FORENAME:

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DEPARTMENT:

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SIGNATURE:

Question	Points	Score
1	15	
2	35	
3	20	
4	30	
Total:	100	

- The time limit is 75 minutes.
- Give your answers in exact form (for example $\frac{\pi}{3}$ or $5\sqrt{3}$), except as noted in particular problems.
- Calculators, mobile phones, smart watches, etc. are not allowed.
- In order to receive credit, you must **show all of your work**. If you do not indicate the way in which you solved a problem, you may get little or no credit for it, even if your answer is correct.
- Place [a box around your answer] to each question.
- Use a **BLUE ball-point pen** to fill the cover sheet. Please make sure that your exam is complete.
- Do not write in the table above.

Elementary Laplace Transforms: Suppose that $a, b \in \mathbb{R}$, $n \in \mathbb{N}$, and $\mathcal{L}\{f(t)\}$ exists and $F(s) = \mathcal{L}\{f(t)\}$

- $\mathcal{L}\{1\} = \frac{1}{s}, s > 0$
- $\mathcal{L}\{e^{at}\} = \frac{1}{s-a}, \ s > a,$
- $\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}, \ s > 0,$
- $\mathcal{L}\{t^n e^{at}\} = \frac{n!}{(s-a)^{n+1}}$
- $\mathcal{L}\{e^{at}\sin bt\} = \frac{b}{(s-a)^2+b^2}$
- $f * g = \int_0^t f(t-\tau)g(\tau)d\tau$
- $\mathcal{L}\{\cos at\} = \frac{s}{s^2+a^2}, s > 0$
- $\mathcal{L}\{\sin at\} = \frac{a}{s^2+a^2}, \ s > 0$
- $\mathcal{L}\{\cosh at\} = \frac{s}{s^2-a^2}, \ s > |a|$
- $\mathcal{L}\{\sinh at\} = \frac{a}{s^2-a^2}, \ s > |a|$
- $\mathcal{L}\{e^{at}\cos bt\} = \frac{s-a}{(s-a)^2+b^2}$
- $f * g = g * f = \int_0^t g(t-\tau)f(\tau)d\tau$
- $\mathcal{L}\{f(ct)\} = \frac{1}{c}F(\frac{s}{c}), \ c > 0$
- $\mathcal{L}\{u_c(t)f(t-c)\} = e^{-cs}\mathcal{L}\{f(t)\}$
- $\mathcal{L}\{u_c(t)\} = \frac{e^{-cs}}{s}, \ s > 0$
- $\mathcal{L}\{e^{at}f(t)\} = F(s-a)$
- $\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n F(s)}{ds^n}$
- $\mathcal{L}\{f * g\} = F(s)G(s)$

1. [15 points] Find the general solution of the $y'' - 2y' + y = \frac{e^t}{t^2+1}$ with $y_h(t) = c_1 e^t + c_2 t e^t$.

Solution: $y_c(t) = u_1 e^t + u_2 t e^t$

$$\begin{aligned} u_1' e^t + u_2' t e^t &= 0 \\ u_1' e^t + u_2' e^t + u_2' t e^t &= \frac{e^t}{t^2+1} \end{aligned} \Rightarrow \begin{aligned} u_1' + u_2' t &= 0 \\ u_1' + u_2' + u_2' t &= \frac{1}{t^2+1} \end{aligned}$$

$$u_1' = -\frac{t}{t^2+1} \Rightarrow u_1 = \int -\frac{t}{t^2+1}dt = -\frac{1}{2}\ln(t^2+1) + c_1$$

$$u_2' = \frac{1}{t^2+1} \Rightarrow u_2 = \int \frac{1}{t^2+1}dt = \arctan t + c_2$$

$$y_c(t) = -\frac{1}{2}\ln(t^2+1)e^t + te^t \arctan t$$

$$y(t) = c_1 e^t + c_2 t e^t - \frac{e^t \ln(t^2+1)}{2} + te^t \arctan t$$

2. (a) 10 points Calculate the Laplace Transform of $f(t) = 3e^{-t} + 4t^3 - 2e^{2t}\cos 3t$.

