12A + 4B - 12A - 8B + 4B)x^2e^{2x} + (6A + 8B - 8B)xe^{2x} + 2Be^{2x}$$

$$+(-C - 4D + 4C)\cos x(-D + 4C + 4D)\sin x = xe^{2x} + \cos x$$

$$6Axe^{2x} + 2Be^{2x} + (3C - 4D)\cos x(4C + 3D)\sin x = xe^{2x} + \cos x$$

$$A = \frac{1}{6}, B = 0, C = \frac{3}{25}, D = \frac{4}{25}$$

$$y_p = \frac{1}{6}x^3e^{2x} + \frac{3}{25}\cos x + \frac{4}{25}\sin x$$

$$y(x) = y_h + y_p = c_1e^{2x} + c_2xe^{2x} + \frac{1}{6}x^3e^{2x} + \frac{3}{25}\cos x + \frac{4}{25}\sin x$$