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DEPARTMENT OF MATHEMATICS
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Math 352
Complex Calculus II
Worksheet 1

Problems
Evaluate the following integrals, where C is the circle $|z| = 5$ in the positive sense

1.

$$\int_{\mathcal{C}} \frac{\sin z}{(z^2-1)(z+10)}\, dz$$

2.

$$\int_{\mathcal{C}} \frac{e^{\frac{1}{z}}\left(4z^3-3\right)}{z\left(z^3+1\right)}\, dz$$

3.

$$\int_{\mathcal{C}} \frac{z+2}{e^{2z}-1}\, dz$$

Use residues and contours to evaluate the following improper integral

4.

$$\int_{,}^{\infty} \frac{1}{(x^2+1)(x^2+4)}\, dz$$

Compute the Cauchy Principal Value for the following integral.

5.

$$\int_{-\infty}^{\infty} \frac{1}{(x^2+1)(x^2+4)}\, dz$$

Find the Taylor series of the given function, about the given point z_0.

(a)

$$\mathsf{f}(z) = \mathsf{e}^z,\ \mathsf{z}_0 = \frac{i\pi}{2}$$

(b)

$$\mathsf{f}(z) = \frac{z}{4+z^2}, \mathsf{z}_0 = 0$$

(a) Find the Laurent series which represents

$$f(z) = z \sin\left(\frac{1}{z^2}\right)$$

in the region $0 < |z| < \infty$.

(b) Evaluate the following integral

$$\int_C z \sin\left(\frac{1}{z^2}\right) dz$$

6. Evaluate the integral

$$\int_C \frac{z e^{\frac{1}{z}}}{1-z} dz$$

where C is the circle of radius 2, center 0, positively oriented.

7. Find the residue at $z = 0$ for the following function

$$f(z) = \frac{1}{z^2(3-z)}$$

Use this residue to evaluate

$$\int_C \frac{1}{z^2(3-z)} dz$$

where C is the circle of radius 2, center 0, positively oriented.

8. Show that

$$\mathbf{I} = \int_0^\infty \frac{x^6}{\left(a^4 + x^4\right)^2} dx = \frac{3\sqrt{2}}{16a}$$

where a is real and positive.

9. For a real and positive show that

$$\mathbf{I}_1 = \int_0^\infty \frac{\left(1+x^2\right)\cos ax}{1+x^2+x^4} dx = \frac{\pi}{\sqrt{3}} \exp\left(-\sqrt{3}a/2\right) \cos\left(a/2\right)$$

and

$$\mathbf{I}_2 = \int_0^\infty \frac{x \sin ax}{1+x^2+x^4} dx = \frac{\pi}{\sqrt{3}} \exp\left(-\sqrt{3}a/2\right) \sin\left(a/2\right)$$

Hint: Do both parts together. Note that

$$\mathbf{I}_1 = \frac{1}{2} \int_0^\infty \left[g(x) + g(-x)\right] \cos ax dx = \int_{-\infty}^\infty g(x) \cos ax dx$$

and similarly

$$\mathbf{I}_2 = \int_{-\infty}^\infty g(x) \sin ax dx$$

where

$$g(x) = \frac{1}{1-x+x^2}$$

Consider therefore

$$\mathbf{J} = \int_C g(z) e^{iaz} dz$$

on an appropriate contour.

10. Find the circle of convergence of the following series:

(a) $\sum_{k=1}^{\infty} \mathsf{k}^k \mathsf{z}^k$

(b) $\sum_{k=1}^{\infty} \frac{k!}{k^k} \mathsf{z}^k$

(c) $\sum_{k=1}^{\infty} (z+5i) \frac{k!}{k^k} z^k$

11. Sum the series

$$1 + \cos\theta + \frac{\cos 2\theta}{2!} + \frac{\cos 3\theta}{3!} + \cdots$$

for any real number θ. What if θ is complex?

12. Expand the function

$$e^{\sin z}$$

in a Taylor series about $z = 0$ up tp and including the term with z^5..

13. Let C be the quadrilateral with vertices ± 3, $\pm i$ traversed counterclockwise. Compute

$$\int_C e^{2/z} dz$$

by developing the integrand into a Laurent series.

14. Let $f(z) = \left(z^2+1\right)^{-3}$.

(a) Find all singularities of $f(z)$. Find the order of each pole.

(b) Find the residue of $f(z)$ at each isolated singularity.

(c) Evaluate

$$\int_C f(z)\, dz,$$

where C is the circle $|z-i| = 1$ with clockwise orientation.

15. Let

$$f(z) = \frac{z^2(z-\pi)}{\sin^2 z}$$

(a) Find all singularities of $f(z)$. Find the order of each pole.

(b) Find the residue of $f(z)$ at each isolated singularity.

(c) Evaluate

$$\int_C f(z)\, dz,$$

where C is the rectangular contour with vertices $-4 \pm 2i$, and $1 \pm 2i$ with counterclockwise orientation.

16. Compute

$$\int_{-\infty}^{\infty} \frac{dx}{(x^2+1)^2 (x^2+4)}$$

17. Compute

$$\int_{-\infty}^{\infty} \frac{xdx}{(x^2+9)^2}$$

18. Compute

$$\int_{-\infty}^{\infty} \frac{x^2 dx}{(x^2+16)^2}$$

19. Find al the Laurent series for

$$f(z) = \frac{1}{z^2 - 1}$$

expanded about $z_0 = 1$ and describe the domain of convergence for each series.

20. Consider the mapping

$$w = f(z) = \frac{\sin(\pi z)}{\sin(\pi / z)}$$

(a) Classify all singular points of it.

(b) Consider the positively oriented contour $C : |z - 2| = \frac{11}{10}$.Compute

$$\int_C f(z)\, dz.$$

21. Find the order of the pole at 0 of the function

$$f(z) = \frac{1}{\left(e^{-z^2} - 1 + z^2\right)^2}$$

22. Find and classify all singular points of

$$h(z) = \frac{1}{\sin z} - \frac{1}{z} + \frac{1}{z - \pi} + \frac{1}{z + \pi}$$

23. Evaluate

$$\int_{|z|=3} e^{1/z} \cos \frac{1}{z} dz$$

24. Find the residue at $z = 0$ of

$$f(z) = \frac{e^z}{\cos z - 1}$$

25. Find and classify all isolated singularities of

$$f(z) = \sin\left(\frac{z}{z + 1}\right)$$

26. Obtain all Laurent expansions of

$$f(z) = z^{-1} + (z - 1)^{-2} + (z + 2)^{-1}$$

about $z = 0$ and indicate where each is valid.

27. Find the first few terms in the Laurent expansion of

$$\frac{1}{z^2 (e^z - e^{-z})}$$

valid for $0 < |z| < \pi$.

28. Obtain three different Laurent expansion of

$$\frac{7z - 2}{z(z + 1)(z - 2)}$$

about $z = -1$.