Your Name / Adınız - Soyadınız

Your Signature / İmza

Student ID # / Öğrenci No

Professor's Name / Öğretim Üyesi

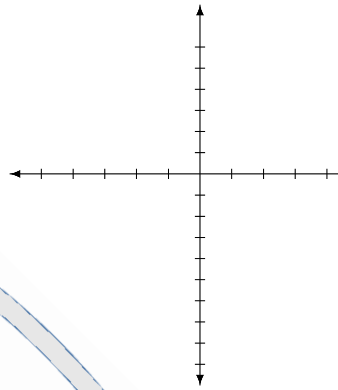
Your Department / Bölüm

- This exam is closed book.
- Give your answers in exact form (for example $\frac{\pi}{3}$ or $5\sqrt{3}$), except as noted in particular problems.
- Calculators, cell phones are not allowed.
- In order to receive credit, you must **show all of your work**. If you do not indicate the way in which you solved a problem, you may get little or no credit for it, even if your answer is correct. **Show your work in evaluating any limits, derivatives**.
- Place a box around your answer to each question.
- If you need more room, use the backs of the pages and indicate that you have done so.
- Do not ask the invigilator anything.
- Use a **ball-point pen** to fill the cover sheet. Please make sure that your exam is complete.
- Time limit is 90 min.

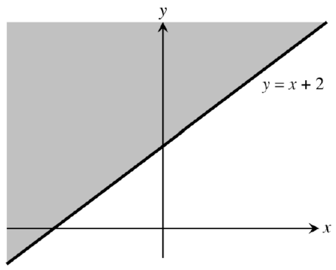
Do not write in the table to the right.

Problem	Points	Score
1	25	
2	25	
3	25	
4	25	
Total:	100	

1. (a) 8 points Find and sketch the domain for $f(x,y) = \sqrt{y-x-2}$.



Solution:
Domain: all points (x, y) on or above the line
y = x + 2



p.753, pr.5

(b) 8 points $\lim\limits_{\substack{(x,y)\to(1,1)\\ x\neq 1}} \dfrac{xy-y-2x+2}{x-1} = ?$

Solution:

$$\begin{aligned}\lim_{\substack{(x,y)\to(1,1)\\ x\neq 1}} \frac{xy-y-2x+2}{x-1} &= \lim_{\substack{(x,y)\to(1,1)\\ x\neq 1}} \frac{(x-1)(y-2)}{x-1}\\ &= \lim_{(x,y)\to(1,1)} (y-2)\\ &= (1-2) = -1\end{aligned}$$

p.762, pr.15

(c) 9 points $\lim\limits_{(x,y)\to(0,0)} g(x,y) = \dfrac{x-y}{x+y} = ?$

Solution: Along $y = kx$, we have

$$\begin{aligned}\lim_{(x,y)\to(0,0)} \frac{x-y}{x+y} &= \lim_{(x,kx)\to(0,0)} \frac{x-(kx)}{x+(kx)}\\ &= \lim_{x\to 0} \frac{1-k}{1+k}\\ &= \frac{1-k}{1+k}\end{aligned}$$

Different limits for different values of k so the original limit DOES NOT EXIST.

p.762, pr.45

2. (a) 10 points Suppose $w = \frac{x}{z} + \frac{y}{z}$, $x = \cos^2 t$, $y = \sin^2 t$, and $z = \frac{1}{t}$. Find $\left[\frac{dw}{dt}\right]_{t=3}$.

Solution:
$\frac{\partial w}{\partial x} = \frac{1}{z}$, $\frac{\partial w}{\partial y} = \frac{1}{z}$, $\frac{\partial w}{\partial z} = \frac{-(x+y)}{z^2}$, $\frac{dx}{dt} = -2\cos t \sin t$, $\frac{dy}{dt} = 2\sin t \cos t$, $\frac{dz}{dt} = -\frac{1}{t^2}$,

$\Rightarrow \frac{dw}{dt} = -\frac{2}{z}\cos t \sin t + \frac{2}{z}\sin t \cos t + \frac{x+y}{z^2 t^2} = \frac{\cos^2 t + sin^2 t}{\left(\frac{1}{t^2}\right)(t^2)}$;

