Your Name / Adınız - Soyadınız

Your Signature / İmza

Student ID # / Öğrenci No

Professor's Name / Öğretim Üyesi

Your Department / Bölüm

- Calculators, cell phones off and away!.
- In order to receive credit, you must **show all of your work**. If you do not indicate the way in which you solved a problem, you may get little or no credit for it, even if your answer is correct. **Show your work in evaluating any limits, derivatives**.
- Place $\boxed{\text{a box around your answer}}$ to each question.
- Use a **BLUE ball-point pen** to fill the cover sheet. Please make sure that your exam is complete.
- Time limit is 80 min.

Do not write in the table to the right.

Problem	Points	Score
1	22	
2	25	
3	23	
4	30	
Total:	100	

1. (a) $\boxed{10 \text{ Points}}$ *Evaluate* the integral $\displaystyle\int \frac{\sin(5t)}{1+(\cos(5t))^2}\,dt$.

Solution: Let $u=\cos(5t)$. Then $du=-5\sin(5t)\,dt$. Hence the integral becomes

$$\begin{aligned}\int \frac{\sin(5t)}{1+(\cos(5t))^2}\,dt &= -\frac{1}{5}\int \frac{1}{1+(\cos(5t))^2}\underbrace{(-5)\sin(5t)\,dt}_{du}\\ &= -\frac{1}{5}\int\frac{1}{1+u^2}\,du\\ &= -\frac{1}{5}\tan^{-1}u+c\\ &= \boxed{-\frac{1}{5}\tan^{-1}(\cos(5t))+c}\end{aligned}$$

p.652, pr.3

(b) $\boxed{12 \text{ Points}}$ Investigate the convergence of the *series* $\displaystyle\sum_{n=1}^{\infty}(-1)^n\tanh n$. Name the test you use.

Solution: Let $a_n=(-1)^n\tanh n$. Then

$$a_n=(-1)^n\tanh n = a_n=(-1)^n\frac{e^n-e^{-n}}{e^n+e^{-n}}=(-1)^n\frac{e^n-e^{-n}}{e^n+e^{-n}}\frac{e^{-n}}{e^{-n}}=(-1)^n\frac{1-e^{-2n}}{1+e^{-2n}}.$$

If we take the limit, then

$$\lim_{n\to\infty}e^{-2n}=0\Rightarrow\lim_{n\to\infty}\frac{1-e^{-2n}}{1+e^{-2n}}=\frac{1-0}{1+0}=1.$$

Therefore, we have

$$\lim_{n\to\infty}a_n=\lim_{n\to\infty}(-1)^n\tanh n=\lim_{n\to\infty}(-1)^n\,()=\underbrace{\lim_{n\to\infty}(-1)^n}_{\text{D. N. E.}}\underbrace{\left(\lim_{n\to\infty}\frac{1-e^{-2n}}{1+e^{-2n}}\right)}_{1}=\pm1$$

which does not exist. Hence by the *nth Term Test*, the series *diverges*.

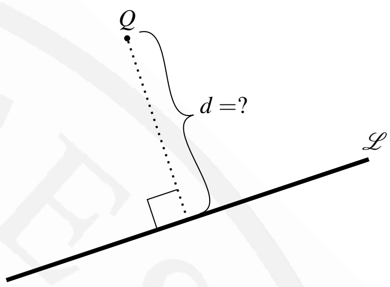
p.533, pr.95

2. Given the point $Q(0,4,1)$ and the line $\mathscr{L}: \begin{cases} x = 2+t, \\ y = 2+t, \\ z = t \end{cases}$.

(a) $\boxed{\text{12 Points}}$ Find the distance from the point Q to the line $\mathscr{L}$.