Solution:

$$\mathcal{L}\{f(t)\} = \mathcal{L}\left\{3e^{-t} + 4t^3 - 2e^{2t}\cos 3t\right\} = 3\mathcal{L}\left\{e^{-t}\right\} + 4\mathcal{L}\left\{t^3\right\} - 2\mathcal{L}\left\{e^{2t}\cos 3t\right\}$$
$$F(s) = 3\frac{1}{s+1} + 4\frac{3!}{s^4} - 2\frac{s-2}{(s-2)^2+9}$$
$$F(s) = \frac{3}{s+1} + \frac{24}{s^4} - \frac{2s-4}{s^2-4s+13}$$

(b) 15 points Find the inverse Laplace Transfonm of $F(s) = \dfrac{6s^3+13s^2+2s+10}{s(s+2)(s^2+1)}$.

Solution:

$$\frac{6s^3+13s^2+2s+10}{s(s+2)(s^2+1)} = \frac{A}{s} + \frac{B}{s+2} + \frac{Cs+D}{s^2+1}$$
$$6s^3+13s^2+2s+10 = A(s+2)(s^2+1) + Bs(s^2+1) + (Cs+D)s(s+2)$$
$$s=0 \Rightarrow 10 = 2A \Rightarrow A = 5$$
$$s=-2 \Rightarrow -48+52-4+10 = -10B \Rightarrow B = -1$$
$$s=1 \Rightarrow 6+13+2+10 = 6A+2B+3(C+D) \Rightarrow C+D = 1$$
$$s=-1 \Rightarrow -6+13-2+10 = 2A-2B+C-D \Rightarrow C-D = 3 \Rightarrow C = 2, D = -1$$
$$\frac{6s^3+13s^2+2s+10}{s(s+2)(s^2+1)} = \frac{5}{s} + \frac{-1}{s+2} + \frac{2s-1}{s^2+1}$$
$$\mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{6s^3+13s^2+2s+10}{s(s+2)(s^2+1)}\right\} = \mathcal{L}^{-1}\left\{\frac{5}{s} - \frac{1}{s+2} + \frac{2s}{s^2+1} - \frac{1}{s^2+1}\right\}$$
$$f(t) = 5 - e^{-2t} + 2\cos t - \sin t$$

(c) 10 points Find the Laplace Transform of $f(t) = \int_0^t (t-\tau)^3 e^{3\tau}\,d\tau$.

Solution:

$$\mathcal{L}\{f*g\} = \mathcal{L}\{f(t)\}.\mathcal{L}\{g(t)\}$$
$$t^3 * e^{3t} = \int_0^t (t-\tau)^3 e^{3\tau}\,d\tau$$
$$\Rightarrow \mathcal{L}\left\{\int_0^t (t-\tau)^3 e^{3\tau}\,d\tau\right\} = \mathcal{L}\left\{t^3 * e^{3t}\right\} = \mathcal{L}\left\{t^3\right\}\mathcal{L}\left\{e^{3t}\right\} = \frac{3!}{s^4}.\frac{1}{s-3} = \frac{6}{s^4(s-3)}$$

3. 20 points Find the solution of the following initial value problem by using Laplace Transform.

$$y'' - 3y' - 18y = 0, \quad y(0) = 5, \quad y'(0) = 12$$

Solution:

$$\mathcal{L}\{y'' - 3y' - 18y\} = \mathcal{L}\{0\}$$
$$\left[s^2\mathcal{L}\{y\} - sy(0) - y'(0)\right] - 3\left[s\mathcal{L}\{y\} - y(0)\right] - 18\mathcal{L}\{y\} = 0$$
$$\left[s^2\mathcal{L}\{y\} - 5s - 12\right] - 3\left[s\mathcal{L}\{y\} - 5\right] - 18\mathcal{L}\{y\} = 0$$
$$(s^2 - 3s - 18)\mathcal{L}\{y\} = 5s - 3$$
$$\mathcal{L}\{y\} = \frac{5s-3}{(s-6)(s+3)}$$
$$\frac{5s-3}{(s-6)(s+3)} = \frac{A}{s-6} + \frac{B}{s+3}$$
$$5s - 3 = A(s+3) + B(s-6)$$
$$s = 6 \Rightarrow 27 = 9A \Rightarrow A = 3$$
$$s = -3 \Rightarrow -18 = -9B \Rightarrow B = 2$$
$$y(t) = \mathcal{L}^{-1}\left\{\frac{5s-3}{(s-6)(s+3)}\right\} = \mathcal{L}^{-1}\left\{\frac{3}{s-6} + \frac{2}{s+3}\right\}$$
$$y(t) = 3e^{6t} + 2e^{-3t}$$