$w = \frac{x}{z} + \frac{y}{z} = \frac{\cos^2 t}{\left(\frac{1}{t}\right)} + \frac{\sin^2 t}{\left(\frac{1}{t}\right)} = t \Rightarrow \frac{dw}{dt} = 1$

$\Rightarrow \frac{dw}{dt}(3) = 1$ p.790, pr.15

(b) 15 points If $w = \ln(x^2 + y^2 + z^2)$, $x = ue^v \sin u$, $y = ue^v \cos u$, $z = ue^v$, find $\frac{\partial w}{\partial u}$ and $\frac{\partial w}{\partial v}$ at $(u, v) = (-2, 0)$.

Solution: : Note that

$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial u},$$

and

$$x^2 + y^2 + z^2 = u^2e^{2v}\sin^2 u + u^2e^{2v}\cos^2 u + u^2e^{2v} = 2u^2e^{2v}.$$

Then, we get

$$\begin{aligned}
\frac{\partial w}{\partial x}\frac{\partial x}{\partial u} &= \frac{2x}{x^2+y^2+z^2}(e^v \sin u + ue^v \cos u) = \frac{2ue^v \sin u}{2u^2e^{2v}}(e^v \sin u + ue^v \cos u) \\
&= \frac{e^v \sin^2 u + ue^v \sin u \cos u}{ue^v}, \\
\frac{\partial w}{\partial y}\frac{\partial x}{\partial u} &= \frac{2y}{x^2+y^2+z^2}(e^v \cos u - ue^v \sin u) = \frac{e^v \cos^2 u - ue^v \cos u \sin u}{ue^v}, \\
\frac{\partial w}{\partial z}\frac{\partial z}{\partial u} &= \frac{2z}{x^2+y^2+z^2}(e^v) = \frac{2ue^v}{2u^2e^{2v}}(e^v) = \frac{1}{u}.
\end{aligned}$$

This implies that

$$\frac{\partial w}{\partial u} = \frac{e^v}{ue^v} + \frac{1}{u} = \frac{2}{u} \Longrightarrow \frac{\partial w}{\partial u}|_{u=-2} = -1.$$

Similarly, we have

$$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial v}.$$

Each term can be calculated as follows.

$$\begin{aligned}
\frac{\partial w}{\partial x}\frac{\partial x}{\partial v} &= \frac{2x}{x^2+y^2+z^2}(ue^v \sin u) = \frac{2ue^v \sin u}{2u^2e^{2v}}(ue^v \sin u) = \sin^2 u. \\
\frac{\partial w}{\partial y}\frac{\partial x}{\partial v} &= \frac{2y}{x^2+y^2+z^2}(ue^v \cos u) = \frac{\cos u}{ue^v}(ue^v \cos u) = \cos^2 u. \\
\frac{\partial w}{\partial z}\frac{\partial z}{\partial u} &= \frac{2z}{x^2+y^2+z^2}(ue^v) = \frac{2ue^v}{2u^2e^{2v}}(ue^v) = 1.
\end{aligned}$$

This implies that

$$\frac{\partial w}{\partial v} = 2 \Longrightarrow \frac{\partial w}{\partial v}|_{v=0} = 2.$$

p.753, pr.5

3. (a) 8 points Find the derivative of $f(x,y,z) = xy + yz + xz$ at $P_0(1,-1,2)$ in the direction of $\mathbf{A} = 3\mathbf{i} + 6\mathbf{j} - 2\mathbf{k}$.

Solution:
$\mathbf{u} = \frac{\mathbf{A}}{|\mathbf{A}|} = \frac{3\mathbf{i} + 6\mathbf{j} - 2\mathbf{k}}{\sqrt{(3)^2 + 6^2 + (-2)^2}} = \frac{3}{7}\mathbf{i} + \frac{6}{7}\mathbf{j} - \frac{2}{7}\mathbf{k}$; $f_x(x,y,z) = y + z \Rightarrow f_x(1,-1,2) = 1$; $f_y(x,y,z) = x + z \Rightarrow$

$f_y(1,-1,2) = 3$; $f_z(x,y,z) = y + x \Rightarrow f_z(1,-1,2) = 0 \Rightarrow \nabla f = \mathbf{i} + 3\mathbf{j} \Rightarrow (D_{\mathbf{u}} f)_{P_0} = \nabla f \cdot \mathbf{u} = \frac{3}{7} + \frac{18}{7} = 3.$

p.790, pr.15

(b) 9 points Find the directions in which $f(x,y) = x^2 + xy + y^2$ increases and decreases most rapidly at $P_0(-1,1)$. Then find the derivatives in these directions.