Solution: We shall use the distance formula $d = \dfrac{|\vec{PQ}\times \mathbf{v}|}{|\mathbf{v}|}$. Here (by letting $t=0$) $P(2,2,0)$ is a point on $\mathscr{L}$ and $\mathbf{v} = \mathbf{i}+\mathbf{j}+\mathbf{k}$ is a vector that is parallel to $\mathscr{L}$. Now we have $\vec{PQ} = -2\mathbf{i}+2\mathbf{j}+\mathbf{k}$ and so

$$\vec{PQ}\times\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 2 & 1 \\ 1 & 1 & 1 \end{vmatrix} = \mathbf{i}\begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} - \mathbf{j}\begin{vmatrix} -2 & 1 \\ 1 & 1 \end{vmatrix} + \mathbf{k}\begin{vmatrix} -2 & 2 \\ 1 & 1 \end{vmatrix}$$
$$= \mathbf{i}+3\mathbf{j}-4\mathbf{k}$$

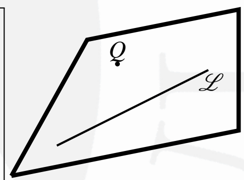


Therefore, we have

$$d = \frac{|\vec{PQ}\times\mathbf{v}|}{|\mathbf{v}|} = \frac{\sqrt{1+9+16}}{\sqrt{1+1+1}} = \frac{\sqrt{26}}{\sqrt{3}} = \boxed{\frac{\sqrt{78}}{3}}$$

p.695, pr.37

(b) $\boxed{\text{13 Points}}$ Find the equation of the plane which contains both the point Q and the line $\mathscr{L}$.



Solution: We know from part (a) that the points $P(2,2,0)$ and $Q(0,4,1)$ are on the plane. Setting $t=1$, we get another point $R(3,3,1)$ which is also on the plane. Let $\mathbf{a}$ be the vector from $R(3,3,1)$ to $P(2,2,0)$;

$$\mathbf{a} = (2-3)\mathbf{i}+(2-3)\mathbf{j}+(0-1)\mathbf{k} = -\mathbf{i}-\mathbf{j}-\mathbf{k}.$$

Let $\mathbf{b}$ be the vector from $R(3,3,1)$ to $Q(0,4,1)$;

$$\mathbf{b} = (0-3)\mathbf{i}+(4-3)\mathbf{j}+(1-1)\mathbf{k} = -3\mathbf{i}+\mathbf{j}+0\mathbf{k}.$$

A normal vector $\mathbf{n}$ for the plane may be found by means of cross products.

$$\mathbf{n} = \mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & -1 & -1 \\ -3 & 1 & 0 \end{vmatrix} = \mathbf{i}\begin{vmatrix} -1 & -1 \\ 1 & 0 \end{vmatrix} - \mathbf{j}\begin{vmatrix} -1 & -1 \\ -3 & 0 \end{vmatrix} + \mathbf{k}\begin{vmatrix} -1 & -1 \\ -3 & 1 \end{vmatrix}$$
$$= \mathbf{i}+3\mathbf{j}-4\mathbf{k}$$

The general equation of a plane, as we know, is:

$$\underbrace{(\mathbf{i}+3\mathbf{j}-4\mathbf{k})}_{\mathbf{n}}\cdot\underbrace{((x-0)\mathbf{i}+(y-4)\mathbf{j}+(z-1)\mathbf{k})}_{\vec{QP}} = 0$$
$$\Rightarrow (x-0)+3(y-4)-4(z-1) = 0$$
$$\Rightarrow x+3y-12-4z+4 = 0$$
$$\Rightarrow \boxed{x+3y-4z = 8}$$

p.695, pr.37

3. (a) $\boxed{\text{13 Points}}$ Find the largest value that the directional derivetive of $f(x,y,z) = xyz$ can have at the point $(1,1,1)$.

Solution: First we find the partial derivatives.

$$
\begin{aligned}
f_x(x,y,z) &= yz \\
f_y(x,y,z) &= xz \\
f_z(x,y,z) &= xy \\
\nabla f(x,y,z) &= f_x\,\mathbf{i} + f_y\,\mathbf{j} + f_z\,\mathbf{k} \\
&\Rightarrow \nabla f(x,y,z) = yz\,\mathbf{i} + xz\,\mathbf{j} + xy\,\mathbf{k} \\
\text{at } (1,1,1) &\Rightarrow \nabla f(1,1,1) = \mathbf{i} + \mathbf{j} + \mathbf{k} \\
&\Rightarrow \text{the maximum value of } D_{\mathbf{u}} f(1,1,1) = |\nabla f(1,1,1)| = |\mathbf{i}+\mathbf{j}+\mathbf{k}| = \sqrt{1^2+^2+1^2} = \boxed{\sqrt{3}}
\end{aligned}
$$

p.879, pr.42

(b) $\boxed{\text{10 Points}}$ Show that if $w = f(s)$ is any differentiable function of s and if $s = y + 5x$, then

