ÇANKAYA UNIVERSITY
DEPARTMENT OF MATHEMATICS
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Math 352
Complex Calculus II
Worksheet 2

Problems

1. Show that the function $P(z) = z^5 + 15z + 1$ has precisely four zeros inside the annulus $3/2 < |z| < 2$.
2. Determine the number of solutions $e^z = 2z + 1$, with $|z| < 1$.
3. Show that the equation $z^6 + 4z^2 = 1$ has exactly two roots in the open disk $D = \{z \in \mathbb{C} : |z| < 1\}$.
4. Find the number of zeros of $f(z) = z^4 + z^3 + 5z^2 + 2z + 4$ in the first quadrant of the complex plane.
5. Using either the Argument Principle or Rouche's Theorem determine how many solutions
$$z^4 + z^3 = 2z^2 - 2z - 4$$
has in the first quadrant.
6. Consider the equation $2z^5 + 8z - 1 = 0$.
   (a) Show that all the roots of this equation lie in the open disk $D = \{z \in \mathbb{C} : |z| < 2\}$.
   (b) Show that this equation has exactly one root in the open dics $D = \{z \in \mathbb{C} : |z| < 1\}$.
   (c) How many roots does this equation have in the open annulus $D = \{z \in \mathbb{C} : 1 < |z| < 2\}$. Justify your assertion.
7. Show that the polynomial $g(z) = z^4 - 7z - 1$ has one zero in the disk $D = \{z \in \mathbb{C} : |z| < 1\}$.
8. Find a linear fractional transformation $T$ which maps the points $1, -i, -1$ to $i, 0, \infty$. What happens to the exterior of the unit circle under the mapping?
9. Find the most general linear fractional transformation that fixes the two points $1$, and $-1$.
10. Find the linear fractional transformation
$$w = \frac{az + b}{cz + d}$$
that maps $-1, 0, 1$ onto $-1,\ i,\ 1$ respectively and determine what the lower half plane is mapped onto.
11. Find a linear fractional transformation that maps onto such that $|z| \leq 1$ onto $|w| \leq 1$ such that $z = i/3$ is mapped onto $w = 0$. (You need to show that the transformation maps $|z| \leq 1$ onto $|w| \leq 1$.
12. Find and sketch the image of the region $\pi/4 < \text{Arg}(z) < \pi/2$ under the mapping $w = z^2$.
13. Find and sketch the image of the region $0 < \ y < 2$ under the mapping $w = e^z$.
14. Show that the transformation
$$w = \frac{i(1 - z)}{1 + z}$$
maps the unit disc $D = \{z \in \mathbb{C} : |z| < 1\}$ one-to-one and onto the upper half plane $H = \{w \in \mathbb{C} : \operatorname{Im} w > 0\}$.
15. Show that the mapping
$$w = \frac{(1 - i)z + 2}{(1 + i)z + 2}$$
maps the disk $D = \{z \in \mathbb{C} : |z + 1| < 1\}$ one-to-one and onto the upper half plane $H = \{w \in \mathbb{C} : \operatorname{Im} w > 0\}$.

16. Find all linear fractional transformations whose (only) fixed points are $-1$, $1$.

17. Find all linear fractional transformations without fixed points in the finite plane $\mathbb{C}$.

18. Find all linear fractional transformations that map the $x$-axis onto the $u$-axis.

19. Find the image of the semi-infinite strip $\{(x,y) : x > 0, 0 < y < 2\}$when $w = iz + 1$. Sketch the strip and its image.

20. Find a linear fractional transformation that maps $D = \{(r,\theta) : 2 \le r \le 3, \pi/4 \le \theta \le \pi/2\}$ under the mapping $w = Logz$.

21. Find and sketch the image of $D == \{(r,\theta) : 2 \le r \le 3, \pi/4 \le \theta \le \pi/2\}$ under the mapping $w =$Log$z$.

22. The branch $F$ of $\left(z^2-1\right)^{1/2}$ is defined in class (or see the textbook page 330)

$$F(z) = \sqrt{r_1 r_2}\exp\frac{i(\theta_1+\theta_2)}{2}$$

$r_k > 0, \quad 0 \le \theta_k < 2\pi, \quad (k = 1,2)$ and $r_1 + r_2 > 2$.

Explain geometrically why the conditions $r_1 > 0,\ 0 < \theta_1 + \theta_2 < \pi$ describe the quadrant $\{(u,v) : x > 0, y > 0\}$ of the plane. Then show that the transformation $w = F(z)$ maps that quadrant onto the quadrant $\{(u,v) : u > 0, v > 0\}$.

23. For the transformation $w = F(z)$ of the first quadrant of the $z$-plane onto the first quadrant of the $w$-plane in the Exercise 20, show that

$$u = \frac{1}{\sqrt{2}}\sqrt{r_1 r_2 + x^2 - y^2 - 1} \quad and\ v = \frac{1}{\sqrt{2}}\sqrt{r_1 r_2 - x^2 + y^2 + 1}$$

where

$$(r_1 r_2)^2 = \left(x^2 + y^2 + 1\right)^2 - 4x^2,$$

and image of the portion of the hyperbola $x^2 - y^2 = 1$ in the first quadrant is the ray $v = u$ $(u > 0)$.

24. Show that in Exercise 21 the domain $D$ that lies under the hyperbola and in the first quadrant of the $z$ plane is described by the conditions $r_1 > 0,\ 0 < \theta_1 + \theta_2 < \pi/2$. Then that the image of image of $D$ is the octant $\{(u,v) : 0 < v < u\}$. Sketch the domain $D$ and its image.

25. Write $z - 1 = r_1 \exp(i\theta_1)$ and $z + 1 = r_2 \exp(i\vartheta_2)$; where

$$0 < \theta_1 < 2\pi \quad and \quad -\pi < \vartheta_2 < \pi,$$

to define a branch of the function

(a) $\left(z^2-1\right)^{1/2}$

(b) $\left(\frac{z-1}{z+1}\right)^{1/2}$

In each case, the branch cut should consist of the two rays $\theta_1 = 0$ and $\vartheta_2 = \pi$.