Solution: $\nabla f = (2x+y)\mathbf{i} + (x+2y)\mathbf{j} \Rightarrow \nabla f(-1,1) = -\mathbf{i} + \mathbf{j} \Rightarrow \mathbf{u} = \frac{\nabla f}{|\nabla f|} = \frac{-\mathbf{i}+\mathbf{j}}{\sqrt{(-1)^2+1^2}} = -\frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}$; f increases most rapidly in the direction $\mathbf{u} = -\frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}$ and decreases most rapidly in the direction $-\mathbf{u} = \frac{1}{\sqrt{2}}\mathbf{i} - \frac{1}{\sqrt{2}}\mathbf{j}$;

$(D_{\mathbf{u}} f)_{P_0} = \nabla f \cdot \mathbf{u} = |\nabla f| = \sqrt{2}$ and $(D_{-\mathbf{u}} f)_{P_0} = -\sqrt{2}$

p.791, pr.19

(c) 8 points In what direction is the derivative of $f(x,y) = xy + y^2$ at $P(3,2)$ equal to zero? Give your reasons.

Solution: $\nabla f = y\mathbf{i} + (x+2y)\mathbf{j} \Rightarrow \nabla f(3,2) = 2\mathbf{i} + 7\mathbf{j}$; a vector orthogonal to ∇f is $\mathbf{v} = 7\mathbf{i} - 2\mathbf{j} \Rightarrow \mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{7\mathbf{i} - 2\mathbf{j}}{\sqrt{(7)^2+(-2)^2}} = \frac{7}{\sqrt{53}}\mathbf{i} - \frac{2}{\sqrt{53}}\mathbf{j}$ and $-\mathbf{u} = -\frac{7}{\sqrt{53}}\mathbf{i} + \frac{2}{\sqrt{53}}\mathbf{j}$ are the directions where the derivative is zero.

p.791, pr.31

4. (a) 12 points Suppose $x^2 + 2xy - y^2 + z^2 = 7$. Find equations for the **(a)** tangent plane and **(b)** normal line at the point $P_0(1,-1,3)$ on the surface.

Solution: Let

$$F(x,y,z) := x^2 + 2xy - y^2 + z^2 - 7$$

so that

$$\nabla F(x,y,z) = (2x+2y)\mathbf{i} + (2x-2y)\mathbf{j} + 2z\mathbf{k}$$

and $\nabla F(1,-1,3) = (2(1)+2(-1)+2(3))\mathbf{i} + (2(-3)-2(2)+3)\mathbf{j} + (-1)\mathbf{k} = 4\mathbf{j} + 6\mathbf{k}$. Hence

1. the equation of the tangent plane at $P_0(1,-1,3)$ is

$$0(x-1) + 4(y+1) + 6(z-3) = 0$$

   or $2y + 3z = 7$.

2. Similarly, the equations of the normal line through $P_0(1,-1,3)$ are

$$x = 1, y = -1 + 4t, z = 3 + 6t.$$

p.799, pr.4

(b) 13 points Find all the local maxima, local minima, and the saddle points of $f(x,y) = x^2 - y^2 - 2x + 4y + 6$.

Solution: We first compute the partial derivatives: $f_x(x,y) = 2x - 2 = 0$ and $f_y(x,y) = -2y + 4 = 0$. Since both partial derivatives are defined for all (x,y), the critical points are solutions for the two equations

$$f_x = 2x - 2 = 0$$

and

$$f_y = -2y + 4 = 0.$$

Hence $x = 1$ $y = 2$, critical point is $(1,2)$;
for $(1,2)$: $f_{xx}(1,2) = 2$, $f_{yy}(1,2) = -2$,
$f_{xy}(1,2) = 0 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = -4 < 0 \Rightarrow$ SADDLE POINT

p.808, pr.9