$$\frac{\partial w}{\partial x} - 5\frac{\partial w}{\partial y} = 0.$$

Solution: Since

$$\frac{\partial s}{\partial x} = \frac{\partial}{\partial x}(y+5x) = 5$$

and

$$\frac{\partial s}{\partial y} = \frac{\partial}{\partial y}(y+5x) = 1$$

we have, by Chain Rule,

$$
\begin{aligned}
\frac{\partial w}{\partial x} &= \frac{dw}{ds}\frac{\partial s}{\partial x} \\
&= (f'(s))(5) \\
&= 5f'(s)
\end{aligned}
$$

and

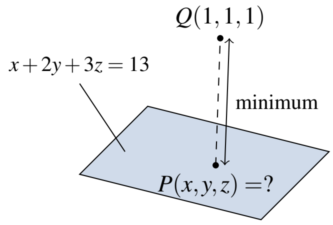
$$
\begin{aligned}
\frac{\partial w}{\partial y} &= \frac{dw}{ds}\frac{\partial s}{\partial y} \\
&= (f'(s))(1) \\
&= f'(s).
\end{aligned}
$$

Therefore

$$\frac{\partial w}{\partial x} - 5\frac{\partial w}{\partial y} = \underbrace{5f'(s)}_{\partial w/\partial x} - 5\underbrace{f'(s)}_{\partial w/\partial y} = 0.$$

p.879, pr.34

4. (a) $\boxed{\text{15 Points}}$ Find the point on the plane $x+2y+3z=13$ *closest* to the point $Q(1,1,1)$.



Solution: We want to give three solutions.

$\boxed{\textit{Solution A}}$: The problem is asking us to find the minimum value of the function $|PQ|$, which is the length of the vector from the origin to P subject to the constraint $x+2y+3z=13$.

A fact that we should keep in mind is that $|PQ|$ will have a minimum value wherever the function $f(x,y,z)=(x-1)^2+(y-1)^2+(z-1)^2$ has a minimum value. So the problem turns into find the minimum value of $f(x,y,z)=(x-1)^2+(y-1)^2+(z-1)^2$ subject to the constraint $g(x,y,z)=x+2y+3z-13=0$.

$$\nabla f = 2(x-1)\mathbf{i}+2(y-1)\mathbf{j}+2(z-1)\mathbf{k}$$
$$\nabla g = \mathbf{i}+2\mathbf{j}+3\mathbf{k}$$
$$\nabla f = \lambda \nabla g \longrightarrow 2(x-1)=\lambda,\ 2(y-1)=2\lambda,\ 2(z-1)=3\lambda$$
$$2(y-1)=2\lambda \longrightarrow 2(y-1)=2(2(x-1)) \longrightarrow 2y-2=4x-4$$

So, we have

$$2y-4x=-2 \longrightarrow y-2x=-1 \longrightarrow y=-1+2x$$
$$2(z-1)=3(2(x-1)) \longrightarrow 2z-2=6x-6 \longrightarrow 2z-6x=-4$$
$$\longrightarrow z-3x-2 \longrightarrow z=3x-2$$
$$x+2y+3z=13 \longrightarrow x+2(2x-1)+3(3x-2)=13 \longrightarrow x+4x-2+9x-6=13$$
$$\longrightarrow 14x=21 \longrightarrow x=\frac{3}{2} \longrightarrow y=2\left(\frac{3}{2}\right)-1=2, \longrightarrow z=3\left(\frac{3}{2}\right)-2=\frac{5}{2}.$$

Therefore, $\boxed{(5/2\,,2\,,3/2)}$ is the point on the plane closest to the point $Q(1,1,1)$.

p.695, pr.37

Solution: $\boxed{\textit{Solution B}}$: We need to minimize the square of the distance function $f(x,y,z)=(x-1)^2+(y-1)^2+(z-1)^2$ with $x=13-2y-3z$. That is, the function

$$g(y,z)=f(13-2y-3z,y,z)=((13-2y-3z)-1)^2+(y-1)^2+(z-1)^2=(12-2y-3z)^2+(y-1)^2+(z-1)^2.$$

will be minimized. Computing partials and setting them equal to zero:

$$g_y=2(12-2y-3z)(-2)+2(y-1)=0 \longrightarrow -48+8y+12z+2y-2=0 \longrightarrow \boxed{5y+6z=25}$$
$$g_z=2(12-2y-3z)(-3)+2(z-1)=0 \longrightarrow -72+12y+18z+2z-2=0 \longrightarrow \boxed{6y+10z=37}.$$