4. 30 points Find the solution of the following initial value problem by using Laplace Transform.

$$y'' + 2y' + y = f(t) \quad , y(0) = 0, \quad y'(0) = 0 \text{ where } f(t) = \begin{cases} 2\cos t, & 0 < t < \pi \\ 0, & t \geq \pi \end{cases} .$$

Solution:

$$f(t) = 2\cos t - 2u_\pi(t)\cos t$$

$$\mathcal{L}\{y'' + 2y' + y\} = \mathcal{L}\{f(t)\}$$

$$\left[s^2\mathcal{L}\{y\} - sy(0) - y'(0)\right] + 2\left[s\mathcal{L}\{y\} - y(0)\right] + \mathcal{L}\{y\} = \mathcal{L}\{2\cos t - 2u_\pi(t)\cos t\}$$

$$(s^2 + 2s + 1)\mathcal{L}\{y\} = \frac{2s}{s^2+1} - e^{-\pi s}\mathcal{L}\{2\cos(t+\pi)\} = \frac{2s}{s^2+1} - e^{-\pi s}\mathcal{L}\{-2\cos t\}$$

$$(s^2 + 2s + 1)\mathcal{L}\{y\} = \frac{2s}{s^2+1} + e^{-\pi s}\frac{2s}{s^2+1}$$

$$\mathcal{L}\{y\} = \frac{2s}{(s^2+1)(s+1)^2} + e^{-\pi s}\frac{2s}{(s^2+1)(s+1)^2}$$

$$\frac{2s}{(s^2+1)(s+1)^2} = \frac{A}{s+1} + \frac{B}{(s+1)^2} + \frac{Cs+D}{s^2+1}$$

$$2s = A(s+1)(s^2+1) + B(s^2+1) + (Cs+D)(s+1)^2$$

$$s = -1 \Rightarrow -2 = 2B \Rightarrow B = -1$$

$$s^2 + 2s + 1 = (s+1)^2 = A(s+1)(s^2+1) + (Cs+D)(s+1)^2$$

$$s + 1 = A(s^2+1) + (Cs+D)(s+1)$$

$$s = -1 \Rightarrow 0 = 2A \Rightarrow A = 0$$

$$s + 1 = (Cs+D)(s+1) \Rightarrow 1 = (Cs+D) \Rightarrow C = 0, D = 1$$

$$\mathcal{L}^{-1}\left\{\frac{2s}{(s^2+1)(s+1)^2}\right\} = \mathcal{L}^{-1}\left\{-\frac{1}{(s+1)^2} + \frac{1}{s^2+1}\right\} = -te^{-t} + \sin t$$

$$y(t) = \mathcal{L}^{-1}\left\{\frac{2s}{(s^2+1)(s+1)^2} + e^{-\pi s}\frac{2s}{(s^2+1)(s+1)^2}\right\}$$

$$y(t) = \mathcal{L}^{-1}\left\{-\frac{1}{(s+1)^2} + \frac{1}{s^2+1}\right\} + \mathcal{L}^{-1}\left\{e^{-\pi s}\left(-\frac{1}{(s+1)^2} + \frac{1}{s^2+1}\right)\right\}$$

$$y(t) = -te^{-t} + \sin t + u_\pi(t)\left(-(t-\pi)e^{-(t-\pi)} + \sin(t-\pi)\right)$$

$$y(t) = -te^{-t} + \sin t + u_\pi(t)\left(-(t-\pi)e^{-(t-\pi)} - \sin t\right)$$