Therefore, we have a system of 2 linear (boxed) equations in the unknowns y, z. We solve this system.

$$y=\frac{\begin{vmatrix}25 & 6\\ 37 & 10\end{vmatrix}}{\begin{vmatrix}5 & 6\\ 6 & 10\end{vmatrix}}=\frac{(25)(10)-(6)(37)}{(5)(10)-(6)(6)}=\frac{250-222}{50-36}=\frac{28}{14}=2$$

$$z=\frac{\begin{vmatrix}5 & 25\\ 6 & 37\end{vmatrix}}{\begin{vmatrix}5 & 6\\ 6 & 10\end{vmatrix}}=\frac{(5)(37)-(6)(25)}{(5)(10)-(6)(6)}=\frac{185-150}{50-36}=\frac{35}{14}=\frac{5}{2}$$

Since $x=13-2y-3z$, we have $x=13-2(2)-3(\frac{5}{2})=9-\frac{15}{2}=\frac{5}{2}$. Therefore $\boxed{\left(\frac{5}{2},2,\frac{3}{2}\right)}$ is the solution to the system and hence is the closest point, we want, on the plane.

Solution: $\boxed{\textit{Solution C}}$: If $\mathbf{n}$ is normal to the plane $x+2y+3z=13$, then $\mathbf{n}=\mathbf{i}+2\mathbf{j}+3\mathbf{k}$. Then the equations

$$\mathscr{L}:\begin{cases} x=4-t, \\ y=3+2t, \\ z=-5+3t \end{cases} .$$

are the parametric equations of the line through Q and parallel to $\mathbf{n}$. This line $\mathscr{L}$ intersects the plane exactly when

$$\underbrace{\underbrace{1+t}_{x}+2\underbrace{(1+2t)}_{y}+3\underbrace{(1+3t)}_{z}}_{x+2y+3z}=13.$$

That is, when
$14t=7$ or $t=\dfrac{1}{2}$, or $(x,y,z)=(1+t,1+2t,1+3t)=\left(1+(\frac{1}{2}),1+2(\frac{1}{2}),1+3(\frac{1}{2})\right)=\left(\frac{3}{2},2,\frac{5}{2}\right)$. Since the length of the segment from Q to $\left(\frac{3}{2},2,\frac{5}{2}\right)$ is the minimal distance from the plane to the point $Q(1,1,1)$, the required closest point is $\boxed{\left(\frac{3}{2},2,\frac{5}{2}\right)}$.

(b) $\boxed{\text{15 Points}}$ Find the local maxima and minima and saddle points for $f(x,y)=x^3+y^3-3xy+15$. Find function's value at these points.

Solution:

$$\begin{aligned} f_x=3x^2-3y&=0 \\ 3(y^2)^2-3y&=0 \\ 3y^4-3y&=0 \\ 3y(y^3-1)&=0 \\ y=0 \quad y&=1 \\ x=0 \quad x&=1 \end{aligned} \qquad\qquad \begin{aligned} f_y=3y^2-3x&=0 \\ 3y^2&=3x \\ x&=y^2 \end{aligned}$$

The critical points for this function are $(0,0)$ and $(1,1)$. Now we have

$$f_{xx}=6x, \qquad f_{yy}=6y, \qquad f_{xy}=-3, \qquad f_{xx}f_{yy}-(f_{xy})^2=(6x)(6y)-(-3y)^2=36xy-9y^2.$$

At $(0,0)$, we have

$$f_{xx}(0,0)f_{yy}(0,0)-(f_{xy}(0,0))^2=(6(0))(6(0))-(-3)^2=36(0)(0)-9=-9<0.$$

So f has a saddle point at $(0,0)$ and $f(0,0)=15$.
At $(1,1)$, we have

$$f_{xx}(1,1)f_{yy}(1,1)-(f_{xy}(1,1))^2=(6(1))(6(1))-(-3)^2=36(1)(1)-9=27>0 \qquad \text{and} \qquad f_{xx}(1,1)=6>0$$

So f has a local minimum at $(1,1)$ and $f(1,1)=14$.

p.880, pr.68